ANALYSIS OF NON-UNIFORM BEAM UNDER BENDING DUE TO INERTIA IMPACT LOADING

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Abstract: This paper evaluates the impact caused by inertia force of beam with non-uniform mass distribution. The impact is made by non-uniform beam falling on supports. Then, stresses of this model are analysed and optimized by using position of the supporting forces, mass distribution and convenient geometry. Results of this study, present the solutions for better design of the beam and having the highest resistance for it, and also have some suggestions for better geometry until the beam get stronger. In this paper, useful relations between bending stress on the beam due to inertia shock, mass distribution and position of supporting forces are demonstrated. The study reveals some interesting findings related to the best bar configuration in terms of geometrical design, design of supports location, mass distribution and use of multi-material components. The studied problem takes place in many of the mechanical systems such as exhaust and suspension systems and etc.

Keywords: inertia impact loading, stepped shaft, multi-material components, exhaust, suspension system

1. INTRODUCTION

In recent decades, prediction of behaviour and optimization of mechanical components under shock stresses has been the main concern of engineers and designers [1, 2]. Shock loads that can be applied to objects, are different [3-6]. A practical type of shock loads is severe inertia impact loading. This load allocates to a mass that has velocity but somehow is prevented to move. In fact, the brake factor creates a force into bar.

Optimization of a bar in bending subjected to inertia impact loading with uniform mass has been investigated and some results have also been obtained [7]. But generally, in real conditions, we encounter with non-uniform mass distribution in components or may be dealing with multi-material components. So, previous achievements cannot be held accountable for the majority of real examples. Among components with non-uniform mass distribution under impact loads, stepped shafts have many applications in industry, especially in mechanical machines, suspension systems etc. [8, 9]. Consequently, proven and useful equations for position of supporting forces and mass distribution of the stepped beams can be very important for designers and researchers, this study discusses about a beam with one symmetric step in centre of it, which is the basis for more complex cases.

The purpose of this study is stress analysis of beam in bending subjected to severe inertia impact loading with non-uniform mass and optimum design of this beam in terms geometry, location of supporting forces and mass distribution. Additionally, the study can be used to a beam made of two materials. This is explained in section 4.

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2. MODEL AND THEORY

2.1. Bar geometry

The studied bar in this paper is symmetric bar with length 3*L* and circular section. That middle and sides diameter of the bar is *D* and *d*, respectively. This subject is shown in Figure 1.

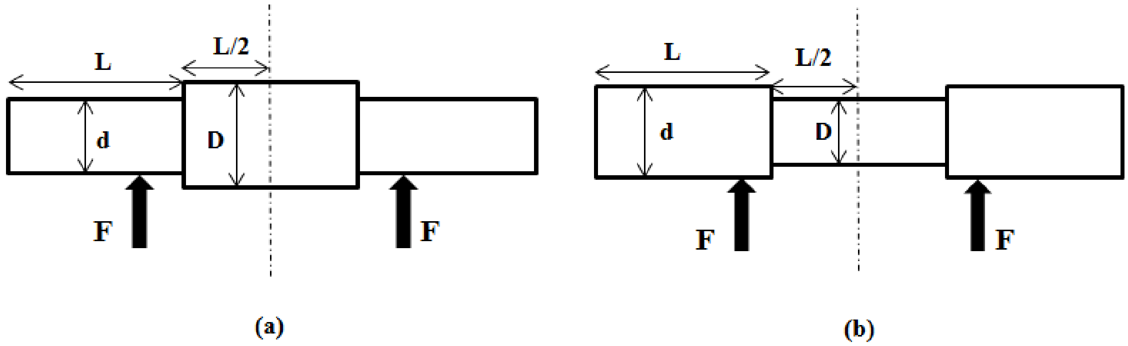


Fig. 1. Beam geometry: (a) for $k>0$ $(D>d)$; (b) for $k<0$ $(D<d)$.

2.2. Strain energy

In this section, is assumed that bar of mass *m* and velocity V_1 hits with stationary object of mass *M* ($M>>m$), that in this case, stationary object acts as support for bar and two symmetric contact points make with bar (see Figure 1).

Also, regarding the relative motion, can be considered that the bar of mass *m* is stationary and the mass *M*, with velocity V_1, hits with bar. Consequently, locations of collision between the bar and supports generate two symmetric supporting forces on bar.

From the momentum balance:

$$MV_1 = (M+m)V \tag{1}$$

where *V* is velocity of mass *m* and *M* after collision, and from the conservation of energy:

$$\frac{1}{2}MV_1^2 = \frac{1}{2}(M+m)V^2 + U \tag{2}$$

with ($M \gg m$) we obtain:

$$U < \frac{1}{2}mV^2 \rightarrow U = \frac{1}{2}mV^2 = \frac{1}{4}\beta\rho\pi d^2 LV^2 \text{ or } \frac{1}{4}\beta\rho\pi d^2 LV^2 \tag{3}$$

where *U* is strain energy and $0<\beta<1/2$.

2.3. Calculation of bending stress

The inertia force is a non-uniformly distributed load *w* acting along the length of the bar, that distribution of this load depends on mass distribution along the bar (Figure 2). If the density of bar is constant in all around of the bar, then the following equations (equations 4 and 5) is true.

$$k = \frac{D}{d}, (k \neq 1) \tag{4}$$

$$w_2 = kw_1 \tag{5}$$

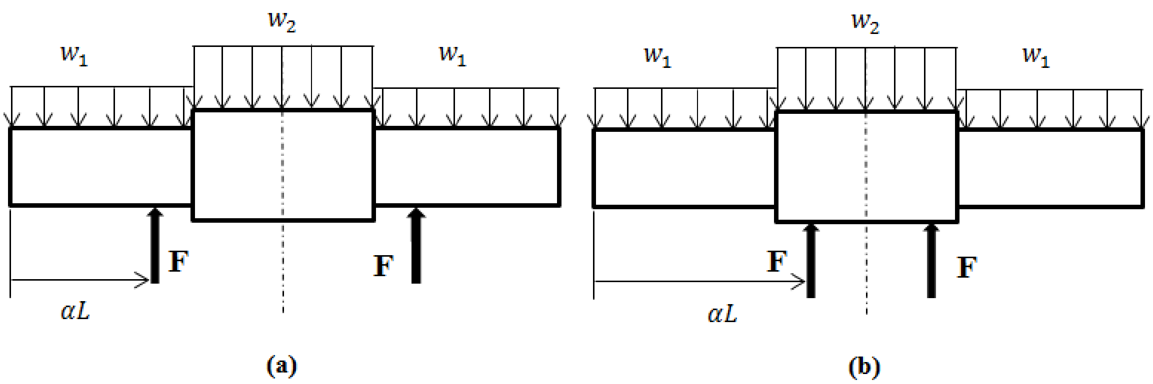


Fig. 2. Distribution of inertia load on bar: (a) status I ($0\leq\alpha\leq1$); (b) status II ($1\leq\alpha\leq3/2$).

Finally, the amount of supporting force F is calculated as follows:

$$F=\frac{(w_2+2w_1)}{2}L=\frac{(k+2)}{2}w_1L \tag{6}$$

Now, two statuses are considered status I, position of supporting forces α is in the range of ($0\leq\alpha\leq1$) and status II, position of supporting forces is in the range of ($1\leq\alpha\leq3/2$) (see Figure 2).

For each range of bar length is calculated bending moment. Status I ($0\leq\alpha\leq1$):

$$M_{1I}=\frac{w_1}{2}x^2,\left(0\leq x\leq\alpha L\right) \tag{7}$$

$$M_{2I}=\frac{w_1}{2}x^2-F(x-\alpha L),\left(\alpha L\leq x\leq L\right) \tag{8}$$

$$M_{3I}=w_1L(x-\frac{L}{2})+\frac{w_2}{2}(x-L)^2-F(x-\alpha L),(L\leq x\leq\frac{3L}{2}) \tag{9}$$

Status II ($1\leq\alpha\leq3/2$):

$$M_{1II}=\frac{w_1}{2}x^2,\left(0\leq x\leq L\right) \tag{10}$$

$$M_{2II}=w_1L(x-\frac{L}{2})+\frac{w_2}{2}(x-L)^2,\left(L\leq x\leq\alpha L\right) \tag{11}$$

$$M_{3II}=w_1L(x-\frac{L}{2})+\frac{w_2}{2}(x-L)^2-F(x-\alpha L),\left(\alpha L\leq x\leq\frac{3L}{2}\right) \tag{12}$$

Now the total strain energy for the bar in bending, as given in any solid mechanics text book [10] and considering the symmetry of the bar is obtained as follows:

$$U=\int_0^{3L/2}\frac{M^2}{2EI}dx \tag{13}$$

This parameter should be calculated for both of statuses I and II separately.

For status I:

$$U_I = \int_0^{\alpha L} \frac{M_{1I}^2}{2EI} dx + \int_L^{\alpha L} \frac{M_{2I}^2}{2EI} dx + \int_L^{3L/2} \frac{M_{3I}^2}{2EI_D} dx \qquad (14)$$

For status II:

$$U_{II} = \int_0^{L} \frac{M_{1I}^2}{2EI} dx + \int_L^{\alpha L} \frac{M_{2I}^2}{2EI} dx + \int_{\alpha L}^{3L/2} \frac{M_{3I}^2}{2EI_D} dx \qquad (15)$$

where E is modulus of elasticity of the bar material and ($I = \frac{\pi d^4}{64}$) is inertia moment of side sections and ($I_D = \frac{\pi D^4}{64}$) is inertia moment of middle section.

Substituting equations (7), (8) and (9) into equation (14) the following relation for status I is reached:

$$U_I = \frac{w_1^2 L^5}{3840EI} P_I(\alpha, k) \qquad (16)$$

where:

$$\begin{aligned} p_I(\alpha,k) = &-80(k+2)\alpha^4 - 4(40k^2+151k+151)\alpha^3 + 240(k^6+4k^5+4k^4+2k^2+8k+8)\alpha^2 \\ &-40(14k^6+39k^5+22k^4+6k^2+40k+32)\alpha + 328k^6+511k^5 \\ &+203k^4+160k^2+400k+256 \end{aligned} \qquad (17)$$

And:

$$w_1 = \sqrt{\frac{3840EIU_I}{P_I(\alpha,k)L^5}} \qquad (18)$$

Also, substituting equations (10), (11) and (12) into equation (15) the following relations for status II is reached:

$$U_{II} = \frac{w_1^2 L^5}{480EI} P_{II}(\alpha, k) \qquad (19)$$

Then $P_{II}(\alpha,k)$ and w_1 is:

$$\begin{aligned} P(\alpha,k)_{II} = &-10(k^6+2k^5)\alpha^4 - 20(k^6+2k^5+8k^4)\alpha^3 + 30(k^6+10k^5+16k^4)\alpha^2 \\ &-90(k^6+4k^5+4k^4)\alpha + 51k^6+110k^5+70k^4+12 \end{aligned} \qquad (20)$$

$$w_1 = \sqrt{\frac{480EIU_{II}}{P_{II}(\alpha,k)L^5}} \qquad (21)$$

Amount of bending moment at critical sections of the bar for both of statuses is obtained as follows:

$$M_{AI} = M_{1I}(x = \alpha L) = \frac{1}{2}\sqrt{\frac{3840EIU_I}{P_I(\alpha,k)L}}\alpha^2 \qquad (22)$$

$$M_{BI} = M_{2I}(x = L) = \frac{1}{2}\sqrt{\frac{3840EIU_I}{P_I(\alpha,k)L}}[\alpha(2+k)-(1+k)] \tag{23}$$

$$M_{CI} = M_{3I}(x = 3L/2) = \frac{1}{2}\sqrt{\frac{3840EIU_I}{P_I(\alpha,k)L}}[\frac{\alpha(2+k)-(1+k)}{2}] \tag{24}$$

By using of relationships ($I = \frac{1}{4}\pi r^4$) and ($\sigma = \frac{Mr}{I}$) bending stresses corresponding to each critical section is obtained:

$$\sigma_{AI} = \frac{M_{AI}d}{2I} = 2\sqrt{\frac{960}{P_I(\alpha,k)}}\alpha^2\sqrt{\beta\rho EV^2} \tag{25}$$

$$\sigma_{BI} = \frac{M_{BI}d}{2I} = 2\sqrt{\frac{960}{P_I(\alpha,k)}}[\alpha(2+k)-(1+k)]\sqrt{\beta\rho EV^2} \tag{26}$$

$$\sigma_{CI} = \frac{M_{CI}D}{2I_D} = 2\sqrt{\frac{960}{P_I(\alpha,k)}}[\frac{\alpha(2+k)-(1+k)}{2k^2}]\sqrt{\beta\rho EV^2} \tag{27}$$

Status II:

$$M_{AII} = M_{1II}(x = L) = \frac{1}{2}\sqrt{\frac{480EIU_{II}}{P_{II}(\alpha,k)L}} \tag{28}$$

$$M_{BII} = M_{2II}(x = \alpha L) = \frac{1}{2}\sqrt{\frac{480EIU_{II}}{P_{II}(\alpha,k)L}}[2\alpha k^2 + \alpha(2-4k)+2k-1] \tag{29}$$

$$M_{CII} = M_{3II}(x = \frac{3L}{2}) = \frac{1}{2}\sqrt{\frac{480EIU_{II}}{P_{II}(\alpha,k)L}}[\alpha(2+k)-(1+\frac{5k}{4})] \tag{30}$$

Then:

$$\sigma_{AII} = \frac{M_{AII}d}{2I} = \sqrt{\frac{480}{P_{II}(\alpha,k)}}\sqrt{\beta\rho EV^2} \tag{31}$$

$$\sigma_{BII} = \frac{M_{BII}D}{2I_D} = \sqrt{\frac{480}{P_{II}(\alpha,k)}}[\frac{2k\alpha^2+\alpha(2-4k)+2k-1}{k^2}]\sqrt{\beta\rho EV^2} \tag{32}$$

$$\sigma_{CII} = \frac{M_{CII}D}{2I_D} = \sqrt{\frac{480}{P_{II}(\alpha,k)}}[\frac{\alpha(2+k)-(1+5K/4)}{k^2}]\sqrt{\beta\rho EV^2} \tag{33}$$

Above equations show that the stresses on the bar are independent of its length.

3. RESULTS AND DISCUSSION

3.1. Results and discussion for status I

In section 2, bending stresses corresponding to critical sections of the bar are reached. These stresses are measured as functions of *α* and *k*. In this section, convenient position of supporting force is determined in order to minimize the maximum stress in the bar. This work is done by plotting the curves of dimensionless stress versus α for each section and different k (Figure 3).

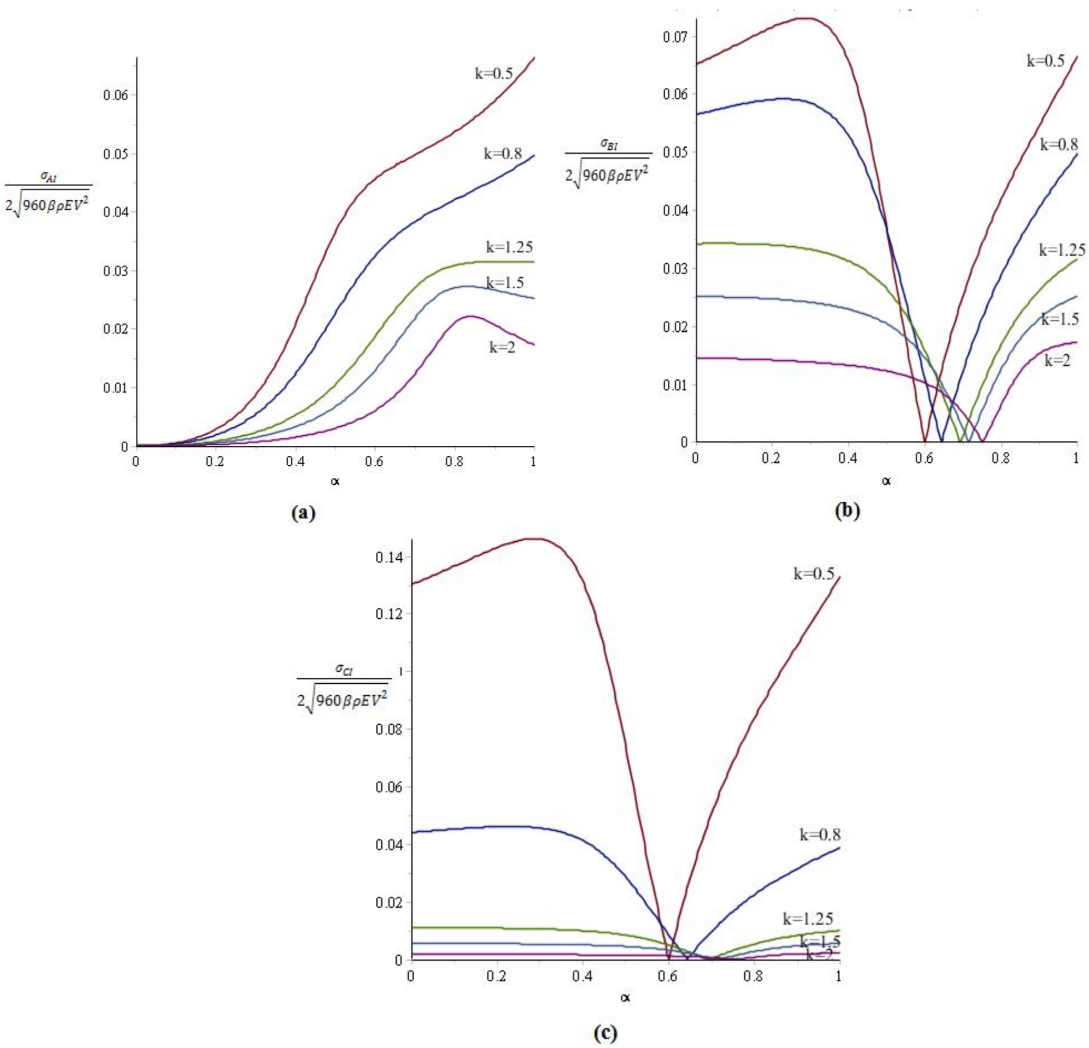


Fig. 3. Dimensionless stress versus α for each section and different k.

The Figure 3.a, show that minimum stress due to bending at first section of studied bar ($x= \alpha L$) occurs when the supporting forces being applied at both ends of the bar (α =0), and the α required to make minimum stresses at second and third sections of bar, adopts certain amount for different k (see Figures 3.b and 3.c).

Now, a place for supporting forces is required that total stress is minimized. For this purpose, diagrams of dimensionless total stress versus α are plotted (Figure 4). These figures show that α required for minimum total stress, adopts certain amount for different k, too.

Where:

$$\sigma_{total,I} = \sigma_{AI} + \sigma_{BI} + \sigma_{CI} \tag{34}$$

Amounts of α for different values of k that minimize the total stress are earned and when this data are evaluated by $\alpha_{\min}$ (value of α that minimizes the $\sigma_{\max}$) versus k diagrams (Figure 5), then mathematical relationships between $\alpha_{\min}$ and k are obtained by data fitting (equations 35 and 36). With these relationships, can be found appropriate position of supporting forces α for each arbitrary amount of k:

$$\alpha_{\min} = -0.0407k^2 + 0.1995k + 0.508 \tag{35}$$

$$\alpha_{\min} = 0.0134k^2 + 0.0386k + 0.0793, (1.89 \le k \le 5) \tag{36}$$

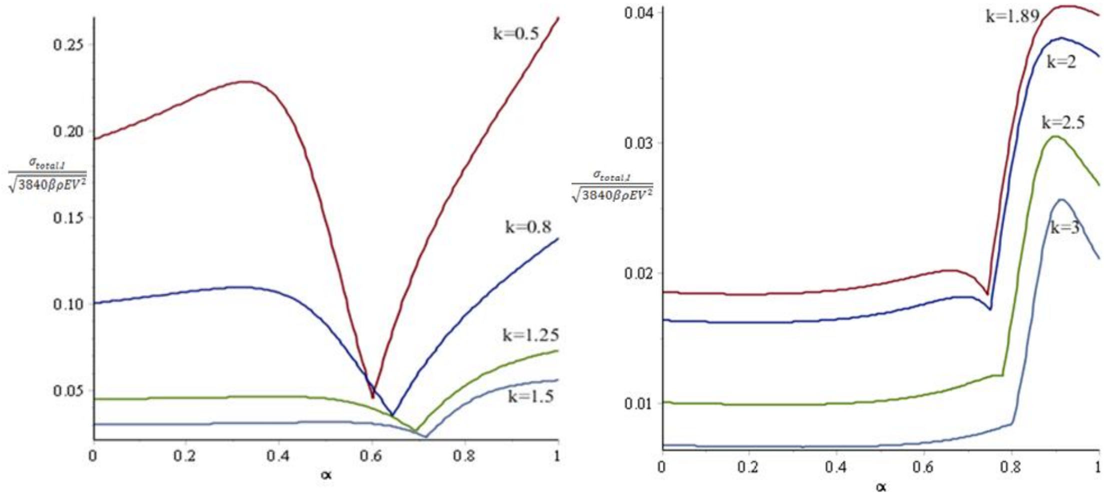
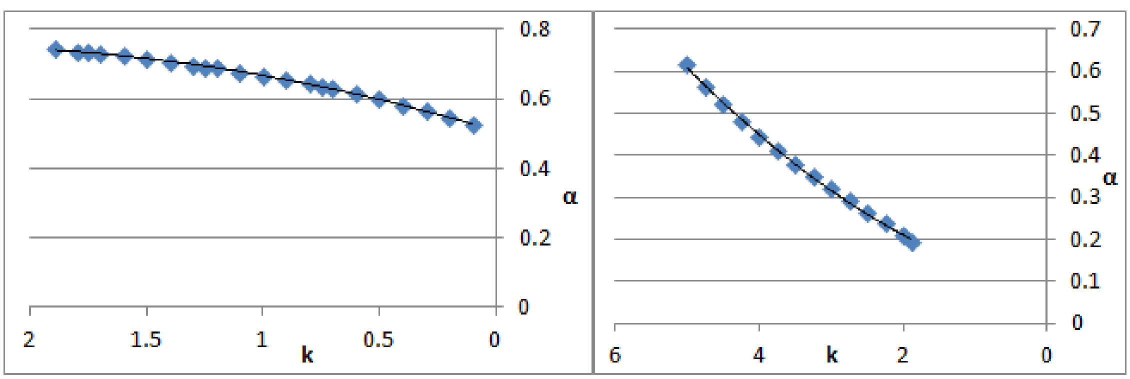


Fig. 4. Dimensionless total stress versus α for different k (status I).

Of course, equation (36) for the k more than 5 is acceptable, too. But since very high k is not so practical and reduces the accuracy of the equations obtained. So, amount of k is assumed less than five.



(a) **(b)**

Fig. 5. $\alpha_{\min}$ vs k arising from data fitting, (a) for 0<k≤1.89 (k≠1), (b) for 1.89≤k≤5.

According to equations (35) and (36) can be seen that for k=1.89, two different values for α is obtained. Namely, there are two optimum values for position of supporting forces at a special diameter proportion (k=1.89).

3.2. Results and discussion for status II

Now, it is assumed that supporting forces are located in the range of ($L \le x \le 3L/2$). In this status, curves of total stress variations versus α are plotted such as status I (Figure 6).

Figure 6 show that total stress in the status II is much greater than status I, so this conclusion tells that it is better that supporting force placed in range of ($0 \le x \le L$). With:

$$\sigma_{total,II} = \sigma_{AII} + \sigma_{BII} + \sigma_{CII} \tag{37}$$

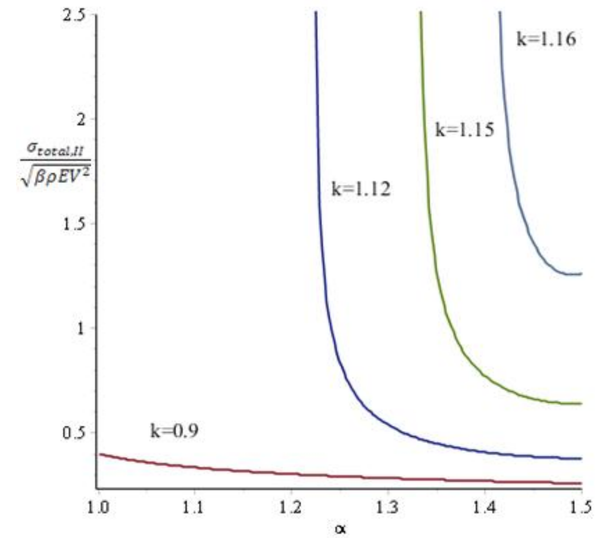


Fig. 6. Dimensionless total stress vs α diagram for different k (status II).

The analysis carried out shows that dimensionless total stress vs. α curve is undefined for some of the k and α, or this curve can be defined only in the certain range of k or α, because $P_{II}(\alpha, k)$ is negative for those k and α. Accordingly, the term under the radical is negative, and this content is unacceptable. Namely, if position of supporting force placed in range of ($L<x<3L/2$) and values of α and k reach a certain limit, then strain energy is negative (equation 19) and in fact, supports are unable to bear the impact and system against the impact will be destroyed.

Evaluation of dimensionless total stress vs. α curves shows that total stress is undefined for k higher than 1.1633, namely when status II is established, this stress can be defined only for $k<1.1633$.

Also, it is shown that dimensionless total stress vs. α curves can be defined just for some of the α even in the range of ($1<k<1.1633$) (Figure 6), minimum α at which the stress is defined, is called as the α_{cr}. Then with plotting α_{cr} versus k curve (Figure 7), a mathematical relation between α_{cr} and k is obtained by data fitting (equation 38).

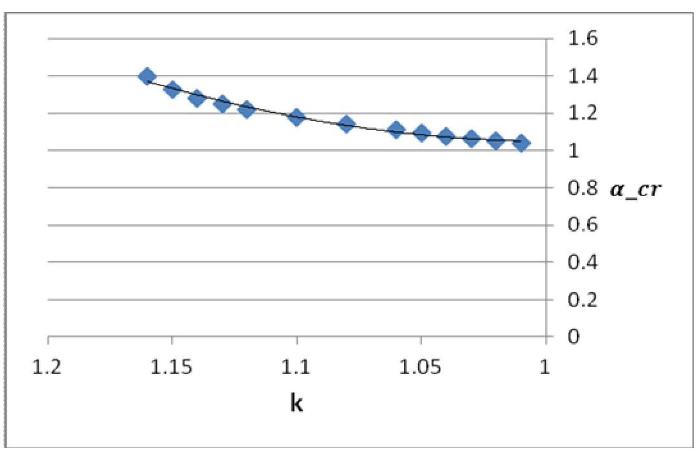


Fig. 7. α_{cr} vs k arising from data fitting

$$\alpha_{cr} = 11.86k^2 - 23.61k + 12.8 \tag{38}$$

With equation (38), can be found the minimum α in order that the system remains stable for each arbitrary value of k.

4. GEOMETRY AND MATERIAL DESIGN

4.1. Geometry design

With considering the bar in the shape of hollow cylinder can be seen that bending stress is reduced by thickness reduction (equations 39 and 40). This phenomenon is due to diminish the inertia of the bar, because the mass decreases due to the thickness reduction and after the mass reduction, the inertia of bar is reduced, naturally. So, the stress due to inertia impact increase on account of thickness increment and the best condition take place when $d_i \approx d$ [7].

$$U = \frac{1}{4}\beta\rho\pi(d^2 - d_i^2)LV^2, I = \frac{\pi}{64}(d^4 - d_i^4) \tag{39}$$

$$\sigma_{AI} = \frac{Md}{2I} = C(\alpha, k)\sqrt{\frac{\beta\rho EV^2 d^2}{(d^2 + d_i^2)}} \tag{40}$$

where $C(\alpha, k)$ is a function of α and k and d_i is inner diameter of tube.

4.2. Material design

Often, we are faced with this issue that there is a need for us to use multi-material components. This status, with considering ($k = \rho_2 / \rho_1$) matches with studied model (ρ is material density). Namely, if we have a bar that middle part of it consists of different material than the sides, then equations of the section 2, can be used to solve this problem. In fact, we can replace density proportion of the two materials instead of diameter proportion in equation (5), because the difference in density between different parts of the bar causes a difference in weight. Therefore, the inertia load distribution on the bar is non-uniform (equation 41).

$$w_2 / w_1 = \rho_2 / \rho_1 = k \tag{41}$$

where ρ_1 and ρ_2 are density in the middle and sides of the bar, respectively.

Thus, all of the equations in section 2 are also true in here, but the appropriate amounts for inertia moment, elasticity modulus and density should be considered for each section of bar.

5. CONCLUSIONS

In this paper, a non-uniform beam under bending which is subjected to a inertia impact loading is studied, and hence determine the subsequent shock stresses that occur due to the inertia beam force created.

Results of performed analysis are very useful, so that by using these results, designers can have easier designs. The results include the relationship between the bending stress on the beam due to inertia shock, mass distribution, position of supporting forces and beam geometry. Also, the results are indicated that the optimal design of the beam is independent of its length and position of supporting forces depends on the mass distribution (*D/d*). Then mathematical relationships between $\alpha_{\min}$ and k was obtained that would be a great achievement for designers. And surprisingly, two amounts for $\alpha_{\min}$ are observed at k=1.89, that implies two optimum values for position of supporting forces at a special diameter proportion (k=1.89).

Additionally, in this paper demonstrated that best place for supporting force is in the range of ($0<x<L$). But if the supporting force located in the range of ($L<x<3L/2$), then total stress is defined only for ($k<1.1633$), also, it is shown that dimensionless total stress vs. α curves can be defined just for some of the α even in the range of ($1<k<1.1633$), otherwise in these conditions, strain energy will be negative and supports will lose its effectiveness, therefore, mathematical relationship between α and k is obtained for designers and researchers.

Finally, it is illustrated that we can substitute density proportion instead of diameter proportion for multi-material components.

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