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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 20 (2010), No. 1, 185 - 214

A UNIFIED THEORY OF ALMOST CONTINUITY FOR MULTIFUNCTIONS

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Abstract. We introduce the notions of upper/lower almost m-continuity for multifunctions from a set satisfying certain minimal condition into a topological space. We obtain their characterizations and properties which unify those of almost continuity, almost quasi-continuity, almost precontinuity, almost α-continuity, almost β-continuity and almost γ-continuity for multifunctions.

1. Introduction

Semi-open sets, preopen sets, α-sets, β-open sets and δ-open sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets several authors introduced and studied various types of weak forms of continuity for functions and multifunctions. In 1968, Singal and Singal [46] introduced the notion of almost continuous functions. In 1982, Popa [27] introduced the concept of upper/lower almost continuous multifunctions. Some properties of upper/lower almost continuous multifunctions are studied in [28]-[33] and other articles.

Keywords and phrases: m-structure, m-space, almost m-continuous, multifunction.
(2000)Mathematics Subject Classification: 54C08, 54C60.

In 1993, Popa and Noiri [34] introduced the concepts of upper/lower almost quasi-continuous multifunctions. Some properties of upper/lower almost quasi-continuous multifunctions are studied in [23]. In 1993, Popa et al. [43] introduced and studied the concepts of upper/lower almost precontinuous multifunctions. Some properties of these multifunctions are studied in [42]. The notions of upper/lower almost α-continuous multifunctions are introduced in [35] and the further properties are studied in [36], [40] and [42]. The notions of upper/lower almost β-continuous multifunctions are introduced by Noiri and Popa [24]. The further properties of these multifunctions are studied in [37]. Recently, Ekici and Park [11] introduced and studied almost γ-continuous multifunctions. Almost quasi-continuity, almost precontinuity, almost β-continuity, almost α-continuity, almost γ-continuity for multifunctions have properties similar to these of almost continuous multifunctions and they hold, in many part, paralel to theory of continuous multifunctions. Further, analogies in their definitions and results suggest the need formulating a unified theory in the setting of multifunctions.

In [41], the present authors introduced and studied the notion of almost m-continuous functions. In this paper, we introduce the notions of upper/lower almost m-continuous multifunctions as multifunctions from a set satisfying some minimal conditions into a topological space. We obtain several characterizations and properties of such multifunctions.

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}(\mathrm{Int}(A)) = A$ (resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$).

A point $x \in X$ is called a δ-cluster point of a subset A if $\mathrm{Int}(\mathrm{Cl}(U)) \cap A \neq \emptyset$ for every open set U containing x. The set of all δ-cluster points of A is called the *δ-closure* of A and is denoted by $\mathrm{Cl}_\delta(A)$. If $A = \mathrm{Cl}_\delta(A)$, then A is said to be *δ-closed* [49]. The complement of a δ-closed set is said to be *δ-open*. The union of all δ-open sets contained in A is called the *δ-interior* of A and is denoted by $\mathrm{Int}_\delta(A)$. It is shown in [49] that $\mathrm{Cl}_\delta(U) = \mathrm{Cl}(U)$ for every open set U of X and $\mathrm{Cl}_\delta(S)$ is closed in X for every subset S of X.

Definition 2.1. A subset A of a topological space (X, τ) is said to be *semi-open* [16] (resp. *preopen* [18], *α-open* [20], *β-open* [1] or *semi-preopen* [4], *b-open* [5] or *γ-open* [3]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(A)) \cup \mathrm{Int}(\mathrm{Cl}(A))$).

The family of all semi-open (resp. preopen, α-open, β-open, γ-open) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\gamma(X)$).

Definition 2.2. The complement of a semi-open (resp. preopen, α-open, β-open,γ-open) set is said to be *semi-closed* [10] (resp. *preclosed* [12], *α-closed* [19], *β-closed* [1], *γ-closed* [3]).

Definition 2.3. The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, γ-closed) sets of X containing A is called the *semi-closure* [10] (resp. *preclosure* [12], *α-closure* [19], *β-closure* [2], *γ-closure* [3]) of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, $\beta\mathrm{Cl}(A)$, $\gamma\mathrm{Cl}(A)$)).

Definition 2.4. The union of all semi-open (resp. preopen, α-open, β-open, γ-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior*, *γ-interior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\beta\mathrm{Int}(A)$, $\gamma\mathrm{Int}(A)$).

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) always denote topological spaces and $F : (X, \tau) \to (Y, \sigma)$ presents a multivalued function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and}$$
$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Definition 2.5. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper almost continuous* [27] (resp. *upper almost quasi-continuous* [34], *upper almost precontinuous* [42], *upper almost α-continuous* [40], *upper almost β-continuous* [24], *upper almost γ-continuous* [11]) at a point $x \in X$ if for each open set V of Y containing $F(x)$, there exists an open (resp. semi-open, preopen, α-open, β-open, γ-open) set U of X containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$,

(2) *lower almost continuous* [27] (resp. *lower almost quasi-continuous* [34], *lower almost precontinuous* [42], *lower almost α-continuous* [40], *lower almost β-continuous* [24], *lower almost γ-continuous* [11]) at a point $x \in X$ if for each open set V of Y such

that $V \cap F(x) \neq \emptyset$, there exists an open (resp. semi-open, pre-open, α-open, β-open, γ-open) set U of X containing x such that $F(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$.

(3) *upper/lower almost continuous* (resp. *upper/lower almost quasi-continuous, upper/lower almost precontinuous, upper/lower almost α-continuous, upper/lower almost β-continuous, upper/lower almost γ-continuous*) if it has the property at each $x \in X$.

3. Almost m-continuous multifunctions

Definition 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [38] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* (or briefly m-open) and the complement of an m_X-open set is said to be *m_X-closed* (or briefly m-closed).

Remark 3.1. Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\gamma(X)$ are all m-structures on X.

Definition 3.2. Let (X, m_X) be an m-space. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [17] as follows:

(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,

(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 3.2. Let (X, τ) be atopological space and A be a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\gamma(X)$), then we have

(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, $\beta\mathrm{Cl}(A)$, $\gamma\mathrm{Cl}(A)$),

(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\beta\mathrm{Int}(A)$), $\gamma\mathrm{Int}(A)$).

Lemma 3.1. (Maki et al.[17]). *Let (X, m_X) an m-space. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X - A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X - A) = X - \mathrm{mCl}(A)$,

(2) If $(X - A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,

(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

(4) If $A \subset B$*, then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$*,*
(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$*,*
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$*.*

Lemma 3.2. (Popa and Noiri [38]). *Let* (X, m_X) *be an m-space and* A *a subset of* X*. Then* $x \in m_X\text{-}\mathrm{Cl}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in m_X$ *containing* x*.*

Definition 3.3. A minimal structure m_X on a nonempty set X is said to have *property* ($\mathcal{B}$) [17] if the union of any family of subsets belonging to m_X belongs to m_X.

Lemma 3.3. (Popa and Noiri [39]). *Let* X *be a nonempty set and* m_X *a minimal structure on* X *satisfying property* ($\mathcal{B}$)*. For a subset* A *of* X*, the following properties hold:*
(1) $A \in m_X$ *if and only if* $\mathrm{mInt}(A) = A$*,*
(2) A *is m-closed if and only if* $\mathrm{mCl}(A) = A$*,*
(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is m-closed.*

Definition 3.4. Let (X, m_X) be an m-space and (Y, σ) a topological space. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be

(1) *upper m-continuous* [25] (resp. *upper almost m-continuous* [25], *upper weakly m-continuous* [25]) at a point $x \in X$ if for each open set V of Y containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \subset V$ (resp. $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$, $F(U) \subset \mathrm{Cl}(V)$),

(2) *lower m-continuous* [25] (resp. *lower almost m-continuous* [25], *lower weakly m-continuous* [25]) at a point $x \in X$ if for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ (resp. $F(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$, $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$) for each $u \in U$,

(3) *upper/lower m-continuous* (resp. *upper/lower almost m-continuous, upper/lower weakly m-continuous*) if it has this property at each point $x \in X$.

Theorem 3.1. *For a multifunction* $F : (X, m_X) \to (Y, \sigma)$*, the following properties are equivalent:*
(1) F *is upper m-continuous;*
(2) $F^+(V) = \mathrm{mInt}(F^+(V))$ *for every open set* V *of* Y*;*
(3) $F^-(K) = \mathrm{mCl}(F^-(K))$ *for every closed set* K *of* Y*;*
(4) $\mathrm{mCl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ *for every subset* B *of* Y*;*
(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(B))$ *for every subset* B *of* Y*.*

Proof. The proof follows from Theorem 3.1 of [25].

Theorem 3.2. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower m-continuous;

(2) $F^-(V) = \mathrm{mInt}(F^-(V))$ for every open set V of Y;

(3) $F^+(K) = \mathrm{mCl}(F^+(K))$ for every closed set K of Y;

(4) $\mathrm{mCl}(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ for every subset B of Y;

(5) $F(\mathrm{mCl}(A)) \subset \mathrm{Cl}(F(A))$ for every subset A of X;

(6) $F^-(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^-(B))$ for every subset B of Y.

Proof. The proof follows from Theorem 3.2 of [25].

Theorem 3.3. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost m-continuous at $x \in X$;

(2) $x \in \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$ for every open set V of Y containing $F(x)$;

(3) $x \in \mathrm{mInt}(F^+(\mathrm{sCl}(V)))$ for every open set V of Y containing $F(x)$;

(4) $x \in \mathrm{mInt}(F^+(V))$ for every regular open set V of Y containing $F(x)$;

(5) for each regular open set V of Y containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \subset V$.

Proof. (1) $\Rightarrow$ (2): Let V be any open set of Y containing $F(x)$. There exists $U \in m_X$ containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Thus, we have $x \in U \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ and hence $x \in \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$.

(2) $\Rightarrow$ (3): This follows from Lemma 3.2 of [22].

(3) $\Rightarrow$ (4): Let V be a regular open set of Y containing $F(x)$. Then it follows from Lemma 3.2 of [22] that $V = \mathrm{Int}(\mathrm{Cl}(V)) = \mathrm{sCl}(V)$.

(4) $\Rightarrow$ (5): Let V be a regular open set of Y containing $F(x)$. By (4), $x \in \mathrm{mInt}(F^+(V))$ and there exists $U \in m_X$ containing x such that $x \in U \subset F^+(V)$; hence $F(U) \subset V$.

(5) $\Rightarrow$ (1): Let V be any open set of Y containing $F(x)$. Since $\mathrm{Int}(\mathrm{Cl}(V))$ is regular open, there exists $U \in m_X$ containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. This shows that F is upper almost m-continuous at $x \in X$.

Theorem 3.4. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost m-continuous at $x \in X$;

(2) $x \in \mathrm{mInt}(F^-(\mathrm{Int}(\mathrm{Cl}(V))))$ for every open set V of Y such that $F(x) \cap V \neq \emptyset$;

(3) $x \in \mathrm{mInt}(F^-(\mathrm{sCl}(V)))$ for every open set V of Y such that $F(x) \cap V \neq \emptyset$;

(4) $x \in \mathrm{mInt}(F^-(V))$ for every regular open set V of Y such that $F(x) \cap V \neq \emptyset$;

(5) for every regular open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_X$ containing x such that $U \subset F^-(V)$.

Proof. The proof is similar to that of Theorem 3.3 and is omitted.

Remark 3.3. Let (X, τ) and (Y, σ) be topological spaces. Let $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\alpha(X), \beta(X)$).

(1) If $F : (X, m_X) \to (Y, \sigma)$ is upper almost m-continuous, then the results established in Theorem 2.3 of [27] (resp. Theorem 3.1 of [34], Theorem 1 of [35], Theorem 1 of [24]) are obtained from Theorem 3.3.

(2) If $F : (X, m_X) \to (Y, \sigma)$ is lower almost m-continuous, then the results established in Theorem 2.1 of [27] (resp. Theorem 3.2 of [34], Theorem 2 of [35], Theorem 2 of [24]) are obtained from Theorem 3.4.

(3) For a function $f : (X, m_X) \to (Y, \sigma)$, by Theorems 3.3 and 3.4 we obtain Theorem 3.1 of [41].

Theorem 3.5. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost m-continuous;

(2) $F^+(V) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$ for every open set V of Y;

(3) $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(K)))) \subset F^-(K)$ for every closed set K of Y;

(4) $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^-(\mathrm{Cl}(B))$ for every subset B of Y;

(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$ for every subset B of Y;

(6) $F^+(V) = \mathrm{mInt}(F^+(V))$ for every regular open set V of Y;

(7) $F^-(K) = \mathrm{mCl}(F^-(K))$ for every regular closed set K of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any open set of Y and $x \in F^+(V)$. Then $F(x) \subset V$. By Theorem 3.3, we have $x \in \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$. This shows that $F^+(V) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$.

(2) $\Rightarrow$ (3): Let K be any closed set of Y. Then $Y - K$ is open in Y and by (2) and Lemma 3.1 we have $X - F^-(K) = F^+(Y - K) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(Y - K)))) = \mathrm{mInt}(X - F^-(\mathrm{Cl}(\mathrm{Int}(K)))) =$

$X - \mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(K))))$. Hence, we obtain $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(K)))) \subset F^-(K)$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then $\mathrm{Cl}(B)$ is a closed set of Y and by (3) we obtain $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^-(\mathrm{Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. Then we have

$$F^+(\mathrm{Int}(B)) = X - F^-(\mathrm{Cl}(Y - B)) \subset$$
$$X - \mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(Y - B))))) =$$
$$X - \mathrm{mCl}(F^-(Y - \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B))))) = \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B))))).$$

Therefore, we have $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$.

(5) $\Rightarrow$ (6): Let V be any regular open set of Y. By (5),we have $F^+(V) \subset \mathrm{mInt}(F^+(V))$. By Lemma 3.1, we obtain $F^+(V) = \mathrm{mInt}(F^+(V))$.

(6) $\Rightarrow$ (7): Let K be any regular closed set of Y. Then $Y - F$ is regular open. By (6) and Lemma 3.1, we obtain $X - F^-(K) = F^+(Y - K) = \mathrm{mInt}(F^+(Y - K)) = \mathrm{mInt}(X - F^-(K)) = X - \mathrm{mCl}(F^-(K))$. Therefore, we obtain $F^-(K) = \mathrm{mCl}(F^-(K))$.

(7) $\Rightarrow$ (1): Let $x \in X$ and V be any regular open set of Y containing $F(x)$. Since $Y - V$ is regular closed, by (7) and Lemma 3.1 we have $X - F^+(V) = F^-(Y - V) = \mathrm{mCl}(F^-(Y - V)) = X - \mathrm{mInt}(F^+(V))$. Therefore, we have $x \in F^+(V) = \mathrm{mInt}(F^+(V))$. Then, there exists $U \in m_X$ containing x such that $F(U) \subset V$. It follows from Theorem 3.3 that F is upper almost m-continuous.

Theorem 3.6. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost m-continuous;

(2) $F^-(V) \subset \mathrm{mInt}(F^-(\mathrm{Int}(\mathrm{Cl}(V))))$ *for every open set V of Y;*

(3) $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^+(\mathrm{Cl}(B))$ *for every subset B of Y;*

(4) $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(K)))) \subset F^+(K)$ *for every closed set K of Y;*

(5) $F^-(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^-(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$ *for every subset B of Y;*

(6) $F^-(V) = \mathrm{mInt}(F^-(V))$ *for every regular open set V of Y;*

(7) $F^+(K) = \mathrm{mCl}(F^+(K))$ *for every regular closed set K of Y.*

Proof. The proof is similar to that of Theorem 3.5.

Corollary 3.1. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, where m_X has property $(\mathcal{B})$, the following properties are equivalent:*

(1) F is upper almost m-continuous,

(2) $F^+(V)$ *is m-open for every regular open set* V *of* Y*,*
(3) $F^-(K)$ *is m-closed for every regular closed set* K *of* Y*.*

Proof. The proof follows from Theorem 3.5 and Lemma 3.3.

Corollary 3.2. *For a multifunction* $F : (X, m_X) \to (Y, \sigma)$*, where* m_X *has property* $(\mathcal{B})$*, the following properties are equivalent:*
(1) F *is lower almost m-continuous,*
(2) $F^-(V)$ *is m-open for every regular open set* V *of* Y*,*
(3) $F^+(K)$ *is m-closed for every regular closed set* K *of* Y*.*

Proof. The proof follows from Theorem 3.6 and Lemma 3.3.

Remark 3.4. Let (X, τ) and (Y, σ) be topological spaces. Let $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X), \beta(X), \gamma(X)$).

(1) If $F : (X, m_X) \to (Y, \sigma)$ is upper almost m-continuous, then the characterizations established in Theorem 2.4 of [27] (resp. Theorem 3.3 of [34], Theorems 1 and 3 of [42], Theorem 3 of [35], Theorem 3 of [24] and Theorem 3.1 of [7], Theorem 3 of [11]) are obtained from Theorem 3.5 and Corollary 3.1.

(2) If $F : (X, m_X) \to (Y, \sigma)$ is lower almost m-continuous, then the characterizations established in Theorem 2.2 of [27] (resp. Theorem 3.4 of [34], Theorems 2 and 4 of [42], Theorem 5 of [35], Theorem 4 of [24], Theorem 8 of [11]) are obtained from Theorem 3.6 and Corollary 3.2.

(3) For a function $f : (X, m_X) \to (Y, \sigma)$, by Theorems 3.5 and 3.6 we obtain Theorem 3.2 of [41].

Theorem 3.7. *For a multifunction* $F : (X, m_X) \to (Y, \sigma)$*, the following properties are equivalent:*
(1) F *is upper almost m-continuous;*
(2) $\mathrm{mCl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ *for every* $V \in \beta(Y)$*;*
(3) $\mathrm{mCl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ *for every* $V \in \mathrm{SO}(Y)$*.*

Proof. (1) $\Rightarrow$ (2): Let V be any β-open set of Y. It follows fromTheorem 2.4 of [4] that $\mathrm{Cl}(V)$ is regular closed. Since F is upper almost m-continuous, by Theorem 3.5 $F^-(\mathrm{Cl}(V)) = \mathrm{mCl}(F^-(\mathrm{Cl}(V))$. By Lemma 3.1, $\mathrm{mCl}(F^-(V)) \subset \mathrm{mCl}(F^-(\mathrm{Cl}(V))) = F^-(\mathrm{Cl}(V))$. Therefore, we have $\mathrm{mCl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$.

(2) $\Rightarrow$ (3): The proof is obvious since $\mathrm{SO}(Y) \subset \beta(Y)$.

(3) $\Rightarrow$ (1): Let K be any regular closed set of Y. Then K is semi-open in Y and hence $\mathrm{mCl}(F^-(K)) \subset F^-(\mathrm{Cl}(V)) = F^-(K)$. It follows

from Lemma 3.1 that $\mathrm{mCl}(F^-(K)) = F^-(K)$. By Theorem 3.5, F is upper almost m-continuous.

Theorem 3.8. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is lower almost m-continuous;
(2) $\mathrm{mCl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ *for every $V \in \beta(Y)$;*
(3) $\mathrm{mCl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ *for every $V \in \mathrm{SO}(Y)$.*

Proof. The proof is similar to that of Theorem 3.7.

Lemma 3.4. (Noiri [22]). *For a subset V of a topological space (Y, σ), the following properties hold:*
(1) $\alpha\mathrm{Cl}(V) = \mathrm{Cl}(V)$ *for every $V \in \beta(Y)$,*
(2) $\mathrm{pCl}(V) = \mathrm{Cl}(V)$ *for every $V \in \mathrm{SO}(Y)$.*

Corollary 3.3. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is upper almost m-continuous;
(2) $\mathrm{mCl}(F^-(V)) \subset F^-(\alpha\mathrm{Cl}(V))$ *for every $V \in \beta(Y)$;*
(3) $\mathrm{mCl}(F^-(V)) \subset F^-(\mathrm{pCl}(V))$ *for every $V \in \mathrm{SO}(Y)$.*

Corollary 3.4. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is lower almost m-continuous;
(2) $\mathrm{mCl}(F^+(V)) \subset F^+(\alpha\mathrm{Cl}(V))$ *for every $V \in \beta(Y)$;*
(3) $\mathrm{mCl}(F^+(V)) \subset F^+(\mathrm{pCl}(V))$ *for every $V \in \mathrm{SO}(Y)$.*

Remark 3.5. Let (X, τ) and (Y, σ) be topological spaces. Let $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X), \beta(X), \gamma(X)$).

(1) If $F : (X, m_X) \to (Y, \sigma)$ is upper almost m-continuous, then the characterizations established in Theorem 1 of [32] (resp. Theorem 1 of [23], Theorem 5 of [42], Theorem 4 of [35], Theorem 5 of [24], Theorem 9 of [11]) are obtained from Theorem 3.7.

(2) If $F : (X, m_X) \to (Y, \sigma)$ is lower almost m-continuous, then the characterizations established in Theorem 2 of [32] (resp. Theorem 2 of [23], Theorem 6 of [42], Theorem 6 of [35], Theorem 6 of [24], Theorem 10 of [11]) are obtained from Theorem 3.8.

(3) For a function $f : (X, m_X) \to (Y, \sigma)$, by Theorems 3.7 and 3.8 we obtain Theorem 3.3 of [41].

Definition 3.5. A subset A of a topological space (X, τ) is said to be

(1) *α-regular* [15] if for each $a \in A$ and each open set U containing

a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$,

(2) *α-paracompact* [50] if every X-open cover of A has an X-open refinement which covers A and is locally finite for each point of X.

Lemma 3.5. (Kovačević [15]) *If A is an α-regular α-paracompact set of a topological space (X, τ) and U is an open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

For a multifunction $F : X \to (Y, \sigma)$, by $\mathrm{Cl}(F) : X \to (Y, \sigma)$ [6] we denote a multifunction defined as follows: $\mathrm{Cl}(F)(x) = \mathrm{Cl}(F(x))$ for each $x \in X$. Similarly, we denote $\mathrm{sCl}(F) : X \to (Y, \sigma)$, $\mathrm{pCl}(F) : X \to (Y, \sigma)$, $\alpha\mathrm{Cl}(F) : X \to (Y, \sigma)$, $\beta\mathrm{Cl}(F) : X \to (Y, \sigma)$, $\gamma\mathrm{Cl}(F) : X \to (Y, \sigma)$.

Lemma 3.6. *If $F : (X, m_X) \to (Y, \sigma)$ is a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$, then $G^+(V) = F^+(V)$ for each regular open set V of Y, where G denotes* $\mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$ *or* $\gamma\mathrm{Cl}(F)$.

Proof. Let V be any regular open set of Y and $x \in G^+(V)$. Then $G(x) \subset V$ and $F(x) \subset G(x) \subset V$. We have $x \in F^+(V)$ and hence $G^+(V) \subset F^+(V)$. Conversely, let $x \in F^+(V)$. Then we have $F(x) \subset V$ and by Lemma 3.5 there exists an open set H of Y such that $F(x) \subset H \subset \mathrm{Cl}(H) \subset V$. Since $G(x) \subset \mathrm{Cl}(F(x)), G(x) \subset V$ and hence $x \in G^+(V)$. Thus we obtain $F^+(V) \subset G^+(V)$. Therefore, $G^+(V) = F^+(V)$.

Lemma 3.7. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, $G^-(V) = F^-(V)$ for each regular open set V of Y, where G denotes* $\mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$ *or* $\gamma\mathrm{Cl}(F)$.

Proof. Let V be any regular open set of Y and $x \in G^-(V)$. Then $G(x) \cap V \neq \emptyset$ and hence $F(x) \cap V \neq \emptyset$ since V is open. We have $x \in F^-(V)$ and hence $G^-(V) \subset F^-(V)$. Conversely, let $x \in F^-(V)$. Then we have $\emptyset \neq F(x) \cap V \subset G(x) \cap V$ and hence $x \in G^-(V)$. Thus we obtain $F^-(V) \subset G^-(V)$. Therefore, $F^-(V) = G^-(V)$.

Theorem 3.9. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then the following properties are equivalent:*

(1) F is upper almost m-continuous;

(2) $\mathrm{Cl}(F)$ *is upper almost m-continuous;*

(3) $\mathrm{sCl}(F)$ *is upper almost* m*-continuous;*
(4) $\mathrm{pCl}(F)$ *is upper almost* m*-continuous;*
(5) $\alpha\mathrm{Cl}(F)$ *is upper almost* m*-continuous;*
(6) $\beta\mathrm{Cl}(F)$ *is upper almost* m*-continuous;*
(7) $\gamma\mathrm{Cl}(F)$ *is upper almost* m*-continuous.*

Proof. We put $G = \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$ or $\gamma\mathrm{Cl}(F)$ in the sequel. Suppose that F is upper almost m-continuous. Then it follows from Theorem 3.5 and Lemmas 3.6 that for every regular open sets V of Y, $G^+(V) = F^+(V) = \mathrm{mInt}(F^+(V)) = \mathrm{mInt}(G^+(V))$. By Theorem 3.5, G is upper almost m-continuous.

Conversely, suppose that G is upper almost m-continuous. Then it follows from Theorem 3.5 and Lemmas 3.6 that for every regular open sets V of Y, $F^+(V) = G^+(V) = \mathrm{mInt}(G^+(V)) = \mathrm{mInt}(F^+(V))$. By Theorem 3.5, F is upper almost m-continuous.

Theorem 3.10. *For a multifunction* $F : (X, m_X) \to (Y, \sigma)$*, the following properties are equivalent:*
(1) F *is lower almost* m*-continuous;*
(2) $\mathrm{Cl}(F)$ *is lower almost* m*-continuous;*
(3) $\mathrm{sCl}(F)$ *is lower almost* m*-continuous;*
(4) $\mathrm{pCl}(F)$ *is lower almost* m*-continuous;*
(5) $\alpha\mathrm{Cl}(F)$ *is lower almost* m*-continuous;*
(6) $\beta\mathrm{Cl}(F)$ *is lower almost* m*-continuous;*
(7) $\gamma\mathrm{Cl}(F)$ *is lower almost* m*-continuous.*

Proof. The proof is similar to that of Theorem 3.9 and is thus omitted.

Remark 3.6. Let (X, τ) and (Y, σ) be topological spaces.

(1) If $F : (X, m_X) \to (Y, \sigma)$ is upper almost m-continuous, where $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\beta(X), \gamma(X)$), then the characterizations established in Theorem 3 of [23] (resp. Theorem 9 of [42], Theorem 8 of [24], Theorem 47 of [11]) are obtained from Theorem 3.9.

(2) If $F : (X, m_X) \to (Y, \sigma)$ is lower almost m-continuous, where $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\beta(X), \gamma(X)$), then the characterizations established in Theorem 6 of [28] (resp. Theorem 4 of [23], Theorem 10 of [42], Theorem 9 of [24], Theorem 49 of [11]) are obtained from Theorem 3.10.

Definition 3.6. Let (X, m_X) be an m-space and A a subset of X. The *m_X-frontier* [39] of a subset A, denoted by m_X-Fr(A), is defined by m_X-Fr$(A) = m_X$-Cl$(A) \cap m_X$-Cl$(X - A) = m_X$-Cl$(A) - m_X$-Int(A).

Theorem 3.11. *Let (X, m_X) be an m-space and (Y, σ) a topological space. The set of all points x of X at which a multifunction $F : (X, m_X) \to (Y, \sigma)$ is not upper almost m-continuous (resp. lower almost m-continuous) is identical with the union of the m_X-frontier of the upper (resp. lower) inverse images of regular open sets containing (resp. meeting) $F(x)$.*

Proof. Suppose that F is not upper almost m-continuous at x. Then, by Theorem 3.3 there exists a regular open set V containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $U \in m_X$ containing x. By Lemma 3.2, we have $x \in \mathrm{mCl}(X - F^+(V))$. Since $x \in F^+(V)$, we have $x \in \mathrm{mCl}(F^+(V))$ and hence $x \in \mathrm{mFr}(F^+(V))$.

Conversely, if F is upper almost m-continuous at x, then for any regular open set V of Y containing $F(x)$ there exists $U \in m_X$ containing x such that $F(U) \subset V$; hence $U \subset F^+(V)$. Therefore, we obtain $x \in U \subset \mathrm{mInt}(F^+(V))$. This contradicts that $x \in \mathrm{mFr}(F^+(V))$. In case F is lower almost m-continuous, the proof is similar.

Remark 3.7. Let (X, τ) and (Y, σ) be topological spaces.

(1) If $F : (X, \tau) \to (Y, \sigma)$ is a multifunction and $m_X = \mathrm{PO}(X)$ (resp. $\beta(X), \gamma(X)$), then by Theorem 3.11 we obtain the result established in Theorem 22 of [42] (resp. Theorem 21 of [24], Theorem 51 of [11]),

(2) If $f : (X, m_X) \to (Y, \sigma)$ is a function, then by Thorem 3.11 we obtain the result established in Theorem 6.7 of [41].

4. Almost m-continuity and δ-open sets

Lemma 4.1. *If $F : (X, m_X) \to (Y, \sigma)$ is lower almost m-continuous, then for each $x \in X$ and each subset B of Y with $F(x) \cap \mathrm{Int}_\delta(B) \neq \emptyset$, there exists $U \in m_X$ containing x such that $U \subset F^-(B)$.*

Proof. Let $x \in X$ and B be a subset of Y with $F(x) \cap \mathrm{Int}_\delta(B) \neq \emptyset$. Since $F(x) \cap \mathrm{Int}_\delta(B) \neq \emptyset$, there exists a nonempty regular open set V of Y such that $V \subset B$ and $F(x) \cap V \neq \emptyset$. Since F is lower almost m-continuous, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$; hence $U \subset F^-(B)$.

Theorem 4.1. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost m-continuous;

(2) $\mathrm{mCl}(F^+(B)) \subset F^+(\mathrm{Cl}_\delta(B))$ for every subset B of Y;

(3) $F(\mathrm{mCl}(A)) \subset \mathrm{Cl}_\delta(F(A))$ for every subset A of X;

(4) $F^+(K) = \mathrm{mCl}(F^+(K))$ for every δ-closed set K of Y;

(5) $F^-(V) = \mathrm{mInt}(F^-(V))$ for every δ-open set V of Y;

(6) $F^-(\mathrm{Int}_\delta(B)) \subset \mathrm{mInt}(F^-(B))$ for each subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Suppose that $x \notin F^+(\mathrm{Cl}_\delta(B))$. Then we have $x \in F^-(Y - \mathrm{Cl}_\delta(B)) = F^-(\mathrm{Int}_\delta(Y - B))$. By Lemma 4.1, there exists $U \in m_X$ containing x such that $U \subset F^-(Y - B) = X - F^+(B)$. Thus we have $U \cap F^+(B) = \emptyset$. By Lemma 3.2, we obtain $x \in X - \mathrm{mCl}(F^+(B))$ and hence $\mathrm{mCl}(F^+(B)) \subset F^+(\mathrm{Cl}_\delta(B))$.

(2) $\Rightarrow$ (3): Let A be any subset of X. By (2) and Lemma 3.1, we have $\mathrm{mCl}(A) \subset \mathrm{mCl}(F^+(F(A))) \subset F^+(\mathrm{Cl}_\delta(F(A)))$. Thus we obtain $F(\mathrm{mCl}(A)) \subset \mathrm{Cl}_\delta(F(A))$.

(3) $\Rightarrow$ (1): Let B be a subset of Y. Then by the hypothesis and Theorem 2.1 of [14], $F(\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B)))))) \subset \mathrm{Cl}_\delta(F(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B)))))) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))) \subset \mathrm{Cl}(B)$. Therefore, $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^+(\mathrm{Cl}(B))$. By Theorem 3.6, F is lower almost m-continuous.

(2) $\Rightarrow$ (4): Let K be a δ-closed set of Y. Then $\mathrm{Cl}_\delta(K) = K$. By (2), we have $\mathrm{mCl}(F^+(K)) \subset F^+(\mathrm{Cl}_\delta(K)) = F^+(K)$. Therefore, $F^+(K) = \mathrm{mCl}(F^+(K))$.

(4) $\Rightarrow$ (5): Let V be any δ-open set of Y. Then $Y - F$ is a δ-closed set of Y. By (4) and Lemma 3.1, we have $X - F^-(V) = F^+(Y - V) = \mathrm{mCl}(F^+(Y - V) = X - \mathrm{mInt}(F^-(V))$. Hence $F^-(V) = \mathrm{mInt}(F^-(V))$.

(5) $\Rightarrow$ (6): Let B be any subset of Y. Then, by (5) $F^-(\mathrm{Int}_\delta(B)) = \mathrm{mInt}(F^-(\mathrm{Int}_\delta(B))) \subset \mathrm{mInt}(F^-(B))$.

(6) $\Rightarrow$ (1): Let V be any regular open set of Y. Then V is δ-open and $\mathrm{Int}_\delta(V) = V$. Thus, by (6) $F^-(V) \subset \mathrm{mInt}(F^-(V))$. By Lemma 3.1 $F^-(V) = \mathrm{mInt}(F^-(V))$ and it follows from Theorem 3.6 F is lower almost m-continuous.

Remark 4.1. Let (X, τ) and (Y, σ) be topological spaces.

(1) If $F : (X, m_X) \to (Y, \sigma)$ is lower almost m-continuous and $m_X = \tau$, then Theorem 5 of [30] follows from Theorem 4.1.

(2) If $f : (X, m_X) \to (Y, \sigma)$ is almost m-continuous, then by Theorem 4.1 we obtain Theorem 3.5 of [41].

Theorem 4.2. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is upper almost m-continuous;
(2) $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(B))))) \subset F^-(\mathrm{Cl}_\delta(B))$ *for every subset B of Y;*
(3) $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^-(\mathrm{Cl}_\delta(B))$ *for every subset B of Y.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. By Lemma 2 of [49], $\mathrm{Cl}_\delta(B)$ is closed in Y. By Theorem 3.5 we obtain $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(B)))) \subset F^-(\mathrm{Cl}_\delta(B))$.
(2) $\Rightarrow$ (3): This is obvious since $\mathrm{Cl}(B) \subset \mathrm{Cl}_\delta(B)$.
(3) $\Rightarrow$ (1): Let K be a regular closed set of Y. Then by (3) and Theorem 2.1 of [14], $\mathrm{mCl}(F^-(K)) = \mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(K)))) = \mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(K))))) \subset F^-(\mathrm{Cl}_\delta(K)) = F^-(K)$. Hence $\mathrm{mCl}F^-(K)) \subset F^-(K)$. It follows from Lemma 3.1 that $\mathrm{mCl}F^-(K)) = F^-(K)$. By Theorem 3.5, F is upper almost m-continuous.

Theorem 4.3. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is lower almost m-continuous;
(2) $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(B))))) \subset F^+(\mathrm{Cl}_\delta(B))$ *for every subset B of Y;*
(3) $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^+(\mathrm{Cl}_\delta(B))$ *for every subset B of Y.*

Proof. The proof is similar to that of Theorem 4.2.

Remark 4.2. If $f : (X, m_X) \to (Y, \sigma)$ is a (single valued) function, then Theorem 3.4 of [41] follows from Theorems 4.2 and 4.3.

Theorem 4.4. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, where (Y, σ) is a semi-regular space, the following properties are equivalent:*
(1) F is upper m-continuous;
(2) $F^-(\mathrm{Cl}_\delta(B)) = \mathrm{mCl}(F^-(\mathrm{Cl}(_\delta(B)))$ *for every subset B of Y;*
(3) $F^-(K) = \mathrm{mCl}(F^-(K))$ *for every δ-closed set K of Y;*
(4) $F^+(V) = \mathrm{mInt}(F^+(V))$ *for every δ-open set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Then $\mathrm{Cl}_\delta(B))$ is closed in Y. By Theorem 3.1, we obtain $F^-(\mathrm{Cl}_\delta(B)) = \mathrm{mCl}(F^-(\mathrm{Cl}(_\delta(B)))$.

(2) $\Rightarrow$ (3): Let K be a δ-closed set of Y. Then $\mathrm{Cl}_\delta(K) = K$. By (2), we have $F^-(K) = \mathrm{mCl}(F^-(K))$.

(3) $\Rightarrow$ (4): Let V be any δ-open set. Then $Y - V$ is δ-closed and $F^-(Y - V) = \mathrm{mCl}(F^-(Y - V))$. Therefore, $X - F^+(V) = \mathrm{mCl}(X - F^+(V)) = X - \mathrm{mInt}(F^+(V))$. Hence we obtain $F^+(V) = \mathrm{mInt}(F^+(V))$.

(4) $\Rightarrow$ (1): Let V be any open set of Y. Since Y is semi-regular, V is δ-open in Y and by (4) we obtain $F^+(V) = \mathrm{mInt}(F^+(V))$. By Theorem 3.1, F is upper m-continuous.

Theorem 4.5. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, where (Y, σ) is a semi-regular space, the following properties are equivalent:*

(1) F is lower m-continuous;

(2) $F^+(\mathrm{Cl}_\delta(B)) = \mathrm{mCl}(F^+(\mathrm{Cl}(_\delta(B)))$ for every subset B of Y;

(3) $F^+(K) = \mathrm{mCl}(F^+(K))$ for every δ-closed set K of Y;

(4) $F^-(V) = \mathrm{mInt}(F^-(V))$ for every δ-open set V of Y;

(5) F is lower almost m-continuous.

Proof. The proofs of the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are similar to those in Theorem 4.4.

(4) $\Rightarrow$ (5): Let V be any regular open set of Y. Then V is δ-open in Y and by (4) $F^-(V) = \mathrm{mInt}(F^-(V))$. By Theorem 3.6, F is lower almost m-continuous.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any open set of Y such that $F(x) \cap V \neq \emptyset$. Since Y is semi-regular, there exists a regular open set W such that $F(x) \cap W \neq \emptyset$ and $W \subset V$. Since F is lower almost m-continuous, there exists $U \in m_X$ containing x such that $F(u) \cap W \neq \emptyset$ for every $u \in U$. Therefore, $F(u) \cap V \neq \emptyset$ for every $u \in U$. This shows that F is lower m-continuous.

Corollary 4.1. *Let (Y, σ) be a semi-regular space and m_X have property $(\mathcal{B})$. Then for a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper m-continuous;

(2) $F^-(\mathrm{Cl}_\delta(B))$ is m-closed in X for every subset B of Y;

(3) $F^-(K)$ is m-closed in X for every δ-closed set K of Y;

(4) $F^+(V)$ is m-open in X for every δ-open set V of Y.

Proof. The proof follows from Theorem 4.4 and Lemma 3.3.

Corollary 4.2. *Let (Y,σ) be a semi-regular space and m_X have property $(\mathcal{B})$. Then for a multifunction $F:(X,m_X)\to(Y,\sigma)$, the following properties are equivalent:*
(1) F is lower m-continuous;
(2) $F^+(\mathrm{Cl}_\delta(B))$ is m-closed in X for every subset B of Y;
(3) $F^+(K)$ is m-closed in X for every δ-closed set K of Y;
(4) $F^-(V)$ is m-open in X for every δ-open set V of Y;
(5) F is lower almost m-continuous.

Corollary 4.3. *Let (Y,σ) be a semi-regular space and m_X have property $(\mathcal{B})$. Then for a function $f:(X,m_X)\to(Y,\sigma)$, the following properties are equivalent:*
(1) f is m-continuous;
(2) $f^{-1}(\mathrm{Cl}_\delta(B))$ is m-closed in X for every subset B of Y;
(3) $f^{-1}(K)$ is m-closed in X for every δ-closed set K of Y;
(4) $f^{-1}(V)$ is m-open in X for every δ-open set V of Y;
(5) f is almost m-continuous.

5. Almost m-continuity and preopen sets

Theorem 5.1. *For a multifunction $F:(X,m_X)\to(Y,\sigma)$, the following properties are equivalent:*
(1) F is upper almost m-continuous;
(2) $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V)))))\subset F^-(\mathrm{Cl}(V))$ for every preopen set V of Y;
(3) $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(V))))\subset F^-(\mathrm{Cl}(V))$ for every preopen set V of Y;
(4) $F^+(V)\subset \mathrm{mInt}(F^+\mathrm{Int}(\mathrm{Cl}(V))))$ for every preopen set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any preopen set of Y. Then $\mathrm{Cl}(V)$ is closed in Y and by Theorem 3.5 we have $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V)))))\subset F^-(\mathrm{Cl}(V))$.

(2) $\Rightarrow$ (3): Let V be any preopen set of Y. Then we have $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(V))))\subset \mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V)))))\subset F^-(\mathrm{Cl}(V))$.

(3) $\Rightarrow$ (4): Let V be any preopen set of Y. By Lemma 3.1 we have

$$\begin{gathered}X\text{-}\mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))=\mathrm{mCl}(X-F^+(\mathrm{Int}(\mathrm{Cl}(V))))=\\ \mathrm{mCl}(F^-(Y-\mathrm{Int}(\mathrm{Cl}(V))))=\mathrm{mCl}(F^-(\mathrm{Cl}(Y-\mathrm{Cl}(V))))=\\ \mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(Y-\mathrm{Cl}(V))))\subset F^-(\mathrm{Cl}(Y-\mathrm{Cl}(V)))=\\ F^-(Y-\mathrm{Int}(\mathrm{Cl}(V)))\subset F^-(Y-V)=X-F^+(V).\end{gathered}$$

Therefore, we obtain $F^+(V) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$.

(4) $\Rightarrow$ (1): Let V be any regular open set of Y. Then V is preopen and $F^+(V) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V)))) = \mathrm{mInt}(F^+(V))$. By Lemma 3.1, $F^+(V) = \mathrm{mInt}(F^+(V))$. It follows from Theorem 3.5 that F is upper almost m-continuous.

Theorem 5.2. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost m-continuous;

(2) $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V))))) \subset F^+(\mathrm{Cl}(V))$ *for every preopen set V of Y;*

(3) $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(V)))) \subset F^+(\mathrm{Cl}(V))$ *for every preopen set V of Y;*

(4) $F^-(V) \subset \mathrm{mInt}(F^-\mathrm{Int}(\mathrm{Cl}(V))))$ *for every preopen set V of Y.*

Proof. The proof is similar to that of Theorem 5.1.

Remark 5.1. Let (X, τ) and (Y, σ) be topological spaces. Let $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X), \beta(X)$).

(1) If $F : (X, m_X) \to (Y, \sigma)$ is upper almost m-continuous, then the characterizations established in Theorem 1 of [23] (resp. Theorem 5 of [42], Theorem 4 of [35], Theorem 5 of [24]) are obtained from Theorem 5.1.

(2) If $F : (X, m_X) \to (Y, \sigma)$ is lower almost m-continuous, then the characterizations established in Theorem 2 of [23] (resp. Theorem 6 of [42], Theorem 6 of [35], Theorem 6 of [24]) are obtained from Theorem 5.2.

(3) If $f : (X, m_X) \to (Y, \sigma)$ is a (single valued) function, then by Theorems 5.1 and 5.2 we obtain Theorem 3.3 of [41].

6. Almost m-continuity and m-continuity

Definition 6.1. A subset A of a topological space (X, τ) is said to be *α-semi-regular* [31] if for each point $a \in A$ and each open set U containing a, there exists a regular open set G such that $a \in G \subset U$.

Lemma 6.1. (Popa [31]). *If A is an α-semi-regular set of a topological space (X, τ), then for every open set U such that $A \cap U \neq \emptyset$, there exists a regular open set G such that $A \cap G \neq \emptyset$ and $G \subset U$.*

Theorem 6.1. *If $F : (X, m_X) \to (Y, \sigma)$ is lower almost m-continuous and F(x) is α-semi-regular in Y for each $x \in X$, then F is lower m-continuous.*

Proof. Let $x \in X$ and V be any open set such that $F(x) \cap V \neq \emptyset$. Since $F(x)$ is α-semi-regular, by Lemma 6.1 there exists a regular open set G such that $F(x) \cap G \neq \emptyset$ and $G \subset V$. Since F is lower almost m-continuous, by Theorem 3.4 there exists $U \in m_X$ containing x such that $x \in U \subset F^-(G) \subset F^-(V)$. Hence $x \in \mathrm{mInt}(F^-(V))$ and hence $F^-(V) \subset \mathrm{mInt}(F^-(V))$. By Lemma 3.1, $F^-(V) = \mathrm{mInt}(F^-(V))$ and by Theorem 3.2 F is lower m-continuous.

Corollary 6.1. *Let (Y, σ) be a semi-regular space. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is lower almost m-continuous if and only if F is lower m-continuous.*

Remark 6.1. Let (X, τ) and (Y, σ) be topological spaces.

(1) If $F : (X, m_X) \to (Y, \sigma)$ is a multifunction, where $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$), then the results established in Theorem 3.1 of [27] (resp. Theorem 5.1 of [34], Theorem 18 of [42], Theorem 10 of [35]) are obtained from Theorem 6.1 and Corollary 6.1

(2) If $f : (X, m_X) \to (Y, \sigma)$ is a (single valued) function, then by Corollary 6.1 we obtain Theorems 4.3 of [41].

Since the intersection of two regular open sets of a topological space (Y, σ) is regular open, the collection of all regular open sets of Y forms a base for a topology σ_s for Y. This is called the semi-regularization of σ.

Theorem 6.2. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction, where m_X has property ($\mathcal{B}$). Then F is lower almost m-continuous if and only is $F : (X, m_X) \to (Y, \sigma_s)$ is lower m-continuous.*

Proof. *Necessity.* Suppose that $F : (X, m_X) \to (Y, \sigma)$ is lower almost m-continuous and let V be an open set in (Y, σ_s). Then there exists a collection $\{V_\alpha : \alpha \in \Delta\}$ of regular open sets such that $V = \cup\{V_\alpha : \alpha \in \Delta\}$. Since F is lower almost m-continuous and m_X has property ($\mathcal{B}$), by Corollary 3.2 $F^-(V_\alpha) \in m_X$ for each $\alpha \in \Delta$. Then $F^-(V) = F^-(\cup\{V_\alpha : \alpha \in \Delta\}) = \cup\{F^-(V_\alpha) : \alpha \in \Delta\} \in m_X$ since m_X is has property ($\mathcal{B}$). Therefore, by Lemma 3.3, for every open set V of σ_s, $F^-(V) = \mathrm{mInt}(F^-(V))$ and by Theorem 3.2 F is lower m-continuouos.

Sufficiency. Suppose that $F : (X, m_X) \to (Y, \sigma_s)$ is lower m-continuous and let V be any regular open set V of (Y, σ). Then V is open in (Y, σ_s) and by Theorem 3.2, $F^-(V) = \mathrm{mInt}(F^-(V))$. By Theorem 3.6, $F : (X, m_X) \to (Y, \sigma)$ is lower almost m-continuous.

Remark 6.2. Let (X, τ) and (Y, σ) be topological spaces. If $F : (X, m_X) \to (Y, \sigma)$ is a lower almost m-continuous, where $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, $\beta(X)$, $\gamma(X)$), then by Theorem 6.2 we obtain the result established in Corollary 4 of [30] (resp. Theorem 5.1 of [34], Theorem 18 of [42], Theorem 10 of [35], Theorem 14 of [24], Corollary 19 of [11]).

Definition 6.2. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be *a*-m-continuous* if $X - F^-(\mathrm{Fr}(V)) \in m_X$ for each open set V of Y, where $\mathrm{Fr}(V)$ denotes the frontier of V.

Theorem 6.3. *Let X be a nonempty set with two minimal structures m_1 and m_2 such that $U \cap V \in m_1$ whenever $U \in m_1$ and $V \in m_2$. If a multifunction $F : X \to (Y, \sigma)$ is upper almost m_1-continuous and a*-m_2-continuous, then F is upper m_1-continuous.*

Proof. Let $x \in X$ and V be any open set of Y containing $F(x)$. It follows that $F(x) \cap \mathrm{Fr}(V) = \emptyset$. Since F is upper almost m_1-continuous, there exists $G \in m_1$ containing x such that $F(G) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Put $U = G \cap (X - F^-(\mathrm{Fr}(V)))$. Since F is a*-m_2-continuous, $(X - F^-(\mathrm{Fr}(V)) \in m_2$. Thus $x \in U, U \in m_1$ and $F(U) \subset F(G) \cap (Y - \mathrm{Fr}(V)) \subset \mathrm{Int}(\mathrm{Cl}(V)) \cap (Y - \mathrm{Fr}(V)) = V$. This shows that F is upper m_1-continuous.

Theorem 6.4. *Let X be a nonempty set with two minimal structures m_1 and m_2 such that $U \cap V \in m_1$ whenever $U \in m_1$ and $V \in m_2$. If a multifunction $F : X \to (Y, \sigma)$ is upper almost m_2-continuous and a*-m_1-continuous, then F is upper m_1-continuous.*

Proof. The proof is similar to that of Theorem 6.3 and is thus omitted.

Remark 6.3. If $f : X \to (Y, \sigma)$ is a function, then by Theorems 6.3 and 6.4 we obtain Theorems 4.1 and 4.2 of [41].

7. Separation axioms and upper almost m-continuity

Definition 7.1. A subset A of an m-space (X, m_X) is said to be *m-dense* in X if $\mathrm{mCl}(A) = X$.

Theorem 7.1. *Let X be a nonempty set with two minimal structures m_1 and m_2 such that $U \cap V \in m_2$ whenever $U \in m_1$ and $V \in m_2$ and (Y, σ) be a Hausdorff space.*

If the following four conditions are satisfied,

(1) a multifunction $F : (X, m_1) \to (Y, \sigma)$ is upper weakly m-continuous,

(2) a multifunction $G : (X, m_2) \to (Y, \sigma)$ is upper almost m-continuous,

(3) F(x) and G(x) are compact sets of (Y, σ) for each $x \in X$,

(4) $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$,

then $A = m_2\mathrm{Cl}(A)$. If $F(x) \cap G(x) \neq \emptyset$ for each point x in an m-dense set D of (X, m_2), then $F(x) \cap G(x) \neq \emptyset$ for each point in X.

Proof. Suppose that $x \in X - A$. Then we have $F(x) \cap G(x) = \emptyset$. Since $F(x)$ and $G(x)$ are compact sets of a Hausdorff space Y, there exist open sests V and W of Y such that $F(x) \subset V, G(x) \subset W$ and $V \cap W = \emptyset$; hence $\mathrm{Cl}(V) \cap \mathrm{Int}(\mathrm{Cl}(W)) = \emptyset$. Since$F$ is upper weakly m-continuous, there exists $U_1 \in m_1$ containing x such that $F(U_1) \subset \mathrm{Cl}(V)$. Since G is upper almost m-continuous, there exists $U_2 \in m_2$ containing x such that $G(U_2) \subset \mathrm{Int}(\mathrm{Cl}(W))$. Now, set $U = U_1 \cap U_2$, then we have $U \in m_2$ and $U \cap A = \emptyset$. Therefore, by Lemma 3.2 we have $x \in X - m_2\mathrm{Cl}(A)$ and hence $A = m_2\mathrm{Cl}(A)$. On the other hand, if $F(x) \cap G(x) \neq \emptyset$ on an m-dense set D of (X, m_2), then we have $X = m_2\mathrm{Cl}(D) \subset m_2\mathrm{Cl}(A) = A$. Therefore, we obtain $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$.

Remark 7.1. Let (X, τ) and (Y, σ) be topological spaces.

(1) If $F : (X, m_X) \to (Y, \sigma)$ is a multifunction, where $m_1 = m_2 = \tau$ (resp $m_1 = \tau^\alpha$ and $m_2 = \mathrm{SO}(X)$), then by Theorems 7.1 we obtain the results established in Theorem 17 of [48] (resp. Theorem 24 of [24]).

(2) If $f : (X, m_X) \to (Y, \sigma)$ a function, then by Theorem 7.1 we obtain the result established in Theorem 6.4 of [41].

Definition 7.2. An m-space (X, m_X) is said to be m-T_2 if for each distinct points $x, y \in X$ there exist $U, V \in m_X$ containing x, y, respectively, such that $U \cap V = \emptyset$.

Definition 7.3. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be *injective* if $x \neq y$ implies that $F(x) \cap F(y) = \emptyset$.

Theorem 7.2. *If $F : (X, m_X) \to (Y, \sigma)$ is an upper almost m-continuous injective multifunction into a Hausdorff space (Y, σ) and F(x) is compact for each $x \in X$, then X is m-T_2.*

Proof. For any distinct points x_1, x_2 of X, we have $F(x_1) \cap F(x_2) = \emptyset$ since F is injective. Since $F(x)$ is compact for each $x \in X$ and Y is Hausdorff, there exist an open set V_i such that $F(x_i) \subset V_i$ for $i = 1, 2$ and $V_1 \cap V_2 = \emptyset$; hence $\mathrm{Int}(\mathrm{Cl}(V_1)) \cap \mathrm{Int}(\mathrm{Cl}(V_2)) = \emptyset$. Since F is upper almost m-continuous, there exists $U_i \in m_X$ containing x_i such that $F(U_i) \subset \mathrm{Int}(\mathrm{Cl}(V_i))$ for $i = 1, 2$. Therefore, we obtain $U_1 \cap U_2 = \emptyset$ and hence X is m-T_2.

Remark 7.2. For a function $f : (X, m_X) \to (Y, \sigma)$, by Theorem 7.2 we obtain the result established in Theorem 6.1 of [41].

Lemma 7.1. (Smithson [48]). *If A and B are disjoint compact subsets of a Urysohn space (Y, σ), then there exist open sets U and V of X such that $A \subset U$, $B \subset V$ and $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$.*

Theorem 7.3. *Let X be a nonempty set with two minimal structures m_1 and m_2 such that $U \cap V \in m_2$ whenever $U \in m_1$ and $V \in m_2$ and (Y, σ) be a Urysohn space.*
If the following four conditions are satisfied,

(1) a multifunction $F_1 : (X, m_1) \to (Y, \sigma)$ is upper almost m-continuous,

(2) a multifunction $F_2 : (X, m_2) \to (Y, \sigma)$ is upper almost m-continuous,

(3) $F_1(x)$ and $F_2(x)$ are compact sets of (Y, σ) for each $x \in X$,

(4) $F_1(x) \cap F_2(x) \neq \emptyset$ for each $x \in X$,

then a multifunction $F : (X, m_2) \to (Y, \sigma)$, defined by $F(x) = F_1(x) \cap F_2(x)$ for each $x \in X$, is upper almost m-continuous.

Proof. Let $x \in X$ and V be an open set of Y such that $F(x) \subset V$. Then, $A = F_1(x) - V$ and $B = F_2(x) - V$ are disjoint compact sets. By Lemma 7.1, there exist open sets V_1 and V_2 such that $A \subset V_1, B \subset V_2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Since F_1 is upper almost m-continuous, there exists $U_1 \in m_1$ containing x such that $F_1(U_1) \subset \mathrm{Int}(\mathrm{Cl}(V_1 \cup V))$. Since F_2 is upper almost m-continuous, there exists $U_2 \in m_2$ containing x such that $F_2(U_2) \subset \mathrm{Int}(\mathrm{Cl}(V_2 \cup V))$. Set $U = U_1 \cap U_2$, then $U \in m_2$ containing x. If $y \in F(x_0)$ for any $x_0 \in U$, then $y \in \mathrm{Int}(\mathrm{Cl}(V_1 \cup V)) \cap \mathrm{Int}(\mathrm{Cl}(V_2 \cup V)) = \mathrm{Int}((\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2)) \cup \mathrm{Cl}(V))$. Since $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$, we have $y \in \mathrm{Int}(\mathrm{Cl}(V))$ and hence $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Therefore, F is upper almost m-continuous.

Definition 7.4. For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the graph $G(F) = \{(x, F(x)) : x \in X\}$ is said to be *strongly almost m-closed* if for each $(x, y) \in (X \times Y) - G(F)$, there exists $U \in m_X$ containing x and a regular open set V containing y such that $(U \times V) \cap G(F) = \emptyset$.

Lemma 7.2. *A multifunction $F : (X, m_X) \to (Y, \sigma)$ has a strongly almost m-closed graph if and only if for each $(x, y) \in (X \times Y) - G(F)$, there exists $U \in m_X$ containing x and a regular open set V containing y such that $F(U) \cap V = \emptyset$.*

Theorem 7.4. *If $F : (X, m_X) \to (Y, \sigma)$ is an upper almost m-continuous multifunction into a Hausdorff space (Y, σ) such that F(x) is α-paracompact for each $x \in X$, then G(F) is strongly almost m-closed.*

Proof. Let $(x_0, y_0) \in (X \times Y) - G(F)$, then $y_0 \in Y - F(x_0)$. Since (Y, σ) is Hausdorff, for each $y \in F(x_0)$ there exist open sets $V(y)$ and $W(y)$ of Y such that $y \in V(y), y_0 \in W(y)$ and $V(y) \cap W(y) = \emptyset$. The family $\{V(y) : y \in F(x_0)\}$ is an open cover of $F(x_0)$ which is α-paracompact. Thus it has a locally finite open refinement $\mathcal{U}$ $= \{U_\alpha : \alpha \in \Delta\}$ which covers $F(x_0)$. Let W_0 be an open neighborhood of y_0 such that W_0 intersects only finitely many member, say, $U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n}$. For each α_k, we choose $y_k \in F(x_0)$ such that $U_{\alpha_k} \subset V(y_k)$ for each $k = 1, 2, ..., n$. Now set $W = W_0 \cap (\cap_{k=1}^{n} W(y_k))$ and $U = \cup\{U_\alpha : \alpha \in \Delta\}$. Then W is an open neighborhood of y_0 with $W \cap U = \emptyset$ which implies $\mathrm{Int}(\mathrm{Cl}(W)) \cap \mathrm{Int}(\mathrm{Cl}(U)) = \emptyset$. Since F is upper almost m-continuous, there exists $U_0 \in m_X$ containing x_0 such that $F(U_0) \subset \mathrm{Int}(\mathrm{Cl}(U))$. Therefore, we have $F(U_0) \cap \mathrm{Int}(\mathrm{Cl}(W)) = \emptyset$ and $G(F)$ has a strongly almost m-closed graph.

Theorem 7.5. *Let (X, m_X) be an m-space. If for each pair of distinct points x_1 and x_2 in X, there exists a multifunction $F : (X, m_X) \to (Y, \sigma)$, where (Y, σ) is a normal space, such that*
(1) F is punctually closed,
(2) F is upper weakly m-continuous at x_1,
(3) F is upper almost m-continuous at x_2, and
(4) $F(x_1) \cap F(x_2) = \emptyset$,
then (X, m_X) is an m-T_2-space.

Proof. Let x_1 and x_2 be distinct points of X. Then, since (Y, σ) is a normal space, F is punctually closed and $F(x_1) \cap F(x_2) = \emptyset$, there exist open sets V_1 and V_2 containing $F(x_1)$ and $F(x_2)$, respectively, such that $V_1 \cap V_2 = \emptyset$; hence $\mathrm{Cl}(V_1) \cap \mathrm{Int}(\mathrm{Cl}(V_2)) = \emptyset$. Since F is upper weakly m-continuous at x_1, there exists $U_1 \in m_X$ containing x_1 such that $F(U_1) \subset \mathrm{Cl}(V_1)$. Since F is upper almost m-continuous at x_2, there exists $U_2 \in m_X$ containing x_2 such that $F(U_2) \subset \mathrm{Int}(\mathrm{Cl}((V_2))$. Therefore, we have $F(U_1) \cap F(U_2) = \emptyset$ which implies $U_1 \cap U_2 = \emptyset$. This shows that (X, m_X) is an m-T_2-space.

Remark 7.3. Let (X, τ) and (Y, σ) be topological spaces. If $F : (X, m_X) \to (Y, \sigma)$ is a multifunction, where $m_X = \tau$ (resp. $\gamma(X)$), then by Theorem 7.5 we obtain the result established in Theorem 2.7 of [29] (resp. Theorem 45 of [11]).

8. New forms of almost continuity for multifunctions

There are many modifications of open sets in topological spaces. It is well-known that the following relationships hold among generalized open sets defined in Section 2:

DIAGRAM I

$$\begin{array}{ccccc} \text{open} \Rightarrow & \alpha\text{-open} & \Rightarrow & \text{preopen} & \\ & \Downarrow & & \Downarrow & \\ & \text{semi-open} & \Rightarrow & \gamma\text{-open} & \Rightarrow \beta\text{-open} \end{array}$$

Recently, many researchers are interested in δ-preopen sets [44] and δ-semi-open sets [26]. First, let recall θ-open sets due to Veličko [49] which play an important role in this section. A point $x \in X$ is a θ-cluster point of a subset A of X if $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set V containing x. The set of all θ-cluster points of A is called the *θ-closure* of A and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then A is said to *θ-closed* [49]. The complement of a θ-closed set is said to *θ-open*. The union of all θ-open sets contained in A is called the *θ-interior* of A and is denoted by $\mathrm{Int}_\theta(A)$. The family of all θ-open sets of a topological space (X, τ) is a topology for X and is denoted by τ_θ.

Definition 8.1. A subset of a topological space (X, τ) is said to be

(1) *δ-semiopen* [26] (resp. *θ-semiopen* [9] if $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}_\theta(A))$),

(2) *δ-preopen* [44] (resp. *θ-preopen*) if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}_\theta(A))$),

(3) *δ-β-open* [13] (resp. *θ-β-open*) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\theta(A)))$).

By $\delta\mathrm{SO}(X)$ (resp. $\delta\mathrm{PO}(X)$, $\delta\beta(X)$, $\theta\mathrm{SO}(X)$, $\theta\mathrm{PO}(X)$, $\theta\beta(X)$), we denote the collection of all δ-semiopen (resp. δ-preopen, δ-β-open, θ-semiopen, θ-preopen, θ-β-open) sets of a topological space (X, τ).

Lemma 8.1. *For a subset of a topological space (X, τ), the following properties hold:*

(1) Every θ-semiopen set is δ-semiopen and every δ-semiopen set is semiopen,

(2) Every preopen set is δ-preopen and every δ-preopen set is θ-preopen,

(3) Every β-open set is δ-β-open and every δ-β-open set is θ-β-open.

Proof. This follows from the fact that $\mathrm{Cl}(A) \subset \mathrm{Cl}_\delta(A) \subset \mathrm{Cl}_\theta(A)$ for any subset A of X [49].

By Lemma 9.1 and Definitions 2.1 and 9.1, the following relationships hold:

DIAGRAM II

$$\begin{array}{ccccccccccc}
\theta\text{-open} & \Rightarrow & \delta\text{-open} & \Rightarrow & \text{open} & \Rightarrow & \text{preopen} & \Rightarrow & \delta\text{-preopen} & \Rightarrow & \theta\text{-preopen} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\theta\text{-semiopen} & \Rightarrow & \delta\text{-semiopen} & \Rightarrow & \text{semiopen} & \Rightarrow & \beta\text{-open} & \Rightarrow & \delta\text{-}\beta\text{-open} & \Rightarrow & \theta\text{-}\beta\text{-open}
\end{array}$$

In Diagram II, none of implications is reversible as shown by the following examples:

Example 8.1. (Caldas et al. [9] and Park et al. [26]).

(1) Let X be the real numbers with the usual topology and $A = (0, 1]$, then A is a θ-semiopen set which is not open.

(2) Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $A = \{a, b\}$. Then A is a δ-open set which is not θ-semi-open.

(3) Let $X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $A = \{a, c, d\}$. Then A is an open set of (X, τ) which is not δ-semiopen.

Example 8.2. (1) Let X be the real numbers with the usual topology and $A = (0, 1]$, then A is a β-open set which is not θ-preopen.

(2) Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $A = \{c\}$. Then A is a θ-preopen set which is not δ-β-open.

(3) Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$ and $A = \{c\}$. Then A is a δ-preopen set which is not β-open.

For topological spaces (X, τ) and (Y, σ), we can define many new types of generalized almost continuity for a multifunction $F : (X, \tau) \to (Y, \sigma)$. For example, in case $m_X = \tau_\theta$, $\delta\mathrm{SO}(X)$, $\delta\mathrm{PO}(X))$, $\delta\beta(X)$, $\theta\mathrm{SO}(X)$, $\theta\mathrm{PO}(X)$, $\theta\beta(X)$ we can define new types of generalized almost continuity for multifunctions as follows:

Definition 8.2. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper δ-continuous* [8] (resp. *upper almost δ-semi-continuous, upper almost δ-precontinuous*, *upper almost δ-β-continuous*) if for each $x \in X$ and each open set V of Y containing $F(x)$, there exists a δ-open (resp. δ-semiopen, δ-preopen, δ-β-open) set U of X containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$,

(2) *lower δ-continuous* [8] (resp. *lower almost δ-semi-continuous, lower almost δ-precontinuous*, *lower almost δ-β-continuous*) if for each $x \in X$ and each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a δ-open (resp. δ-semiopen, δ-preopen, δ-β-open) set U of X containing x such that $F(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$.

Definition 8.3. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper almost θ-continuous* (resp. *upper almost θ-semi-continuous, upper almost θ-precontinuous*, *upper almost θ-β-continuous*) if for each $x \in X$ and each open set V of Y containing $F(x)$, there exists a θ-open (resp. θ-semiopen, θ-preopen, θ-β-open) set U of X containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$,

(2) *lower almost θ-continuous* (resp. *lower almost θ-semi-continuous, lower almost θ-precontinuous*, *lower almost θ-β-continuous*) if for each $x \in X$ and each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a θ-open (resp. θ-semiopen, θ-preopen, θ-β-open) set U of X containing x such that $F(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$.

For the multifunctions defined in Definitions 9.2 and 9.3, we have the following relationship:

DIAGRAM III

u/l aθ-c	$\Rightarrow$	u/l δ-c	$\Rightarrow$	u/l ac	$\Rightarrow$	u/l apc	$\Rightarrow$	u/l aδ-pc	$\Rightarrow$	u/l aθ-pc
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$
u/l aθ-sc	$\Rightarrow$	u/l aδ-sc	$\Rightarrow$	u/l aqc	$\Rightarrow$	u/l aβ-c	$\Rightarrow$	u/l aδ-β-c	$\Rightarrow$	u/l aθ-β-c

In Diagram III, we use the abbreviations as follows: u = upper, l = lower, a = almost, c = continuous, p = pre, s = semi and q = quasi.

Definition 8.4. A topological space (X, τ) is said to be *almost-regular* [45] if for each regular closed set F of X and each point $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $F \subset V$.

Lemma 8.2. (Noiri [21]). *For a topological space (X, τ), the following characterizations hold:*

(1) (X, τ) is semi-regular if and only if $\mathrm{Cl}_\delta(A) = \mathrm{Cl}(A)$ for any subset A of X,

(2) (X, τ) is almost-regular if and only if $\mathrm{Cl}_\theta(A) = \mathrm{Cl}_\delta(A)$ for any subset A of X.

Theorem 8.1. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties hold:*

(1) If (X, τ) is semi-regular, then upper/lower almost δ-semicontinuity (resp. upper/lower almost δ-precontinuity, upper/lower almost δ-β-continuity) is equivalent to upper/lower almost quasi-continuity (resp. upper/lower almost precontinuity, upper/lower almost β-continuity),

(2) If (X, τ) is almost-regular, then upper/lower almost θ-semicontinuity (resp. upper/lower almost θ-precontinuity, upper/lower almost θ-β-continuity) is equivalent to upper/lower almost δ-semicontinuity (resp. upper/lower almost δ-precontinuity, upper/lower almost δ-β-continuity).

Proof. This is an immediate consequence of Lemma 8.1.

Corollary 8.1. *Let (X, τ) be a regular space. For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties hold:*

(1) upper/lower almost θ-semicontinuity, upper/lower almost δ-semicontinuity and upper/lower almost quasicontinuity are equivalent,

(2) upper/lower almost θ-precontinuity, upper/lower almost δ-precontinuity and upper/lower almost precontinuity are equivalent,

(3) upper/lower almost θ-β-continuity, upper/lower almost δ-β-continuity and upper/lower almost β-continuity are equivalent.

Proof. This is an immediate consequence of Theorem 8.1.

Conclusions. Let (X,τ) be a topological space. Then, it is well-know that τ_θ and τ_δ are topologies. And also $\delta\mathrm{SO}(X)$, $\delta\mathrm{PO}(X)$, $\delta\beta(X)$, $\theta\mathrm{SO}(X)$, $\theta\mathrm{PO}(X)$, and $\theta\beta(X)$ are all m-spaces with property $(\mathcal{B})$. Therefore, by the results established in Sections 3-8 we can obtain properties of the multifunctions defined in Definitions 8.2 and 8.3.

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