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LIPSCHITZ ESTIMATES FOR MULTILINEAR COMMUTATOR OF SINGULAR INTEGRAL WITH VARIABLE CALDERÓN-ZYGMUND KERNEL

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Abstract: In this paper, we will study the continuity of a multilinear commutator generated by the singular integral with variable Calderón-Zygmund kernel and the functions b_j, on Triebel-Lizorkin space, Hardy space and Herz-Hardy space, where the functions b_j belong to the Lipschitz space $Lip_\beta(R^n)$.

1. Introduction

Let T be the Calderón-Zygmund operator. Coifman, Rochberg and Weiss (see [5]) proved that the commutator $[b,T](f) = bT(f) - T(bf)$, where $b \in BMO(R^n)$, is bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proved a similar result when T is replaced by the fractional operator.

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In [8][16], Janson and Paluszynski study these results for the Triebel-Lizorkin spaces and the case $b \in Lip_\beta(R^n)$, where $Lip_\beta(R^n)$ is the Lipschitz space of order β Our work is motivated by these papers. The main purpose of this paper is to discuss the boundedness of a multilinear commutator generated by the singular integral with variable Calderón-Zygmund kernel and by Lipschitz functions on Triebel-Lizorkin space, Hardy space and Herz-Hardy space on the spaces of homogeneous type, where $b_j \in Lip_\beta(R^n)$.

2. Preliminaries and Definitions

Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, and write $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$. Q will denote a cube of R^n with sides parallel to the axes. Let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^\#(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. Denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)(0 < p \le 1)$ has the atomic decomposition characterization(see [18]). For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(R^n)$ be the homogeneous Tribel-Lizorkin space. The Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y\in R^n \\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty.$$

Lemma 1.(see [16]) *For* $0 < \beta < 1,\ 1 < p < \infty$, *we have*

$$\begin{aligned} ||f||_{\dot{F}_p^{\beta,\infty}} &\approx \left|\left|\sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |f(x) - f_Q|dx\right|\right|_{L^p} \\ &\approx \left|\left|\sup_Q \inf_c \frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |f(x) - c|dx\right|\right|_{L^p}. \end{aligned}$$

Lemma 2.(see [16]) *For* $0 < \beta < 1,\ 1 \le p \le \infty$, *we have*

$$\begin{aligned} ||f||_{Lip_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |f(x) - f_Q|dx \\ &\approx \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}}\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx\right)^{1/p}. \end{aligned}$$

Lemma 3.(see [2]) *For* $1 \le r < \infty$ and $\beta > 0$, *let*

$$M_{\beta,r}(f)(x) = \sup_{x\in Q}\left(\frac{1}{|Q|^{1-\frac{\beta r}{n}}}\int_Q |f(y)|^r dy\right)^{1/r}.$$

Suppose that $r < p < n/\beta$, and $1/q = 1/p - \beta/n$. *Then*

$$||M_{\beta,r}(f)||_{L^q} \le C||f||_{L^p}.$$

Lemma 4.(see [16]) *If* $Q_1 \subset Q_2$, *then*

$$|f_{Q_1} - f_{Q_2}| \le C||f||_{Lip_\beta}|Q_2|^{\beta/n}.$$

Definition 1. Let $0 < p,\ q < \infty$, $\alpha \in R$, $B_k = \{x \in R^n, |x| \le 2^k\}$, $A_k = B_k \backslash B_{k-1}$ and $\chi_k = \chi_{Ak}$ for $k \in \mathbf{Z}$.

1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \{f \in L^q_{Loc}(R^n\backslash\{0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q}\right]^{1/p};$$

2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f \in L^q_{Loc}(R^n) : ||f||_{K_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q} + ||f\chi_{B_0}||^p_{L^q}\right]^{1/p}.$$

Definition 2. Let $\alpha \in R$, $0 < p, q < \infty$.

(1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(R^n) = \{f \in S'^{n}) : G(f) \in \dot{K}_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{H\dot{K}_q^{\alpha,p}} = ||G(f)||_{\dot{K}_q^{\alpha,p}};$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'^{n}) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{HK_q^{\alpha,p}} = ||G(f)||_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 3. Let b_i $(i = 1, \cdots, m)$ be a locally integrable function and $0 < p \le 1$. A bounded measurable function a on R^n is called a $(p, \vec{b})$ atom, if

(1) $\quad supp\ a \subset B = B(x_0, r)$

(2) $\quad ||a||_{L^\infty} \le |B(x_0, r)|^{-1/p}$

(3) $\quad \int_B a(y)dy = \int_B a(y)\prod_{l\in\sigma} b_l(y)dy = 0$ for any $\sigma \in C_j^m$, $1 \le j \le m$.

Definition 4. Let $\alpha \in R$, $1 < q < \infty$. A function $a(x)$ on R^n is called a central (α, q)-atom (or a central (a, q)-atom of restricted type) if

1) $\quad$ supp$a \subset B(0, r)$ for some $r > 0$ (or for some $r \ge 1$),

2) $\quad ||a||_{L^q} \le |B(0, r)|^{-\alpha/n}$,

3) $\quad \int_{R^n} a(x)x^\eta dx = 0$ for $|\eta| \le [\alpha - n(1 - 1/q)]$.

Lemma 5.(see [7][13][14]) *Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \ge n(1 - 1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$(or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α, q)-atoms (or central (α, q)-atoms of restricted type), a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j|^p < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the S'^n sense, and*

$$||f||_{H\dot{K}_q^{\alpha,p}}(\text{or } ||f||_{HK_q^{\alpha,p}}) \approx \left(\sum_j |\lambda_j|^p\right)^{1/p}.$$

Definition 5.(see [1][15]) Let $K:\ R^n\backslash\{0\} \to R$, $K(x) = \Omega(x)/|x|^n$. K is said to be a Calderón-Zygmund kernel if

(a) $\Omega \in C^\infty(R^n \setminus \{0\})$;

(b) Ω is homogeneous of degree zero;

(c) $\int_\Sigma \Omega(x)x^\alpha d\sigma(x) = 0$ for all multi-indices $\alpha \in (N \cup \{0\})^n$ with $|\alpha| = N$, where $\Sigma = \{x \in R^n : |x| = 1\}$ is the unit sphere of R^n.

Definition 6.(see [1][15]) Let $K(x,y) = \Omega(x,y)/|y|^n : R^n \times (R^n \setminus \{0\}) \to R$. K is said to be a variable Calderón-Zygmund kernel if

(d) $K(x,\cdot)$ is a Calderón-Zygmund kernel for a.e. $x \in R^n$;

(e) $max_{|\gamma|\le 2n} \left|\left|\frac{\partial^{|\gamma|}}{\partial^\gamma y}\Omega(x,y)\right|\right|_{L^\infty(R^n\times\Sigma)} = M < \infty$.

Suppose b_j $(j = 1, \cdots, m)$ are fixed locally integrable functions on R^n. Let T be the singular integral operator with variable Calderón-Zygmund kernel defined by

$$T(f)(x) = \int_{R^n} K(x, x-y)f(y)d(y),$$

where $K(x,x-y) = \dfrac{\Omega(x,x-y)}{|x-y|^n}$ is a variable Calderón-Zygmund kernel.

The multilinear commutator of singular integral with variable Calderón-Zygmund kernel is defined by

$$T_{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y))K(x,x-y)f(y)dy.$$

If $b_1 = ... = b_m$, then $T_{\vec{b}}$ is just the m order commutator. Commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [2-6][8-12][16-17][19-20]). Our main purpose is to establish the boundedness of the multilinear commutator $T_{\vec{b}}$ on Triebel-Lizorkin space, Hardy space and Herz-Hardy space.

Given a positive integer m and $1 \le j \le m$, we set $||\vec{b}||_{Lip_\beta} = \prod_{j=1}^m ||b_j||_{Lip_\beta}$ and denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), ..., \sigma(j)\}$ of $\{1, ..., m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, ..., m\} \setminus \sigma$. For $\vec{b} = (b_1, ..., b_m)$ and $\sigma = \{\sigma(1), ..., \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, ..., b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)}...b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{Lip_\beta} = ||b_{\sigma(1)}||_{Lip_\beta}...||b_{\sigma(j)}||_{Lip_\beta}$.

3. Theorems and Proofs

Theorem 1. *Let* $0 < \beta < 1/m$, $1 < p < \infty$, $\vec{b} = (b_1, \cdots, b_m)$ *with* $b_j \in Lip_\beta(R^n)$ *for* $1 \le j \le m$ *and* $T_{\vec{b}}$ *be the multilinear commutator of singular integral with variable Calderón-Zygmund kernel as in Definition 6. Then*

(a) $T_{\vec{b}}$ *is bounded from* $L^p(R^n)$ *to* $\dot{F}_p^{m\beta,\infty}(R^n)$.

(b) $T_{\vec{b}}$ *is bounded from* $L^p(R^n)$ to $L^q(R^n)$ *for* $1/p - 1/q = m\beta/n$ *and* $1/p > m\beta/n$.

Proof. (a). Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Set $\vec{b}_Q = ((b_1)_Q, \cdots, (b_m)_Q)$, where $(b_j)_Q = |Q|^{-1}\int_Q b_j(y)dy,\ 1 \le j \le m$. Write $f = f_1 + f_2,$ where $f_1 = f\chi_Q, f_2 = f\chi_{R^n\setminus Q}$, we have

$T_{\vec{b}}(f)(x) = \int_{R^n}(b_1(x) - b_1(y)) \cdots (b_m(x) - b_m(y))K(x, x - y)f(y)dy$
$= (b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q)T(f)(x)$
$+ (-1)^m T((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f)(x)$
$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma \int_{R^n}(\vec{b}(y)-\vec{b}_Q)_{\sigma^c}K(x, x-y)f(y)dy$
$= (b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q)T(f)(x)$
$+ (-1)^m T((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f_1)(x)$
$+ (-1)^m T((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f_2)(x)$
$+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x) - \vec{b}_Q)_\sigma T((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x),$

then

$|T_{\vec{b}}(f)(x) - T(((b_1)_Q - b_1) \cdots ((b_m)_Q - b_m)f_2)(x_0)|$
$\le ||(b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q)T(f)(x)||$
$+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(\vec{b}(x) - \vec{b}_Q)_\sigma T((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)||$
$+ ||T((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f_1)(x)||$
$+ ||T((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f_2)(x) - T((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f_2)(x_0)||$
$= I_1(x) + I_2(x) + I_3(x) + I_4(x),$

thus

$\frac{1}{|Q|^{1+m\beta/n}}\int_Q |T_{\vec{b}}(f)(x) - T(((b_1)_Q - b_1) \cdots ((b_m)_Q - b_m)f_2)(x_0)|dx$
$\le \frac{1}{|Q|^{1+m\beta/n}}\int_Q I_1(x)dx + \frac{1}{|Q|^{1+m\beta/n}}\int_Q I_2(x)dx + \frac{1}{|Q|^{1+m\beta/n}}\int_Q I_3(x)dx +$

$\frac{1}{|Q|^{1+m\beta/n}}\int_Q I_4(x)dx$
$= I + II + III + IV.$

For I, by using **Lemma 2**, we have

$I \le \frac{1}{|Q|^{1+m\beta/n}} \sup_{x\in Q} |b_1(x) - (b_1)_Q| \cdots |b_m(x) - (b_m)_Q| \int_Q |T(f)(x)|dx$
$\le C||\vec{b}||_{Lip_\beta} \frac{1}{|Q|^{1+m\beta/n}} |Q|^{m\beta/n} \int_Q |T(f)(x)|dx$
$\le C||\vec{b}||_{Lip_\beta} M(T(f))(\tilde{x}).$

For II, fix $1 < r < p$ and let μ, μ' be the integers such that $\mu + \mu' = m$, $0 \le \mu < m$, $0 < \mu' \le m$. By using the Hölder's inequality, the boundedness of $T_{\vec{b}}$ on L^r and **Lemma 2**, we get

$II \le \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} \frac{1}{|Q|^{1+m\beta/n}} \int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma||T((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)|dx$
$\le C \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} \frac{1}{|Q|^{1+m\beta/n}} \left(\int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{r'} dx\right)^{1/r'}$
$\left(\int_Q |T((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)|^r dx\right)^{1/r}$
$\le C \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} \frac{1}{|Q|^{1+m\beta/n}} \left(\int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{r'} dx\right)^{1/r'}$
$\left(\int_Q |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c} f(x)|^r dx\right)^{1/r}$
$\le C \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} \frac{1}{|Q|^{1+m\beta/n}} ||\vec{b}_\sigma||_{Lip_\beta} |Q|^{1/r} |Q|^{\mu\beta/n} ||\vec{b}_{\sigma^c}||_{Lip_\beta} |Q|^{|\mu'\beta/n}$
$\left(\int_Q |f(x)|^r dx\right)^{1/r}$
$\le C||\vec{b}||_{Lip_\beta} M_r(f)(\tilde{x}).$

For III, by Hölder's inequality, we have

$III = \frac{1}{|Q|^{1+m\beta/n}} \int_Q |T((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f_1)(x)|dx$
$\le C \frac{1}{|Q|^{1+m\beta/n}} \left(\int_{R^n} |T(\prod_{j=1}^m (b_j(x) - (b_j)_Q) f_1)(x)|^r dx\right)^{1/r} |Q|^{1-1/R}$
$\le C \frac{1}{|Q|^{1+m\beta/n}} |Q|^{1-1/r} \left(\int_{2Q} | \prod_{j=1}^m (b_j(x) - (b_j)_Q) f(x)|^r dx\right)^{1/r}$
$\le C \frac{1}{|Q|^{1+m\beta/n}} |Q|^{1-1/r} ||\vec{b}||_{Lip_\beta} |Q|^{m\beta/n} \left(\int_{2Q} |f(x)|^r dx\right)^{1/r}$
$\le C||\vec{b}||_{Lip_\beta} M_r(f)(\tilde{x}).$

For IV, since $|x_0 - y| \approx |x - y|$ for $y \in (2Q)^c$, by **Lemma 4**, we have

$I_4(x) = \left|\int_{R^n} (K(x, x - y) - K(x_0, x_0 - y)) \prod_{j=1}^m (b_j(y) - (b_j)_Q) f_2(y) dy\right|$
$\le C \int_{(2Q)^c} \left|\frac{\Omega(x,x-y)}{|x-y|^n} - \frac{\Omega(x_0,x_0-y)}{|x_0-y|^n}\right| | \prod_{j=1}^m (b_j(y) - (b_j)_Q)||f(y)|dy$
$\le C \sum_{l=1}^\infty \int_{2^{l+1}Q\setminus 2^l Q} \left[\sum_{k=1}^\infty \sum_{h=1}^{g_k} |a'_{hk}(x)| \left|\frac{Y_{hk}(x-y)}{|x-y|^n} - \frac{Y_{hk}(x_0-y)}{|x_0-y|^n}\right|\right] | \prod_{j=1}^m (b_j(y) -$

$(b_j)_Q)||f(y)|dy$
$\leq C\sum_{k=1}^{\infty} k^{-2n} \cdot k^{n/2}\sum_{l=1}^{\infty}\int_{2^{l+1}Q\setminus 2^lQ}\frac{|x-x_0|}{|x_0-y|^{n+1}}|\prod_{j=1}^m(b_j(y) - (b_j)_Q)||f(y)|dy$
$\leq C\sum_{k=1}^{\infty} k^{-3n/2}\sum_{l=1}^{\infty}\frac{d}{(2^ld)^{n+1}}\int_{2^{l+1}Q}|\prod_{j=1}^m(b_j(y)-(b_j)_Q)||f(y)|dy$
$\leq C\sum_{l=1}^{\infty} 2^{-l}|2^{l+1}Q|^{-1}\int_{2^{l+1}Q}|f(y)|\prod_{j=1}^m(|b_j(y) - (b_j)_{2^{l+1}Q}| + |(b_j)_{2^{l+1}Q}-(b_j)_Q|)dy$
$\leq C\sum_{k=1}^{\infty} 2^{-l}|2^{l+1}Q|^{m\beta/n}||\vec{b}||_{Lip_\beta}M(f)(x)$
$\leq C||\vec{b}||_{Lip_\beta}|Q|^{m\beta/n}M(f)(x)\sum_{l=1}^{\infty}2^{(m\beta-1)l}$
$\leq C||\vec{b}||_{Lip_\beta}|Q|^{m\beta/n}M(f)(\tilde{x}),$
thus

$$IV \leq C||\vec{b}||_{Lip_\beta}M(f)(\tilde{x}).$$

We put these estimates together, by using **Lemma 1** and taking the supremum over all Q such that $x \in Q$, we obtain

$$||T_{\vec{b}}(f)||_{\dot{F}_p^{m\beta,\infty}} \leq C||\vec{b}||_{Lip_\beta}||f||_{L^p}.$$

This completes the proof of (a).

(b). By the same argument as in proof of (a), we have
$\frac{1}{|Q|}\int_Q|T_{\vec{b}}(f)(x) - T(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|dx$
$\leq \frac{1}{|Q|}\int_Q I_1(x)dx + \frac{1}{|Q|}\int_Q I_2(x)dx + \frac{1}{|Q|}\int_Q I_3(x)dx + \frac{1}{|Q|}\int_Q I_4(x)dx$
$\leq C||\vec{b}||_{Lip_\beta}(M_{m\beta,1}(T(f)) + M_{m\beta,r}(f) + M_{m\beta,r}(f) + M_{m\beta,1}(f)),$
thus

$$(T_{\vec{b}}(f))^{\#} \leq C||\vec{b}||_{Lip_\beta}(M_{m\beta,1}(T(f)) + M_{m\beta,r}(f) + M_{m\beta,1}(f)).$$

By using **Lemma 3** and the boundedness of T, we have
$||T_{\vec{b}}(f)||_{L^q} \leq C||(T_{\vec{b}}(f))^{\#}||_{L^q}$
$\leq C||\vec{b}||_{Lip_\beta}(||M_{m\beta,1}(T(f))||_{L^q} + ||M_{m\beta,r}(f)||_{L^q} + ||M_{m\beta,1}(f)||_{L^q})$
$\leq C||\vec{b}||_{Lip_\beta}||f||_{L^p}.$
This completes the proof of (b) and the theorem.

Theorem 2. *Let $0<\beta\leq 1$, $n/(n+1)<p\leq 1$, $1/q = 1/p - m\beta/n$, $\vec{b} = (b_1,\cdots,b_m)$ with $b_j \in Lip_\beta(R^n)$ for $1\leq j\leq m$. Then $T_{\vec{b}}$ is bounded from $H^p(R^n)$ to $L^q(R^n)$.*

Proof. It suffices to show that there exists a constant $C > 0$ such that for every H^p-atom a,

$$||T_{\vec{b}}(a)||_{L^q} \leq C.$$

Let a be a H^p-atom, such that a supported on a cube $B = B(x_0, r)$, $||a||_{L^\infty} \leq |B|^{-1/p}$ and $\int_{R^n} a(x)x^\gamma dx = 0$ for $|\gamma| \leq [n(1/p-1)]$.

Write

$||T_{\vec{b}}(a)(x)||_{L^q} \leq \left(\int_{|x-x_0|\leq 2r} |T_{\vec{b}}(a)(x)|^q dx\right)^{1/q} +$
$\left(\int_{|x-x_0|>2r} |T_{\vec{b}}(a)(x)|^q dx\right)^{1/q}$
$= I + II.$

For I, choose $1 < p_1 < n/m\beta$ and q_1 such that $1/q_1 = 1/p_1 - m\beta/n$. By the boundedness of $T_{\vec{b}}$ from $L^{p_1}(R^n)$ to $L^{q_1}(R^n)$(see **Theorem 1**), we get

$$I \leq C||T_{\vec{b}}(a)||_{L^{q_1}} r^{n(1/q-1/q_1)} \leq C||a||_{L^{p_1}} r^{n(1/q-1/q_1)} \leq C.$$

For II, let $\tau, \tau' \in N$ such that $\tau + \tau' = m$, and $\tau' \neq 0$. We get

$|T_{\vec{b}}(a)(x)| \leq |(b_1(x) - b_1(x_0)) \cdots (b_m(x) - b_m(x_0)) \int_B (K(x, x-y) - K(x, x-x_0))a(y)dy|$
$+ \sum_{j=1}^m \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(x_0))_{\sigma^c} \int_B (\vec{b}(y) - \vec{b}(x_0))_\sigma K(x, x-y)a(y)dy|$
$\leq C||\vec{b}||_{Lip_\beta} |x-x_0|^{m\beta} \cdot \int_B |K(x, x-y) - K(x, x-x_0)||a(y)|dy$
$+ C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x-x_0|^{\tau\beta} \int_B |y-x_0|^{\tau'\beta} |K(x, x-y)||a(y)|dy$
$\leq C||\vec{b}||_{Lip_\beta} |x-x_0|^{m\beta} \cdot \int_B \left|\frac{\Omega(x,x-y)}{|x-y|^n} - \frac{\Omega(x,x-x_0)}{|x-x_0|^n}\right| |a(y)|dy$
$+ C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x-x_0|^{\tau\beta} \int_B |y-x_0|^{\tau'\beta} |K(x, x-y)||a(y)|dy$
$\leq C||\vec{b}||_{Lip_\beta} |x-x_0|^{m\beta} \int_B \frac{|x_0-y|}{|x-x_0|^{n+1}} |a(y)|dy$
$+ C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x-x_0|^{\tau\beta} \int_B |y-x_0|^{\tau'\beta} \frac{1}{|x-y|^n} |a(y)|dy$
$\leq C||\vec{b}||_{Lip_\beta} |x-x_0|^{m\beta-n-1} \cdot |B|^{1+1/n-1/p} + C||\vec{b}||_{Lip_\beta} |x-x_0|^{m\beta-n} \cdot |B|^{1-1/p}$
$\leq C||\vec{b}||_{Lip_\beta} |x-x_0|^{m\beta-n-1} \cdot |B|^{1+1/n-1/p},$

so

$II \leq C||\vec{b}||_{Lip_\beta} \cdot |B|^{1+1/n-1/p} \left(\int_{|x-x_0|>2r} |x-x_0|^{(m\beta-n-1)q} dx\right)^{1/q}$
$\leq C||\vec{b}||_{Lip_\beta} \cdot |B|^{1+1/n-1/p} \left(\int_{|x-x_0|>2r} |x-x_0|^{-(n+1-m\beta)q} dx\right)^{1/q}$

$\leq C||\vec{b}||_{Lip_\beta} \cdot |B|^{1+1/n-1/p} \sum_{k=1}^{\infty} |2^k B|^{-(n+1-m\beta)/n} |2^{k+1}B|^{1/q}$
$\leq C||\vec{b}||_{Lip_\beta} |B|^{1/q-1/p+m\beta/n} \sum_{k=1}^{\infty} 2^{k(1/q+m\beta/n-1-1/n)}$
$\leq C||\vec{b}||_{Lip_\beta}$.
This completes the proof of Theorem 2.

Theorem 3. *Let $0 < \beta \leq 1$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = m\beta/n$, $n(1-1/q_1) \leq \alpha < n(1-1/q_1) + m\beta$, $\vec{b} = (b_1, \cdots, b_m)$ with $b_j \in Lip_\beta(R^n)$ for $1 \leq j \leq m$. Then $T_{\vec{b}}$ is bounded from $H\dot{K}_{q_1}^{\alpha,p}(R^n)$* to *$\dot{K}_{q_2}^{\alpha,p}(R^n)$.*

Proof. By **Lemma 5**, let $f \in H\dot{K}_{q_1}^{\alpha,p}(R^n)$ and $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, $supp a_j \subset B_j = B(0, 2^j)$, a_j be a central $(\alpha, q)-$atom, and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$. We have
$||T_{\vec{b}}(f)||_{\dot{K}_{q_2}^{\alpha,p}}^p \leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| ||T_{\vec{b}}(a_j)\chi_k||_{L^{q_2}} \right)^p$
$+ C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j| ||T_{\vec{b}}(a_j)\chi_k||_{L^{q_2}} \right)^p$
$= I + II$.
For II, by the boundedness of $T_{\vec{b}}$ on (L^{q_1}, L^{q_2}), we have
$II \leq C||\vec{b}||_{Lip_\beta}^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\sum_{j=k-1}^{\infty} |\lambda_j| ||a_j||_{L^{q_1}})^p$
$\leq C||\vec{b}||_{Lip_\beta}^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\sum_{j=k-1}^{\infty} |\lambda_j| \cdot 2^{-j\alpha})^p$
$\leq C||\vec{b}||_{Lip_\beta}^p \begin{cases} \sum_{k=-\infty}^{\infty} \sum_{j=k-1}^{\infty} |\lambda_j|^p \cdot 2^{(k-j)\alpha p}, & 0 < p \leq 1 \\ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\sum_{j=k-1}^{\infty} |\lambda_j|^p \cdot 2^{-j\alpha p/2}) (\sum_{j=k-1}^{\infty} 2^{-j\alpha p'/2})^{p/p'}, & 1 < p < \infty \end{cases}$
$\leq C||\vec{b}||_{Lip_\beta}^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p$.
For I, we have the estimates
$|T_{\vec{b}}(a)(x)| \leq |(b_1(x) - b_1(0)) \cdots (b_m(x) - b_m(0)) \int_{B_j} (K(x, x-y) - K(x, x-0)) a_j(y) dy|$
$+ \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(0))_{\sigma^c} \int_{B_j} (\vec{b}(y) - \vec{b}(0))_\sigma K(x, x-y) a_j(y) dy|$
$\leq C||\vec{b}||_{Lip_\beta} |x|^{m\beta} \cdot \int_{B_j} |K(x, x-y) - K(x, x-0)||a_j(y)| dy$
$+ C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x|^{\tau\beta} \int_{B_j} |y|^{\tau'\beta} |K(x, x-y)||a_j(y)| dy$
$\leq C||\vec{b}||_{Lip_\beta} |x|^{m\beta} \cdot \int_{B_j} \left| \frac{\Omega(x, x-y)}{|x-y|^n} - \frac{\Omega(x, x-0)}{|x-0|^n} \right| |a_j(y)| dy$
$+ C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x|^{\tau\beta} \int_{B_j} |y-0|^{\tau'\beta} |K(x, x-y)||a_j(y)| dy$
$\leq C||\vec{b}||_{Lip_\beta} |x|^{m\beta} \int_{B_j} \frac{|0-y|}{|x-0|^{n+1}} |a_j(y)| dy$

$+ C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x|^{\tau\beta} \int_{B_j} |y|^{\tau'\beta} \frac{1}{|x-y|^n} |a_j(y)|dy$
$\leq C||\vec{b}||_{Lip_\beta}|x|^{m\beta-n-1} \cdot |B|^{1+1/n-1/q_1-\alpha/n} + C||\vec{b}||_{Lip_\beta}|x|^{m\beta-n} \cdot |B|^{1-1/q_1-\alpha/n}$
$\leq C||\vec{b}||_{Lip_\beta}|x|^{m\beta-n-1} \cdot |B|^{1+1/n-1/q_1-\alpha/n}$
$\leq C||\vec{b}||_{Lip_\beta}|x|^{m\beta-n-1} \cdot 2^{j(1+n(1-1/q_1)-\alpha)}$
thus
$||T_{\vec{b}}(a_j)\chi_k||_{L^{q_2}} \leq C||\vec{b}||_{Lip_\beta} \cdot 2^{j(1+n(1-1/q_1)-\alpha)} \left(\int_{B_k} |x|^{(m\beta-n-1)q_2} dx\right)^{1/q_2}$
$\leq C||\vec{b}||_{Lip_\beta} \cdot 2^{j(1+n(1-1/q_1)-\alpha)} \cdot 2^{k(m\beta-n-1+n/q_2)}$
$\leq C||\vec{b}||_{Lip_\beta} \cdot 2^{j(1+n(1-1/q_1)-\alpha)-k(1+n(1-1/q_1))}$
so
$I \leq C||\vec{b}||^p_{Lip_\beta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \cdot 2^{[j(1+n(1-\frac{1}{q_1})-\alpha)-k(1+n(1-\frac{1}{q_1}))]}\right)^p$
$\leq C||\vec{b}||^p_{Lip_\beta} \begin{cases} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} |\lambda_j|^p \cdot 2^{(j-k)(1+n(1-\frac{1}{q_1})-\alpha)p}, & 0 < p \leq 1 \\ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p \cdot 2^{\frac{p}{2}[j(1+n(1-\frac{1}{q_1})-\alpha)-k(1+n(1-\frac{1}{q_1}))]}\right) & \\ \quad \times \left(\sum_{j=-\infty}^{k-2} 2^{\frac{p'}{2}[j(1+n(1-\frac{1}{q_1})-\alpha)-k(1+n(1-\frac{1}{q_1}))]}\right)^{p/p'}, & 1 < p < \infty \end{cases}$
$\leq C||\vec{b}||^p_{Lip_\beta} \begin{cases} \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+2}^{\infty} 2^{(j-k)(1+n(1-\frac{1}{q_1})-\alpha)p}, & 0 < p \leq 1 \\ \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+2}^{\infty} 2^{\frac{p}{2}[(j-k)(1+n(1-\frac{1}{q_1})-\alpha)]}, & 1 < p < \infty \end{cases}$
$\leq C||\vec{b}||^p_{Lip_\beta} \sum_{j=-\infty}^{\infty} |\lambda_j|^p.$
From the estimates for I and II it follows

$$||T_{\vec{b}}(f)||_{\dot{K}^{\alpha,p}_{q_2}} \leq C||\vec{b}||_{Lip_\beta} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p} \leq C||f||_{H\dot{K}^{\alpha,p}_{q_1}}.$$

This completes the proof of Theorem 3.

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