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INFINITESIMAL MOTIONS OF THE METRICAL $2-\pi$ STRUCTURES ON THE TANGENT BUNDLE

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Abstract. We define the notion of almost metrical $2-\pi$ structure on the tangent bundle and study the main properties of such a structure. We study here the existence and arbitrariness of a d-connection $F\Gamma(N)$ and determine all these connections. Finally we determine the infinitesimal motions of this structure and study the properties of these motions.

INTRODUCTION

In [7] M. Yawata studied the infinitesimal transformations and motions on a vector bundle establishing the main properties of these. We define an almost metrical $2-\pi$ structure on *TM* as a pair of d-tensor fields $\left(\varphi^i_j, g_{ij}\right)$ where $\varphi^i_j(x, y)$ is a *d*-tensor field of type (1.1) with the property $\varphi^i_j \varphi^j_k = \lambda^2 \delta^i_k$, and $g_{ij}(x, y)$ is a *d*-tensor field of type (0, 2), symmetric and nonsingular. [1], [2]

For $\lambda = \pm\sqrt{-1}$, we have an almost Hermitian $2-\pi$ d-structure and for $\lambda = \pm 1$, we have an almost metrical product d-structure $2-\pi$ d-structure on TM.

We study the existence and arbitrariness of a d-connection $F\Gamma(N)$ for which $\varphi^i_{j|k} = 0, \varphi^i_j |_k = 0, g_{ij|k} = 0, g_{ij} |_k = 0$ and determine all *d*-connections with these properties [1], [2]. These connections and the composition of the mapping give us a group $G_{2-\pi,m}$.

In [3] we have determined the infinitesimal motions of φ^j_i establishing the fundamental equations and the main consequences.

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Here, we determine the infinitesimal motions of the metrical $2-\pi$ structures $\left(\varphi_j^i, g_{ij}\right)$ and give the main properties of these structures.

Some notations and fundamental result are taken from the book [6].

1. PRELIMINARIES

Let *M* be an *n*-dimensional differentiable manifold of the class C^∞ and (TM, π, M) its tangent bundle. Let $\left(x^i\right)$ be the local coordinates of a point *x* in *M*. Then a point $u \in M$ has the canonical coordinate system $(x, y) \equiv \left(x^i, y^i\right)$ where $\pi(u) = x$

A local coordinate transformation on *TM* is given by

$$(1.1)\quad \begin{cases} \widetilde{x}^i = \widetilde{x}^i\left(x^1, x^2, \ldots, x^n\right), \det\left\|\dfrac{\partial \widetilde{x}^i}{\partial x^j}\right\| \neq 0, \\ \widetilde{y}^i = \dfrac{\partial \widetilde{x}^i}{\partial x^j} y^j \end{cases}.$$

The vertical distribution $V : u \in TM \to V_u \in T_u TM$ is the kernel of the differential of the projection $\pi : TM \to M$.

Denoting by $\dfrac{\partial}{\partial x_i}$ and $\dfrac{\partial}{\partial y^a}$ $\left(i, j, k, \ldots a, b, c, \ldots = 1, 2, \ldots, n\right)$ the local natural basis of the module of the vector field *x*(*TM*), we observe that $\dfrac{\partial}{\partial y^a}$ is a local basis of *V*. Hence *V* is an integrable *n*-dimensional distribution on TM.

A nonlinear connection *N* on *TM* is a distribution of the class C^∞ given by $N : u \in TM \to N_u C T_u(TM)$ such that

$$(1.2)\ T_u TM = N_u \oplus V_u;$$

N is called a horizontal distribution on *TM* and locally it is spanned by the adapted basis

$$(1.3)\ \delta_i = \partial_i - N_i^j \dot{\partial}_j$$

where $\partial_i = \dfrac{\partial}{\partial x_i}$, $\dot{\partial}_i = \dfrac{\partial}{\partial y^i}$ and N_j^i, are C^∞ functions called the coefficients of the nonlinear connection *N*. $\left(\delta_i, \dot{\partial}_i\right)$ is a local basis of $F(M)$-module of the vector fields $X(TM)$ adapted to the supplementary distributions *N* and *V*. Its dual basis $\left(dx^i, \delta y^i\right)$ is given by

$$(1.4)\ \left(dx^i, \delta y^i\right) = dy^i + N_j^i dx^j.$$

A vector field $X \in X(TM)$ is uniquely expressed in the form

$$(1.5)\ X = X^H + X^V,\ X^H \in N,\ X^V \in V$$

and 1-form ω also given by

$$(1.6)\ \omega = \omega^H + \omega^V;\ \omega^H\left(X^V\right) = 0,\ \omega^V\left(X^H\right) = 0$$

A tensor field $t \in T_S^r(TM)$ is called *distinguished tensor field* (shortly *d-tensor field*) if it has the properties:

$$(1.7)\ t(\omega_1,\dots,\omega_r,\ X_1,\dots X_s) = 0$$

for any $\omega_r = \omega^H$ or $\omega_a = \omega^V$ $(i = 1,\dots,r)$ and $X_b = X_b^H$ or $X_b = X_b^V$, $(b = 1, 2,\dots,n)$

For example X^H, X^V are *d*-tensor fields and ω^V, ω^V are *d*-fields 1- forms.

Let $J : X(TM) \to X(TM)$ be the tangent structure defined by

$$(1.8)\ J(\delta_i) = \dot{\partial}^i,\ J\left(\dot{\partial}^i\right) = 0$$

Then a linear connection ∇ on *TM* is called *distinguished connection* (shortly *d*-connection) if it satisfies

$$(1.9)\ \left(\nabla_x Y^H\right)^V = 0,\ \left(\nabla_x Y^V\right)^H = 0 \text{ and } \nabla_x J = 0;$$

for any vector fields *X, Y* on *TM*.

We write $\nabla_X^H = \nabla_{X^N}$, $\nabla_X^V = \nabla_{X^V}$, ∇^H and ∇^V being the *h*-and *v*-covariant derivatives in the algebra of *d*-tensor fields $T_d(TM) \subset T(TM)$, respectively.

A *d*-connection ∇ on *TM* is determined by a triad $\nabla\Gamma = \left(N_j^i, F_{jk}^i, C_{jk}^i\right)$ which is called the coefficients of ∇:

$$(1.10)\ \nabla_{\delta_k}\delta_j = F_{jk}^i\delta_i;\nabla_{\delta_k}\dot{\partial}_j = F_{jk}^i\dot{\partial}_i;\quad \nabla_{\dot{\partial}_k}\delta_j = C_{jk}^i\dot{\partial}_i;\nabla_{\dot{\partial}_k}\dot{\partial}_i = C_{jk}^i\dot{\partial}_i$$

With respect to the adapted basis $\{\delta_i, \dot{\partial}_i\}$ the torsion tensor *T* has the following components

$$(1.11)\ T(\delta_k, \delta_j) = T_{jk}^i\delta_i + R_{jk}^i\dot{\partial}_i,\ T\left(\dot{\partial}_k, \delta_j\right) = C_{jk}^i\delta_i + P_{jk}^i\dot{\partial}_i,\ T\left(\dot{\partial}_k, \dot{\partial}_j\right) = \delta_{jk}^i\dot{\partial}_i$$

where the coefficients $T_{jk}^i, R_{jk}^i, P_{jk}^i, \textit{and } S_{jk}^i$ are given by

$$(1.12)\ T_{jk}^i = L_{jk}^i - L_{kj}^i,\ R_{jk}^i = \delta_k N_j^i - \delta_j N_k^i,\ P_{jk}^i = \dot{\partial}_k N_j^i - L_{kj}^i,\ \delta_{jk}^i = C_{jk}^i - C_{kj}^i$$

The curvature *d*-tensor fields have the following components:

$$(1.13)\begin{cases} R(\delta_h, \delta_j)\delta_k = R^i_{hjk}\delta_i,\ R(\delta_h, \delta_j) = R^i_{hjk}\dot{\partial}_i \\ R(\ddot{\partial}_h, \delta_j)\delta_k = P^i_{hjk}\delta_i,\ R(\dot{\partial}_h\delta_j)\dot{\partial}_k = P^i_{hjk}\dot{\partial}_i, \\ R(\dot{\partial}_h, \dot{\partial}_j)\delta_k = S^i_{hjk},\ R(\dot{\partial}_h\dot{\partial}_j)\dot{\partial}_k = S^i_{hkj}\dot{\partial}_i \end{cases}$$

2. ALMOST METRICAL $2-\pi$ STRUCTURES ON TM

Definition 2.1 A d-tensor field g of type (0,2) on TM is called metrical structure if it is symmetric and non singular.

Let $g = (g_{ij}(x, y))$ be a metrical structure, then we have

(2.1) $g_{ij}(x, y) = g_{ij}(y, x)$ and $\det(g_{ij}(x, y)) \neq 0$

We denote by g^{ij} reciprocal *d*-tensor field of g_{ij} and define the *d*-tensor fields: O^{ij}_{hk} and ${}^*O^{ij}_{hk}$ by

(2.2) $O^{ij}_{hk} = \frac{1}{2}(\delta^i_h\delta^i_k + g_{hk}g^{ij}),\ {}^*O^{ij}_{hk} = \frac{1}{2}(\delta^i_h\delta^i_k - g_{hk}g^{ij})$

which are called Obata's operators of metrical *d*-structures g_{ij}

Definition 2.2 A d-connection $F\Gamma = (N^i_j, F^i_{jk}, C^i_{jk})$ is called compatible with metrical d-structure g_{ij} (shortly metrical d-connection) if it satisfies

$$(2.3)\ g_{ij|k} = 0;\ g_{ij}\,|_k = 0,$$

where | and .| are the h-, respectively the v-covariant derivatives with respect to a d-connection $F\Gamma(N) = (F^i_{jk}, C^i_{jk})$.

Definition 2.3. An almost $2-\pi$ *d* structure on *TM* is a tensor field $\varphi^i_j(x, y)$ of type (1.1) satisfying

$$(2.4)\ \varphi^i_k\varphi^k_j = \lambda^2\delta^i_j$$

where λ is a complex number different from zero. Generally we assume $u = 2p\cdot\tau_0$ are almost $2-\pi$ structure $\varphi^i_j(x, y)$ we can associate the *d*-tensor field of type (2, 2) on *TM*.

$$(2.5)\,\Phi^{ij}_{hk} = \frac{1}{2}\left(\delta^i_h\delta^j_k - \frac{1}{\lambda^2}\varphi^i_h\varphi^j_k\right),\ {}^*\Phi^{ij}_{hk} = \frac{1}{2}\left(\delta^i_h\delta^j_k --+ \frac{1}{\lambda^2}\varphi^i_h\varphi^j_k\right).$$

Definition 2.4. A *d*-connection $F\Gamma = (N^i_j, F^i_{jk}, C^i_{jk})$ is called a $2-\pi$ connection compatible with almost $2-\pi$ structure φ^i_j if it satisfies

(2.6) $\varphi^i_{j|k} = 0,\ \varphi^i_j\,|_k = 0$

Definition 2.5. Let φ^i_j be an almost $2-\pi$ structure and g_{ij} a metrical *d*-structure on *TM* satisfying

(2.7) $g_{hk} = \frac{1}{\lambda^2} g_{ij}\varphi^i_h\varphi^j_k$

Then a pair of *d*-tensor fields φ^i_j, g_{ij} on *TM* is called an almost metrical $2-\pi$ *d*-structure.

Definition 2.6. A *d*-connection $F\Gamma = \left(N^i_j, F^i_{jk}, C^i_{jk}\right)$ is called *compatible with almost metrical* $2-\pi$ *d-structure* (shortly $2-\pi$ metrical *d*-connection) if it satisfies:

(2.8) $g_{ij|k} = 0,\ g_{ij}\,|_k = 0, \varphi^i_{j|k} = 0,\ \varphi^i_j\,|_k = 0$

We have the following theorem on the existence of $2-\pi$ metrical connections

Theorem 2.1. *Let* $\overset{\circ}{F\Gamma} = \left(\overset{\circ}{N}{}^i_j, \overset{\circ}{F}{}^i_{jk}, \overset{\circ}{C}{}^i_j\right)$ *be an arbitrary d-connection, then the d-connection with the coefficients*

$$(2.9.)\quad \begin{aligned} {}^*F^h_{jk} &= \overset{\circ}{F}{}^h_{jk} + \frac{1}{4}\left\{g^{rh}g_{rs}\,|_k + \frac{1}{\lambda^2}\left(\varphi^r_j\varphi^h_j\,|_k - \varphi^{rh}\varphi_{rj}\,|_k\right)\right\} \\ {}^*C^h_{jk} &= \overset{\circ}{C}{}^h_{jk} + \frac{1}{4}\left\{g^{rh}g_{rs}\,|_k + \frac{1}{\lambda^2}\left(\varphi^r_j\varphi^h_r\,|_k - \varphi^{rh}\varphi_{rj}\,|_k\right)\right\} \end{aligned}$$

is a $2-\pi$ *metrical d-connection, where I | are the h-and-v-covariant derivatives with respect to* $\overset{\circ}{F\Gamma}$.

We shall determine a $2-\pi$ metrical connection using the method given by R. Miron and M. Hashiguchi [6]

Theorem 2.2. *Let* $\overset{\circ}{F\Gamma} = \left(\overset{\circ}{N}{}^i_j, \overset{\circ}{F}{}^i_{jk}, \overset{\circ}{C}{}^i_j\right)$ *be a fixed d-connection on TM and* $\left({}^*F^i_{jk},\ {}^*C^i_{jk}\right)$ *the coefficients of the d-connection given by*

$$(2.10)\quad \begin{aligned} N^k_j &= \overset{\circ}{N}{}^h_j - X^h_j, \\ F^h_{jk} &= {}^*F^h_{jk} + \overset{\circ}{C}{}^h_{jl}X^l_k + \Phi^{hl}_{jl}O^{lr}_{mb}Y^m_{rk}, \\ C^i_{jk} &= {}^*C^i_{jk} + \Phi^{hb}_{ji}O^{lr}_{mb}Z^m_{hk} \end{aligned}$$

where $X^h_j, Y^m_{rk}, Z^m_{rk}$ *are arbitrary d-tensor fields.*

Let $F\Gamma(N)$ be a $2-\pi$ metrical *d*-connection and $F\overline{\Gamma}(N)$ another $2-\pi$ metrical *d*-connection. Using Theorem 2.2 we may write the relations between the coefficients of $F\Gamma(N)$ and $F\overline{\Gamma}(N)$ in the form:

$$(2.11)\ \overline{F}^{\,h}_{jk} = F^{\,h}_{jk} + \Phi^{hl}_{mj} O^{mr}_{pl} X^{\,p}_{rk};\ \overline{C}^{\,h}_{jk} = C^{\,h}_{jk} + \Phi^{hl}_{mj} O^{mr}_{pl} Y^{\,p}_{rk}$$

where $X^{\,p}_{rk}, Y^{\,p}_{rk}$ are arbitrary α–tensor fields.

The mapping $F\Gamma(N) \to \overline{F}\Gamma(N)$ given by (2.11) is called a transformation of $2-\pi$ metrical *d*-connections.

Theorem 2.3. *The set $G_{2-\pi}(N)$ of the transformations of $2-\pi$ metrical d-connections (2.11) is Abelian group with the composition of the mappings*

3. INFINITESIMAL MOTIONS OF A METRICAL ALMOST $2-\pi$ d-STRUCTURE

Let $\varphi^I_J(X,Y)$ be an almost $2-\pi$ *d*-structure on *TM*. In the paper [3] we have studied the infinitesimal transformation on *TM*.

$$(3.1)\ \begin{aligned} `x^i &= x^i + v^i(x)dt, \\ `y^i &= y^i + y^m \partial m v^i dt \end{aligned}$$

which have the property $\underset{v}{L}\varphi^i_j = 0$.

Now , we can give

Definition 3.1. The infinitesimal transformation (3.1) is called an *infinitesimal motion* of a metrical almost $2-\pi$ d-structure if it satisfies

$$(3.2)\ \underset{v}{L}\varphi^i_j(x,y) = 0,\ \underset{v}{L} g_{ij}(x,y) = 0$$

Applying the results from the papers [3], [6] we get:

Theorem 3.1. *An infinitesimal transformation (3.1) is an infinitesimal motion for d-structure (φ^i_j, g_{ij}), if and only if the following equations are satisfied:*

$$(3.3)\ \begin{cases} \theta_v \varphi^i_j - \varphi^m_j \partial_m v^i + \varphi^i_m \partial_j v^m = 0 \\ \theta_v g_{ij} + g_{mj}\partial_i v^m + g_{im}\partial_j v^m = 0 \end{cases}$$

where θ_v is the operator

$$(3.4)\ \theta_v = v^h \partial_h + y^h \partial_h v^i \dot{\partial}_i$$

Theorem 3.2 *If (3.2) are satisfied, then the d-tensors have the properties*

$$(3.5)\ \underset{v}{L} O^{ij}_{hk} = \underset{v}{L}^{*} O^{ij}_{hk} = \underset{v}{L} \Phi^{ij}_{hk} = 0.$$

Consequently, we can use the *d*-connection $F\Gamma$ which have the properties (2.8). One oh these *d*-connections is given by (2.9).

In this case we have:

Theorem 3.3. *With respect to the d-connection (2.1) the equations (3.3) are equivalent to the following equations:*

$$(3.6)\quad \begin{aligned} &-\varphi_j^h v^i\,|_k + \varphi_h^i v_{|j}^h + \left(y^h v_{|h}^m - v^h y_{|h}^m\right)\dot{\partial}_m \varphi_j^i = 0 \\ &g_{hj} v_{|i}^h + g_{ih} v_{|j}^h + \left(y^h v_{|h}^m - v^h y_{|h}^m\right)\dot{\partial}_m g_{ij} = 0 \end{aligned}$$

where $\dot{\partial}_m = \dfrac{\partial}{\partial y^m}$.

Theorem 3.4. *If* $y^h v_{|h}^m - v^h y_{|h}^m = 0$ *then (3.4) reduce to the classical system of equations:*

$$(3.7)\quad -\varphi_j^h v^i\,|_h + \varphi_h^i v_{|j}^h = 0;\ g_{hj} v_{|i}^h + g_{ih} v_{|j}^h + g_{ij} v_{|j}^h = 0$$

Proof. By a straightforward calculation we get

$$\left(\underset{v}{L} t_j^i\right)_{Ik} = \delta_k\left(\theta_v t_j^i - t_j^r \partial_r v^r + t_r^i \partial_j v^r\right) + F_{mk}^i \underset{v}{L} t_j^m - F_{jk}^m \underset{v}{L} t_m^i$$

$$\underset{v}{L} t_{jIk}^i = \theta_v\left(\delta_k t_j^i + F_{mk}^i t_j^m - F_{jk}^m t_m^i\right) - t_{jIk}^m \partial_m v^i + t_{mIk}^i \partial_j v^m + t_{j\,\mathrm{Im}}^i \partial_k v^m$$

Taking account of theorem (3.4)in our paper [3] we get:

Theorem 3.5. *If (3.1) is an infinitesimal motion of the* $2-\pi$ *d-structure* $\left(\varphi_j^i, g_{ij}\right)$ *then for a* d-*connection compatible with* $\left(\varphi_j^i, g_{ij}\right)$, *we have*

$$(3.14)\quad \begin{cases} -\varphi_j^r \underset{v}{L} F_{rk}^i + \varphi_r^i \underset{v}{L} F_{jk}^r + \left\{v^m R_{km}^r + \left(y^m v_{|m}^r - v^m y_{|m}^r\right)_{\|k}\right\}\dot{\partial}_r \varphi_j^i = 0 \\ g_{rj} \underset{v}{L} F_{ik}^r + g_{ir} \underset{v}{L} F_{jk}^r + \left\{v^m R_{mk}^r + \left(y^m v_{|m}^r - v^m y_{|m}^r\right)_{\|k}\right\}\dot{\partial}_r \varphi_j^i = 0 \\ -\varphi_j^r \underset{v}{L} C_{rk}^i + \varphi_r^i \underset{v}{L} C_{jk}^r + \left\{v_{\|k}^r + \dot{\partial}_k\left(y^m v_{|m}^r - v^m y_{|m}^r\right)\right\}\dot{\partial}_r \varphi_j^i = 0 \\ g_{rj} \underset{v}{L} C_{rk}^i + g_{ir} \underset{v}{L} C_{jk}^r = \left\{v_{r\|k} + \dot{\partial}_k\left(y^m v_{|m}^r - v^m y_{|m}^r\right)\right\}\dot{\partial}_r g_{ij} = 0 \end{cases}$$

where the motion "||" denote *h*-covariant derivation with respect to the Berwald connection, that is, $v_{\|k}^r = \delta_k v^r + B_{mk}^r v^m$, $B_{mk}^r = \dot{\partial}_m N_k^r$.

Remark. The equations (3.8) are the first conditions of integrability of the system of the equations (3.4).

Theorem 3.6 *Let assume the conditions:*

1. *Nonliniar connection N is integrable.*
2. $y^m y_{|m}^r - v^m y_{|m}^r = 0$
3. $v_{|k}^i = 0$

Then, the system of equations (3.8) reduce the classical one:

$$
(3.0)\quad \begin{array}{l} -\varphi^r_j \underset{v}{L} F^j_{rk} + \varphi^i_r \underset{v}{L} F^r_{jk} = 0;\ g_{ir} \underset{v}{L} F^r_{jk} + g_{rj} \underset{v}{L} F^r_{ik} = 0; \\ -\varphi^r_j \underset{v}{L} C^i_{rk} + \varphi^i_r \underset{v}{L} C^r_{jk} = 0;\ g_{ir} \underset{v}{L} C^r_{jk} + g_{rj} \underset{v}{L} C^r_{ik} = 0 \end{array}
$$

All this theory can be particularized in the case $\lambda = \pm\sqrt{-1}$ when the *d*-structure, (φ^i_j, g_{ij}) is an almost Hermitian d-structure on the total space of the tangent bundle and $\lambda = \pm 1$, when the *d*-structure (φ^i_j, g_{ij}) is a metrical almost product *d*-structure on *TM*.

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