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SOME PROPERTIES OF UPPER/LOWER ω-CONTINUOUS MULTIFUNCTIONS

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Abstract. The aim of this paper is to introduce and study upper and lower almost ω-continuous multifunctions as a generalization of upper and lower ω-continuous multifunctions, respectively due to Zorlutuna [21].

1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions [1,6,13,14,16,17,19]. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. Several characterizations and properties of ω-closed sets were provided in [7],[8] and [1]. Recently, Zorlutuna [21] introduced and studied the concept of ω-continuous multifunctions in topological spaces. Also in [14], the theory of almost continuity for multifunctions is unified using certain minimal conditions. In this paper, we introduce and study upper (lower) almost-ω continuous multifunctions and obtain several characterizations of upper (lower) almost ω-continuous multifunctions and basic properties of such functions.

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2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. For a subset A of (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. Recently, as generalization of closed sets, the notion of ω-closed sets were introduced and studied by Hdeib [8]. A point $x \in X$ is called a condensation (resp. θ-cluster) point of A, if $U \cap A$ is uncountable (resp. $Cl(U) \cap A \neq \emptyset$) for each $U \in \tau$ with $x \in U$. The set of all θ-cluster points of A is denoted by $Cl_\theta(A)$. If $A = Cl_\theta(A)$, then A is said to be θ-closed [20]. The complement of a θ-closed set is said to be θ-open. A is said to be ω-closed [8] if it contains all its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \backslash W$ is countable. The family of all ω-open subsets of a topological space (X, τ) forms a topology on X finer than τ. The ω-closure and the ω-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\omega\,\mathrm{Cl}(A)$ and $\omega\,\mathrm{Int}(A)$, respectively. The family of all ω-open subsets of a topological space (X, τ), denoted by τ_ω. τ_ω forms a topology on X finer than τ. We set $\omega O(X, x) = \{A : A \in \tau_\omega \text{ and } x \in A\}$. A subset A is said to be regular open [19] (resp. semiopen [11], preopen [12], semi-preopen [3]) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$). The complement of regular open (resp. semiopen, semi-preopen) set is called regular closed (resp. semiclosed, α-closed, semi pre-closed) set. The intersection (resp. union) of all semiclosed (resp. semiopen) set containing (resp. contained in) $A \subset X$ is called the semiclosure (resp. semiinterior) of A and is denoted by $s\,\mathrm{Cl}(A)$ (resp. $s\,\mathrm{Int}(A)$). The family of all regular open (resp. regular closed, semiopen, semiclosed, preopen, semi-preopen, semi-preclosed) sets of (X, τ) is denoted by $RO(X)$ (resp. $RC(X)$, $SO(X)$, $SC(X)$, $PO(X)$, $SPO(X)$, $SPC(X)$). By a multifunction $F : (X, \tau) \to (Y, \sigma)$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$ and for each $A \subset X$, $F(A) = \bigcup_{x \in A} F(x)$. Then F is said to be surjection if $F(X) = Y$. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be lower ω-continuous [21] (resp. upper ω-continuous) multifunction if $F^-(V) \in \omega O(X,\tau)$ (resp. $F^+(V) \in \omega O(X,\tau)$) for every $V \in \sigma$. A subset N of a topological space (X,τ) is said to be ω-neighborhood of a point $x \in X$, if there exists an ω-open set V such that $x \in V \subset N$.

Lemma 2.1. *The following statements are true:*

(1) *Let A be a subset of a space (X,τ). Then $A \in PO(X)$ if and only if $s\,\mathrm{Cl}(A) = \mathrm{Int}(\mathrm{Cl}(A))$* [9].
(2) *A subset A of a space (X,τ) is semi-preopen if and only if $\mathrm{Cl}(A)$ is regular closed* [3].

Definition 2.2. [6] A multifunction $F : (X,\tau) \to (Y,\sigma)$ is said to be:

(1) lower weakly ω-continuous, if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in \omega O(X,x)$ such that $U \subset F^-(\mathrm{Cl}(V))$,
(2) upper weakly ω-continuous, if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in \omega O(X,x)$ such that $U \subset F^+(\mathrm{Cl}(V))$,
(3) weakly ω-continuous, if it is both upper weakly ω-continuous and lower weakly ω-continuous.

3. On upper and lower almost ω-continuous multifunctions

Definition 3.1. A multifunction $F : (X,\tau) \to (Y,\sigma)$ is said to be:

(1) lower almost ω-continuous, if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in \omega O(X,x)$ such that $U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$,
(2) upper almost ω-continuous, if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in \omega O(X,x)$ such that $U \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$,
(3) almost ω-continuous, if it is both upper almost ω-continuous and lower almost ω-continuous.

Remark 3.2. *Observe that the above Definition is a particular case of Definition 3.4 of* [14].

It is clear that every upper (lower) ω-continuous function is upper (lower) almost ω-continuous. But the converse is not true as shown by the following example.

Example 3.3. *Let $X = \mathbb{R}$ with topologies $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$ and $Y = \{a, b\}$ with topology $\sigma = \{\emptyset, Y, \{a\}\}$. Define $F : (\mathbb{R},\tau) \to (Y,\sigma)$*

as follows:

$$F(x) = \begin{cases} \{a\}, & \textit{if } x \in \mathbb{Q} \\ \{b\}, & \textit{if } x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

It is easy to see that F is upper almost ω-continuous but is not upper ω-continuous.

Theorem 3.4. (1) *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is upper almost ω-continuous if and only if $F : (X, \tau_\omega) \to (Y, \sigma)$ is upper almost continuous.*

(2) *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is lower almost ω-continuous if and only if $F : (X, \tau_\omega) \to (Y, \sigma)$ is lower almost continuous.*

Proof. The proof is obvious from the definitions. □

Theorem 3.5. *The following statements are equivalent for a multifunction $F : (X, \tau) \to (Y, \sigma)$:*

(1) *F is upper almost ω-continuous multifunction,*
(2) *for each $x \in X$ and for each open set V such that $F(x) \subset V$, there exists $U \in \omega O(X, x)$ such that if $y \in U$, then $F(y) \subset \mathrm{Int}(\mathrm{Cl}(V)) = s\,\mathrm{Cl}(V)$,*
(3) *for each $x \in X$ and for each regular open set G of Y such that $F(x) \subset G$, there exists $U \in \omega O(X, x)$ such that $F(U) \subset G$,*
(4) *for each $x \in X$ and for each closed set K such that $x \in F^+(Y \backslash K)$, there exists an ω-closed set H such that $x \in X \backslash H$ and $F^-(\mathrm{Cl}(\mathrm{Int}(K))) \subset H$,*
(5) *$F^+(\mathrm{Int}(\mathrm{Cl}(V))) \in \tau_\omega$ for any open set $V \subset Y$,*
(6) *$F^-(\mathrm{Cl}(\mathrm{Int}(K))) \in \omega C(X)$ for any closed set $K \subset Y$,*
(7) *$F^+(G) \in \tau_\omega$ for any regular open set G of Y,*
(8) *$F^-(K) \in \omega C(X)$ for any regular closed set K of Y,*
(9) *for each point x of X and each neighborhood V of $F(x)$, $F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ is an ω-neighborhood of x,*
(10) *for each point x of X and each neighborhood V of $F(x)$, there exists an ω-neighborhood U of x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$.*

Proof. (1)⇔(2): The proof follows from Definition 3.1 and lemma 2.1.
(2)⇒(3): Let $x \in X$ and G be a regular open set of Y such that $F(x) \subset G$. By (2), there exists $U \in \omega O(X, x)$ such that if $y \in U$, then $F(y) \subset \mathrm{Int}(\mathrm{Cl}(G)) = G$. We obtain $F(U) \subset G$.
(3)⇒(2): Let $x \in X$ and V be an open set of Y such that $F(x) \subset V$. Then, $\mathrm{Int}(\mathrm{Cl}(V)) \in RO(Y)$. By (3), there exists $U \in \omega O(X, x)$ such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$.

(2)⇒(4): Let $x \in X$ and K be a closed set of Y such that $x \in F^+(Y\backslash K)$. By (2), there exists $U \in \omega O(X,x)$ such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(Y\backslash K))$. We have $\mathrm{Int}(\mathrm{Cl}(Y\backslash K)) = Y\backslash \mathrm{Cl}(\mathrm{Int}(K))$ and $U \subset F^+(Y\backslash \mathrm{Cl}(\mathrm{Int}(K))) = X\backslash F^-(\mathrm{Cl}(\mathrm{Int}(K)))$. We obtain $F^-(\mathrm{Cl}(\mathrm{Int}(K))) \subset X\backslash U$. Take $H = X\backslash U$. Then, $x \in X\backslash H$ and H is ω-closed set.
(4)⇒(2): Let $x \in X$ and V be an open set of Y such that $F(x) \subset V$. Then $Y\backslash V$ is closed in Y and $x \in F^+(V) = F^+(Y\backslash(Y\backslash V))$. By (4), there exists an ω-closed set L such that $x \in X\backslash L$ and $F^-(\mathrm{Cl}(\mathrm{Int}(Y\backslash V))) \subset L$. This implies that $X\backslash L \subseteq F^+(\mathrm{Int}(\mathrm{Cl}(V)))$. Put $U = X\backslash L$. Then $U \in \tau_\omega$ and if $y \in U$, then $F(y) \subset \mathrm{Int}(\mathrm{Cl}(V))$.
(1)⇒(5): Let V be any open set of Y and $x \in F^+(\mathrm{Int}(\mathrm{Cl}(V)))$. By (1), there exists $U_x \in \omega O(X,x)$ such that $U_x \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$. Therefore, we obtain $F^+(\mathrm{Int}(\mathrm{Cl}(V))) = \bigcup_{x\in F^+(\mathrm{Int}(\mathrm{Cl}(V)))} U_x$. Hence, $F^+(\mathrm{Int}(\mathrm{Cl}(V))) \in \tau_\omega$.
(5)⇒(1): Let V be any open set of Y and $x \in F^+(V)$. By (5), $F^+(\mathrm{Int}(\mathrm{Cl}(V))) \in \tau_\omega$. Take $U = F^+(\mathrm{Int}(\mathrm{Cl}(V)))$. Then $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Hence, F is upper almost ω-continuous.
(5)⇒(6): Let K be any closed set of Y. Then, $Y\backslash K$ is an open set of Y. By (5), $F^+(\mathrm{Int}(\mathrm{Cl}(Y\backslash K))) \in \tau_\omega$. Since $\mathrm{Int}(\mathrm{Cl}(Y\backslash K)) = Y\backslash \mathrm{Cl}(\mathrm{Int}(K))$, it follows that $F^+(\mathrm{Int}(\mathrm{Cl}(Y\backslash K))) = F^+(Y\backslash \mathrm{Cl}(\mathrm{Int}(K))) = X\backslash F^-(\mathrm{Cl}(\mathrm{Int}(K)))$. We obtain that $F^-(\mathrm{Cl}(\mathrm{Int}(K)))$ is ω-closed in X.
(6)⇒(5): It can be obtained similarly as (5)⇒(6).
(5)⇒(7): Let G be any regular open set of Y. By (5), $F^+(\mathrm{Int}(\mathrm{Cl}(G))) = F^+(G) \in \tau_\omega$.
(7)⇒ (5): Let V be any open set of Y. Then, $\mathrm{Int}(\mathrm{Cl}(V)) \in RO(Y)$. By (7), $F^+(\mathrm{Int}(\mathrm{Cl}(V))) \in \tau_\omega$.
(6)⇒(8): It can be obtained similarly as (5)⇒(7).
(8)⇒(6): It can be obtained similarly as (7)⇒(5).
(5)⇒(9): Let $x \in X$ and V be a neighborhood of $F(x)$. Then there exists an open set G of Y such that $F(x) \subset G \subset V$. Then we have $x \in F^+(G) \subset F^+(V)$. Since $F^+(\mathrm{Int}(\mathrm{Cl}(G))) \in \tau_\omega$, $F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ is an ω-neighborhood of x.
(9)⇒(10): Let $x \in X$ and V be a neighborhood of $F(x)$. By (9), $F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ is an ω-neighborhood of x. Take $U = F^+(\mathrm{Int}(\mathrm{Cl}(V)))$. Then $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$.
(10)⇒(1): Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. Then V is a neighborhood of $F(x)$. By (10), there exists an ω-neighborhood U of x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$.

Therefore, there exists $G \in \tau_\omega$ such that $x \in G \subset U$ and hence $F(G) \subset F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. We obtain that F is upper almost ω-continuous. □

Theorem 3.6. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(1) *F is lower almost ω-continuous multifunction,*
(2) *for each $x \in X$ and for each open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in \omega O(X, x)$ such that if $y \in U$, then $F(y) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$,*
(3) *for each $x \in X$ and for each regular open set G of Y such that $F(x) \cap G \neq \emptyset$, there exists $U \in \omega O(X, x)$ such that if $y \in U$, then $F(y) \cap G \neq \emptyset$,*
(4) *for each $x \in X$ and for each closed set K such that $x \in F^-(Y \backslash K)$, there exists an ω-closed set H such that $x \in X \backslash H$ and $F^+(\mathrm{Cl}(\mathrm{Int}(K))) \subset H$,*
(5) *$F^-(\mathrm{Int}(\mathrm{Cl}(V))) \in \tau_\omega$ for any open set $V \subset Y$,*
(6) *$F^+(\mathrm{Cl}(\mathrm{Int}(K))) \in \omega C(X)$ for any closed set $K \subset Y$,*
(7) *$F^-(G) \in \tau_\omega$ for any regular open set G of Y,*
(8) *$F^+(K) \in \omega C(X)$ for any regular closed set K of Y.*

Proof. We Prove only (1)⇒(2), (2)⇒(3), (3)⇒(4). The other proofs can be obtained similarly as Theorem 3.5.
(1)⇒(2): Let $x \in X$ and V be an open subset of Y such that $F(x) \cap V \neq \emptyset$. Since F is lower almost ω-continuous, there exists $U \in \omega O(X, x)$ such that $U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$. This implies that if $y \in U$, then $F(y) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$.
(2)⇒(3): Let $x \in x$ and G be a regular open subset of Y such that $F(x) \cap G \neq \emptyset$. Then $G = \mathrm{Int}(\mathrm{Cl}(G))$ is open in Y. By (2), there exists $U \in \omega O(X, x)$ such that if $y \in U$, then $F(y) \cap \mathrm{Int}(\mathrm{Cl}(G)) \neq \emptyset$. That is, if $y \in U$, then $F(y) \cap G \neq \emptyset$.
(3)⇒(4): Let $x \in X$ and K be a closed subset of Y such that $x \in F^-(Y \backslash K)$. Then $\mathrm{Int}(\mathrm{Cl}(Y \backslash K))$ is regular open in Y such that $x \in F^-(\mathrm{Int}(\mathrm{Cl}(Y \backslash K)))$. Thus $F(x) \cap \mathrm{Int}(\mathrm{Cl}(Y \backslash K)) \neq \emptyset$. By (3), there exists $U \in \omega O(X, x)$ such that if $y \in U$, then $F(y) \cap \mathrm{Int}(\mathrm{Cl}(Y \backslash K)) \neq \emptyset$. Hence $U \subset F^-(\mathrm{Int}(\mathrm{Cl}(Y \backslash K)))$, and so $U \subset X \backslash F^+(\mathrm{Cl}(\mathrm{Int}(K)))$. Set $L = X \backslash U$. Then L is a ω-closed set such that $x \in X \backslash L$ and $F^+(\mathrm{Cl}(\mathrm{Int}(K))) \subset L$.
(4)⇒(1): Let $x \in x$ and V be an open subset of Y such that $x \in F^-(V)$. Then $Y \backslash V$ is closed in Y such that $x \in F^-(Y \backslash (Y \backslash V))$. By (4), there exists an ω-closed set L such that $x \in X \backslash L$ and

$F^+(\mathrm{Cl}(\mathrm{Int}(Y\backslash V))) \subset L$. Set $U = X\backslash L$. Thus U is ω-open in X such that $x \in U$ and $U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$. Therefore, F is lower almost ω-continuous. □

Theorem 3.7. *The following are equivalent for a multifunction $F : (X, \tau) \to (Y, \sigma)$:*

(1) *F is upper almost ω-continuous;*
(2) $\omega\,\mathrm{Cl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ *for every* $V \in SPO(Y)$*;*
(3) $\omega\,\mathrm{Cl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ *for every* $V \in SO(Y)$*;*
(4) $F^+(V) \subset \omega\,\mathrm{Int}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$ *for every* $V \in PO(Y)$.

Proof. (1),(2),(3) follow from Theorem 3.7 (1),(2),(3) of [14], and (4) follows from Theorem 5.1 (4) of [14]. □

Theorem 3.8. *The following are equivalent for a multifunction $F : (X, \tau) \to (Y, \sigma)$:*

(1) *F is lower almost ω-continuous;*
(2) $\omega\,\mathrm{Cl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ *for every* $V \in \omega O(Y)$*;*
(3) $\omega\,\mathrm{Cl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ *for every* $V \in SO(Y)$*;*
(4) $F^-(V) \subset \omega\,\mathrm{Int}(F^-(\mathrm{Int}(\mathrm{Cl}(V))))$ *for every* $V \in PO(Y)$.

Proof. (1),(2),(3) follow from Theorem 3.7 (1),(2),(3) of [14], and (4) follows from Theorem 5.1 (4) of [14]. □

Definition 3.9. [21] Let (X, τ) be a topological space and let (x_α) be a net in X. It is said that the net (x_α) ω-converges to x, if for each ω-open set G containing x in X, there exists an index $\alpha_0 \in I$ such that $x_\alpha \in G$ for each $\alpha \geq \alpha_0$.

Theorem 3.10. *If $F : (X, \tau) \to (Y, \sigma)$ is a lower (upper) almost ω-continuous multifunction, then for each $x \in X$ and for each net (x_α) which ω-converges to x in X and for each open set $V \subset Y$ such that $x \in F^-(V)$ (resp. $x \in F^+(V)$), the net (x_α) is eventually in* $F^-(\mathrm{Int}(\mathrm{Cl}(V)))$ *(resp.* $F^+(\mathrm{Int}(\mathrm{Cl}(V)))$*).*

Proof. Let (x_α) be a net ω-converges to x in X and let V be any open set in Y such that $x \in F^-(V)$. Since F is lower almost ω-continuous multifunction, there exists an ω-open set U in X containing x such that $U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$. Since (x_α) ω-converges to x, there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. So we obtain that $x_\alpha \in U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$ for all $\alpha \geq \alpha_0$. Thus, the net (x_α) is eventually in $F^-(\mathrm{Int}(\mathrm{Cl}(V)))$.
The proof of the upper almost ω-continuity of F is similar to the above. □

Definition 3.11. Let (X, τ) be a topological space. The collection of all regular open sets forms a base for a topology τ^*. It is called the semiregularization. In case when $\tau = \tau^*$, the space (X, τ) is called semiregular [19].

Theorem 3.12. *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction from a topological space (X, τ) to a semiregular topological space (Y, σ). Then F is lower almost ω-continuous multifunction if and only if F is lower ω-continuous.*

Proof. Let $x \in X$ and let V be an open set such that $x \in F^-(V)$. Since (Y, σ) is a semiregular space, there exist regular open sets U_i for $i \in I$ such that $V = \underset{i \in I}{\cup} U_i$. We have $F^-(V) = F^-(\underset{i \in I}{\cup} U_i) = \underset{i \in I}{\cup} F^-(U_i)$. By Theorem 3.5, $F^-(U_i) \in \tau_\omega$ for $i \in I$. We obtain $F^-(V) \in \tau_\omega$. Hence, by Theorem 2.3 in [21], F is lower ω-continuous. The converse is obvious. □

Corollary 3.13. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is lower almost ω-continuous multifunction if and only if $F : (X, \tau) \to (Y, \sigma^*)$ is lower ω-continuous.*

Suppose that (X, τ), (Y, σ) and (Z, η) are topological spaces. It is known that if $F_1 : (X, \tau) \to (Y, \sigma)$ and $F_2 : (Y, \sigma) \to (Z, \eta)$ are multifunctions, then the composite multifunction $F_2 \circ F_1 : (X, \tau) \to (Z, \eta)$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for each $x \in X$.

Theorem 3.14. *If $F : (X, \tau) \to (Y, \sigma)$ is an upper (lower) semicontinuous multifunction and $G : (Y, \sigma) \to (Z, \eta)$ is an upper (lower) semicontinuous multifunction, then $G \circ F : (X, \tau) \to (Z, \eta)$ is an upper (lower) almost ω-continuous multifunction.*

Proof. Let $V \subset Z$ be any regular open set. From the definition of $G \circ F$, we have $(G \circ F)^+(V) = F^+(G^+(V))$ (resp. $(G \circ F)^-(V) = F^-(G^-(V))$). Since G is upper (lower) semicontinuous multifunction, $G^+(V)$ (resp. $G^-(V)$) is an open set. Since F is upper (lower) ω-continuous multifunction, $F^+(G^+(V))$ (resp. $F^-(G^-(V))$) is an ω-open set. It shows that $G \circ F$ is an upper (resp. lower) almost ω-continuous multifunction. □

Theorem 3.15. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is upper almost ω-continuous if and only if $s \operatorname{Cl} F : (X, \tau) \to (Y, \sigma)$ is upper almost ω-continuous, where $s \operatorname{Cl} F(x) = s \operatorname{Cl}(F(x))$ for each point $x \in X$.*

Proof. Suppose that F is upper almost ω-continuous. Let V be any open set of Y such that $s \operatorname{Cl} F(x) \subset V$. Then $F(x) \subset V$ and by

Theorem 3.5, there exists $U \in \omega O(X, x)$ such that $F(U) \subset s\,\mathrm{Cl}(V)$. For each $u \in U$, $F(U) \subset s\,\mathrm{Cl}(V)$ and hence $(s\,\mathrm{Cl}\,F)^{+}(V) \subset \omega\,\mathrm{Int}(s\,\mathrm{Cl}\,F)^{+}(s\,\mathrm{Cl}(V))$. It follows from Theorem 3.5, that $s\,\mathrm{Cl}\,F$ is upper almost ω-continuous. Conversely, suppose that $s\,\mathrm{Cl}\,F : (X, \tau) \to (Y, \sigma)$ is upper almost ω-continuous. Let V be any open set of Y and $x \in F^{+}(V)$. Then $F(x) \subset V$ and $s\,\mathrm{Cl}\,F(x) \subset s\,\mathrm{Cl}(V)$. There exists $U \in \omega O(X, x)$ such that $s\,\mathrm{Cl}\,F(U) \subset s\,\mathrm{Cl}(V)$. Therefore, we have $U \subset (s\,\mathrm{Cl}\,F)^{+}(s\,\mathrm{Cl}(V)) \subset F^{+}(s\,\mathrm{Cl}(V))$ and hence $x \in U \subset \omega\,\mathrm{Int}(F^{+}(s\,\mathrm{Cl}(V)))$. Thus, we obtain $F^{+}(V) \subset \omega\,\mathrm{Int}(F^{+}(s\,\mathrm{Cl}(V)))$ and by Theorem 3.5, F is upper almost ω-continuous. □

Theorem 3.16. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is lower almost ω-continuous if and only if $s\,\mathrm{Cl}\,F : (X, \tau) \to (Y, \sigma)$ is lower almost ω-continuous.*

Proof. The proof follows from Theorem 3.10 of [14]. □

Definition 3.17. A subset A of a topological space (X, τ) is said to be:

(1) α-regular [10], if for each $a \in A$ and any open set U containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$;
(2) α-paracompact [10], if every X-open cover A has an X-open refinement which covers A and is locally finite for each point of X.

Lemma 3.18. [10] *If A is an α-paracompact and α-regular set of a topological space (X, τ) and U an open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

Lemma 3.19. *If $F : (X, \tau) \to (Y, \sigma)$ is a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$, then we have the following*

(1) *$G^{+}(V) = F^{+}(V)$ for each open set V of Y,*
(2) *$G^{-}(V) = F^{-}(V)$ for each closed set V of Y, where G denotes $\mathrm{Cl}\,F$ or $\omega\,\mathrm{Cl}\,F$.*

Proof. The proof follows from Lemma 3.6 of [14] and Lemma 3.18. □

Theorem 3.20. *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$. Then the following statements are equivalent:*

(1) *F is upper almost ω-continuous;*
(2) *$\omega\,\mathrm{Cl}\,F$ is upper almost ω-continuous;*

(3) $\mathrm{Cl}\, F$ *is upper almost* ω*-continuous.*

Proof. The proof follows from Theorem 3.9 of [14]. □

Theorem 3.21. *Let* $F : (X, \tau) \to (Y, \sigma)$ *be a multifunction such that* $F(x)$ *is* α*-paracompact and* α*-regular for each* $x \in X$*. Then the following statements are equivalent:*

(1) F *is lower almost* ω*-continuous;*
(2) $\omega\, \mathrm{Cl}\, F$ *is lower almost* ω*-continuous;*
(3) $\mathrm{Cl}\, F$ *is lower almost* ω*-continuous.*

Proof. The proof follows from Theorem 3.10 of [14]. □

Theorem 3.22. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$ *such that* $F(x)$ *is an* α*-regular and* α*-paracompact set for each* $x \in X$*, the following are equivalent:*

(1) F *is upper weakly* ω*-continuous,*
(2) F *is upper almost* ω*-continuous,*
(3) F *is upper* ω*-continuous.*

Proof. The proof follows from Theorem 7.1 of [15] and Lemma 3.18.. □

Corollary 3.23. *Let* $F : (X, \tau) \to (Y, \sigma)$ *be a multifunction such that* $F(x)$ *is compact for each* $x \in X$ *and* Y *is regular. Then, the following are equivalent:*

(1) F *is upper weakly* ω*-continuous;*
(2) F *is upper almost* ω*-continuous;*
(3) F *is upper* ω*-continuous.*

Proof. The proof follows from Corollary 7.1 of [15]. □

Lemma 3.24. [17] *If* A *is an* α*-regular set of* X*, then for every open set* G *which intersects* A*, there exists an open set* D *such that* $A \cap D \neq \emptyset$ *and* $\mathrm{Cl}(D) \subset G$*.*

Theorem 3.25. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$ *such that* $F(x)$ *is an* α*-regular set of* Y *for each* $x \in X$*, the following are equivalent:*

(1) F *is lower weakly* ω*-continuous,*
(2) F *is lower almost* ω*-continuous,*
(3) F *is lower* ω*-continuous.*

Proof. The proof follows from Theorem 7.2 of [15] and Lemma 3.24.. □

Theorem 3.26. *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is closed in Y for each $x \in X$ and Y is normal. Then the following are equivalent:*

(1) *F is upper weakly ω-continuous,*
(2) *F is upper almost ω-continuous,*
(3) *F is upper ω-continuous.*

Proof. The proof follows from Theorem 7.3 of [15]. □

Definition 3.27. A space (X, τ) is said to be rimcompact, if each point of X has a base of neighborhoods with compact frontiers.

Theorem 3.28. *If (Y, σ) is a rimcompact space and $F : (X, \tau) \to (Y, \sigma)$ is a compact valued multifunction with the closed graph, then the following are equivalent:*

(1) *F is upper weakly α-continuous;*
(2) *F is upper almost α-continuous;*
(3) *F is upper α-continuous.*

Proof. Suppose that F is upper weakly α-continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since Y is rimcompact, for each $z \in F(x)$. Since Y is rimcompact, for each $z \in F(x)$ there exists an open set $W(z)$ such that $z \in W(z) \subset V$ and the frontier $Fr(W(z))$ is compact. The family $\{W(z) : z \in F(x)\}$ is a cover of $F(x)$ by open sets of Y. Since $F(x)$ is compact, there exists a finite number of points, say, $z_1, z_2, ..z_n$ in $F(x)$ such that $F(x) \subset \cup\{W(z_j) : 1 \le j \le n\}$. Let $W = \cup\{W(z_j) : 1 \le j \le n\}$, then we have $Fr(W)$ is compact, $F(x) \subset W \subset V$ and $F(x) \cap Fr(W) = F(x) \cap \mathrm{Cl}(W) \cap \mathrm{Cl}(Y \backslash W) \subset F(x) \cap Y \backslash W = \emptyset$. For each $y \in Fr(W)$, $(x, y) \in X \times Y \backslash G(F)$. Since $G(F)$ is closed, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ containing x and y, respectively, such that $F(U(y)) \cap V(y) = \emptyset$. The family $\{V(y) : y \in Fr(W)\}$ is a cover of $Fr(W)$ by open sets of Y. Since $Fr(W)$ is compact, there exists a finite subset K of $Fr(W)$ such that $Fr(W) \subset \cup\{V(y) : y \in K\}$. Since F is upper weakly ω-continuous, there exists $U_0 \in \omega O(X, x)$ such that $F(U_0) \subset \mathrm{Cl}(W)$. Put $U = U_0 \cap (\cap\{U(y) : y \in K\})$. Then we obtain $U \in \omega O(X, x)$, $F(U) \subset \mathrm{Cl}(W)$ and $F(U) \cap Fr(W) = \emptyset$. Therefore, we obtain $F(U) \subset W \subset V$. This shows that F is upper ω-continuous. □

Corollary 3.29. *If (Y, σ) is a rimcompact space and $f : (X, \tau) \to (Y, \sigma)$ is an almost ω-continuous function with closed graph, then f is ω-continuous.*

Theorem 3.30. *If (Y,σ) is a rimcompact Hausdorff space, then for a multifunction $F:(X,\tau)\to(Y,\sigma)$ the following are equivalent:*

(1) *F is lower weakly ω-continuous;*
(2) *F is lower almost ω-continuous;*
(3) *F is lower ω-continuous.*

Proof. Suppose that F is lower weakly ω-continuous. It follows from Theorem 3.4, that $F:(X,\tau_\omega)\to(Y,\sigma)$ is lower weakly continuous. Since (Y,σ) is a rimcompact, it is regular and hence by Theorem 2 of [18], that $F:(X,\tau_\omega)\to(Y,\sigma)$ is lower continuous. Therefore, $F:(X,\tau)\to(Y,\sigma)$ is lower ω-continuous by Theorem 3.4. □

For a multifunction $F:(X,\tau)\to(Y,\sigma)$, the graph multifunction $G_F:X\Rightarrow X\times Y$ is defined as follows: $G_F(x)=\{x\}\times F(x)$ for every $x\in X$.

Lemma 3.31. *For a multifunction $F:(X,\tau)\to(Y,\sigma)$, the following hold:*

(1) $G_F^+(A\times B)=A\cap F^+(B)$,
(2) $G_F^-(A\times B)=A\cap F^-(B)$

for any subsets $A\subset X$ and $B\subset Y$ [13].

Theorem 3.32. *Let $F:(X,\tau)\to(Y,\sigma)$ be a multifunction such that $F(x)$ is compact for each $x\in X$. Then F is upper almost ω-continuous if and only if $G_F:X\to X\times Y$ is upper almost ω-continuous.*

Proof. Suppose that $G_F:X\to X\times Y$ is upper almost ω-continuous. Let $x\in X$ and V be any open set of Y containing $F(x)$. Since $X\times V$ is open in $X\times Y$ and $G_F(x)\subset X\times V$, there exists $U\in\omega O(X,x)$ such that $G_F(U)\subset \mathrm{Int}(\mathrm{Cl}(X\times V))=X\times \mathrm{Int}(\mathrm{Cl}(V))$. By Lemma 3.31, we have $U\subset G_F^+(X\times \mathrm{Int}(\mathrm{Cl}(V)))=F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ and $F(U)\subset \mathrm{Int}(\mathrm{Cl}(V))$. This shows that F is upper almost ω-continuous. Conversely, suppose that $F:(X,\tau)\to(Y,\sigma)$ is upper almost ω-continuous. Let $x\in X$ and W be any open set of $X\times Y$ containing $G_F(x)$. For each $y\in F(x)$, there exist open sets $U(y)\subset X$ and $V(y)\subset Y$ such that $(x,y)\in U(y)\times V(y)\subset W$. The family of $\{V(y):y\in F(x)\}$ is an open cover of $F(x)$. Since $F(x)$ is compact, it follows that there exists a finite number of points, say $y_1,y_2,y_3,...,y_n$ in $F(x)$ such that $F(x)\subset\cup\{V(y_i):i=1,2,...,n\}$. Take $U=\cap\{U(y_i):i=1,2,....,n\}$ and $V=\cup\{V(y_i):i=1,2,...,n\}$. Then U and V are open sets in X and Y, respectively, and $\{x\}\times F(x)\subset U\times V\subset W$. Since F is upper almost ω-continuous, there exists $U_0\in\omega O(X,x)$ such that $F(U_0)\subset \mathrm{Int}(\mathrm{Cl}(V))$. By Lemma

3.31, we have $U \cap U_0 \subset U \cap F^+(\mathrm{Int}(\mathrm{Cl}(V))) = G_F^+(U \times \mathrm{Int}(\mathrm{Cl}(V))) \subset G_F^+(\mathrm{Int}(\mathrm{Cl}(U \times V))) \subset G_F^+(\mathrm{Int}(\mathrm{Cl}(W)))$. Therefore, we obtain $U \cap U_0 \in \omega O(X, x)$ and $G_F(U \cap U_0) \subset \mathrm{Int}(\mathrm{Cl}(W))$. This shows that G_F is upper almost ω-continuous. □

Theorem 3.33. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is lower almost ω-continuous if and only if $G_F : X \to X \times Y$ is lower almost ω-continuous.*

Proof. Suppose that F is lower almost ω-continuous. Let $x \in X$ and W be any open set of $X \times Y$ such that $x \in G_F^-(W)$. Since $W \cap (\{x\} \times F(x)) \neq \emptyset$, there exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \in U \times V \subset W$ for some open sets U and V of X and Y, respectively. Since $F(x) \cap V \neq \emptyset$, there exists $G \in \omega O(X, x)$ such that $G \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$. By Lemma 3.31, $U \cap G \subset U \cap F^-(\mathrm{Int}(\mathrm{Cl}(V))) = G_F^-(U \times \mathrm{Int}(\mathrm{Cl}(V))) \subset G_F^-(\mathrm{Int}(\mathrm{Cl}(W)))$. Furthermore, $x \in U \cap G \in \tau_\omega$ and hence G_F is lower almost ω-continuous. Conversely, suppose that G_F is lower almost ω-continuous. Let $x \in X$ and V be any open set of Y such that $x \in F^-(V)$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower almost ω-continuous, there exists an ω-open set U containing x such that $U \subset G_F^-(\mathrm{Int}(\mathrm{Cl}(X \times V)))$. Since $G_F^-(\mathrm{Int}(\mathrm{Cl}(X \times V))) = G_F^-(X \times \mathrm{Int}(\mathrm{Cl}(V)))$, by Lemma 3.31, we have $U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$. This shows that F is lower almost ω-continuous. □

Corollary 3.34. [16] *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $g : X \to X \times Y$ the graph function defined as follows: $g(x) = (x, f(x))$ for each $x \in X$. Then f is almost ω-continuous if and only if g is almost ω-continuous.*

Definition 3.35. [21] Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction. The multigraph $G(F)$ is said to be ω-closed graph in $X \times Y$, if for each $(x, y) \in X \times Y \backslash G(F)$, there exist ω-open set U and an open set V containing x and y, respectively, such that $(U \times V) \cap G(F) = \emptyset$.

Theorem 3.36. *Let $F : (X, \tau) \to (Y, \sigma)$ be an upper almost ω-continuous and punctually α-paracompact multifunction into a Hausdorff space (Y, σ). Then the multigraph $G(F)$ of F is an ω-closed graph in $X \times Y$.*

Proof. Suppose that $(x_0, y_0) \notin G(F)$. Then $y_0 \notin F(x_0)$. Since (Y, σ) is a Hausdorff space, then for each $y \in F(x_0)$ there exist open sets $V(y)$ and $W(y)$ containing y and y_0 respectively such that $V(y) \cap W(y) = \emptyset$.

The family $\{V(y) : y \in F(x_0)\}$ is an open cover of $F(x_0)$ which is α-paracompact. Thus, it has a locally finite open refinement $\Phi = \{U_\beta : \beta \in I\}$ which covers $F(x_0)$. Let W_0 be an open neighborhood of y_0 such that W_0 intersects only finitely many members $U_{\beta_1}, U_{\beta_2}, ..., U_{\beta_n}$ of Φ. Choose $y_1, y_2, ..., y_n$ in $F(x_0)$ such that $U_{\beta_i} \subset V(y_i)$ for each $i = 1, 2, ...n$ and set $W = W_0 \cap (\bigcap_{i=1}^{n} W(y_i))$. Then W is an open neighborhood of y_0 with $W \cap (\bigcup_{\beta \in I} U_\beta) = \emptyset$, which implies that $W \cap \mathrm{Int}(\mathrm{Cl}(\bigcup_{\beta \in I} U_\beta)) = \emptyset$. By the upper almost ω continuity of F, there exists $U \in \omega O(X, x_0)$ such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(\bigcup_{\beta \in I} U_\beta))$. It follows that $(U \times W) \cap G(F) = \emptyset$. Therefore, the graph $G(F)$ is an ω-closed graph in $X \times Y$. □

Let $\{X_\alpha : \alpha \in \nabla\}$ and $\{Y_\alpha : \alpha \in \nabla\}$ be any two families of topological spaces with same index set ∇. For each $\alpha \in \nabla$, let $F_\alpha : X_\alpha \to Y_\alpha$ be a multifunction. The product space $\Pi\{X_\alpha : \alpha \in \nabla\}$ will be denoted by ΠX_α and the product multifunction $\Pi F_\alpha : \Pi X_\alpha \to \Pi Y_\alpha$, defined by $F(x) = \Pi\{F_\alpha(x_\alpha) : \alpha \in \nabla\}$ for each $x = \{x_\alpha\} \in \Pi X_\alpha$, is simply denoted by $F : \Pi X_\alpha \to Y_\alpha$.

Theorem 3.37. *Let $F_\alpha : (X, \tau) \to (Y, \sigma)_\alpha$ be a multifunction for each $\alpha \in \nabla$ and $F : X \to \Pi Y_\alpha$ a multifunction defined by $F(x) = \Pi\{F_\alpha(x) : \alpha \in \nabla\}$ for each $x \in X$. If F is upper almost ω-continuous (resp. lower almost ω-continuous), then F_α is upper almost ω-continuous (resp. lower almost ω-continuous) for each $\alpha \in \nabla$.*

Proof. Let $x \in X$, $\alpha \in \nabla$ and V_α any regular open set of Y_α containing $F_\alpha(x)$. Then $P_\alpha^{-1}(V_\alpha) = V_\alpha \times \Pi\{Y_\beta : \beta \in \nabla$ and $\beta \neq \alpha\}$ is a regular open set of ΠY_α containing $F(x)$, where P_α is the natural projection of ΠY_α onto Y_α. Since F is upper almost ω-continuous, there exists $U \in \omega O(X, x)$ such that $F(U) \subset p_\alpha^{-1}(V_\alpha)$. Therefore, we obtain $F_\alpha(U) \subset P_\alpha(F(U)) \subset P_\alpha(P_\alpha^{-1}(V_\alpha)) = V_\alpha$. This shows that $F_\alpha : (X, \tau) \to (Y, \sigma)_\alpha$ is upper almost ω-continuous for each $\alpha \in \nabla$. The proof for lower almost ω-continuous is similar and is thus omitted. □

Theorem 3.38. *If (Y, σ) is a Hausdorff space and $F, G : (X, \tau) \to (Y, \sigma)$ are multifunctions such that*

(1) *$F(x)$ and $G(x)$ are compact for each $x \in X$,*
(2) *G is upper weakly ω-continuous,*
(3) *F is upper almost ω-continuous,*

then the set $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$ is ω-closed in X.

Proof. The proof follows from Theorem 8.3 of [15]. $\square$

Theorem 3.39. *If $F : (X,\tau) \to (Y,\sigma)$ is an upper almost ω-continuous multifunction such that $F(x)$ is α-nearly paracompact for each $x \in X$ and Y is Hausdorff, then for each $(x,y) \in X \times Y \backslash G(F)$, there exist $U \in \omega O(X,x)$ and an open set V containing y such that $(U \times \mathrm{Cl}(V)) \cap G(F) = \emptyset$.*

Proof. Let $(x,y) \in X \times Y\backslash G(F)$, then $y \in Y\backslash F(x)$. Since Y is Hausdorff, for each $a \in F(X)$ there exist open sets $V(a)$ and $W(a)$ containing a and y, respectively, such that $V(a) \cap W(a) = \emptyset$, hence $\mathrm{Int}(\mathrm{Cl}(V(a))) \cap W(a) = \emptyset$. The family $V = \{\mathrm{Int}(\mathrm{Cl}(V(a))) : a \in F(x)\}$ is a cover of $F(x)$ by regular open sets of Y and $F(x)$ is α-nearly paracompact. There exists a locally finite open refinement $H = \{H_\alpha : \alpha \in \nabla\}$ of V such that $F(x) \subset \cup\{H_\alpha : \alpha \in \nabla\}$. Since H is locally finite, there exists an open neighborhood W_0 of Y and a finite subset ∇_0 of ∇ such that $W_0 \cap H_\alpha = \emptyset$ for every $\alpha \in \nabla\backslash\nabla_0$. For each $\alpha \in \nabla_0$, there exists $a(\alpha) \in F(x)$ such that $H_\alpha \subset V(a(\alpha))$. Now, put $W = W_0 \cap (\cap\{W(a(\alpha)) : \alpha \in \nabla_0\})$ and $H = \cup\{H_\alpha : \alpha \in \nabla\}$. Then W is an open neighborhood of y, H is open in Y and $W \cap H = \emptyset$. Therefore, we obtain $F(x) \subset H$ and $\mathrm{Cl}(W) \cap H = \emptyset$ an hence $F(x) \subset Y\backslash \mathrm{Cl}(W)$. Since W is open, $Y\backslash \mathrm{Cl}(W)$ is regular open in Y. Since F is upper almost ω-continuous, there exists $U \in \omega O(X,x)$ such that $F(U) \subset Y\backslash \mathrm{Cl}(W)$, hence $F(U) \cap \mathrm{Cl}(W) = \emptyset$. Therefore, we obtain $(U \times \mathrm{Cl}(V)) \cap G(F) = \emptyset$. $\square$

Corollary 3.40. *If $F : (X,\tau) \to (Y,\sigma)$ is an upper almost ω-continuous multifunction such that $F(x)$ is compact for each $x \in X$ and Y is Hausdorff, then for each $(x,y) \in X \times Y\backslash G(F)$, there exist $U \in \omega O(X,x)$ and an open set V containing y such that $(U \times \mathrm{Cl}(V)) \cap G(F) = \emptyset$.*

Corollary 3.41. *If $f : (X,\tau) \to (Y,\sigma)$ is an ω-continuous function into a Hausdorff space Y, then $G(f)$ is ω-closed.*

Theorem 3.42. *Suppose that (X,τ) and (X_α,τ_α) are topological spaces, where $\alpha \in J$. Let $F : X \to \prod_{\alpha\in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha\in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha\in J} X_\alpha \to X_\alpha$ be the projection for each $\alpha \in J$. If F is upper (lower) almost ω-continuous multifunction, then $P_\alpha \circ F$ is upper (resp. lower) almost ω-continuous multifunction for each $\alpha \in J$.*

Proof. Take any $\alpha_0 \in J$. Let V_{α_0} be an open set in $(X_{\alpha_0},\tau_{\alpha_0})$. Then $(P_{\alpha_0} \circ F)^+(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0}))) = F^+(P^+_{\alpha_0}(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})))) =$

$F^+(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0}))) \times \prod_{\alpha\neq\alpha_0} X_\alpha$ (resp. $(P_{\alpha_0} \circ F)^-(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0}))) = F^-(P^-_{\alpha_0}(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})))) = F^-(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})) \times \prod_{\alpha\neq\alpha_0} X_\alpha))$. Since F is upper (resp. lower) almost ω-continuous multifunction and since $\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})) \times \prod_{\alpha\neq\alpha_0} X_\alpha$ is a regular open set, it follows that $F^+(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})) \times \prod_{\alpha\neq\alpha_0} X_\alpha)$ (resp. $F^-(\mathrm{Int}(\mathrm{Cl}(V_{\alpha_0})) \times \prod_{\alpha\neq\alpha_0} X_\alpha))$ is ω-open in (X, τ). It shows that $P_{\alpha_0} \circ F$ is upper (lower) almost ω-continuous multifunction. Hence, we obtain that $P_\alpha \circ F$ is upper (lower) almost ω-continuous multifunction for each $\alpha \in J$. □

Theorem 3.43. *Suppose that for each $\alpha \in J$, $(X_\alpha, \tau_\alpha), (Y_\alpha, \sigma_\alpha)$ are topological spaces. Let $F_\alpha : X_\alpha \to Y_\alpha$ be a multifunction for each $\alpha \in J$ and let $F : \prod_{\alpha\in J} X_\alpha \to \prod_{\alpha\in J} Y_\alpha$ be defined by $F((x_\alpha)) = \prod_{\alpha\in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha\in J} X_\alpha$ to the product space $\prod_{\alpha\in J} Y_\alpha$. If F is upper (lower) almost ω-continuous multifunction, then each F_α is upper (resp. lower) almost ω-continuous multifunction for each $\alpha \in J$.*

Proof. Let $V_\alpha \subseteq Y_\alpha$ be an open set. Then $\mathrm{Int}(\mathrm{Cl}(V_\alpha)) \times \prod_{\alpha\neq\beta} Y_\beta$ is a regular open set. Since F is upper (lower) almost ω-continuous multifunction, it follows that $F^+(\mathrm{Int}(\mathrm{Cl}(V_\alpha)) \times \prod_{\alpha\neq\beta} Y_\beta) = F^+_\alpha(\mathrm{Int}(\mathrm{Cl}(V_\alpha))) \times \prod_{\alpha\neq\beta} X_\beta$ (resp. $F^-(\mathrm{Int}(\mathrm{Cl}(V_\alpha)) \times \prod_{\alpha\neq\beta} Y_\beta) = F^-_\alpha(\mathrm{Int}(\mathrm{Cl}(V_\alpha))) \times \prod_{\alpha\neq\beta} X_\beta)$ is an ω-open set. Consequently, we obtain that $F^+_\alpha(\mathrm{Int}(\mathrm{Cl}(V_\alpha)))$ (resp. $F^-_\alpha(\mathrm{Int}(\mathrm{Cl}(V_\alpha))))$ is an ω-open set. Thus, we show that F_α is upper (resp. lower) almost ω-continuous multifunction. □

Theorem 3.44. *Suppose that (X, τ), (Y, σ), (Z, η) are topological spaces and $F_1 : (X, \tau) \to (Y, \sigma)$, $F_2 : (X, \tau) \to (Z, \eta)$ are multifunctions. Let $F_1 \times F_2 : (X, \tau) \to (Y, \sigma) \times Z$ be a multifunction which is defined by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is upper (lower) almost ω-continuous multifunction, then F_1 and F_2 are upper (resp. lower) almost ω-continuous multifunctions.*

Proof. Let $x \in X$ and let $K \subset Y$, $H \subset Z$ be open sets such that $x \in F_1^+(K)$ and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and so $F_1(x) \times F_2(x) = (F_1 \times F_2)(x) \subset K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$.Since $F_1 \times F_2$ is upper almost ω-continuous multifunction, there exists an ω-open set U containing x such that $U \subset (F_1 \times F_2)^+(\mathrm{Int}(\mathrm{Cl}(K \times H)))$. We obtain that $U \subset F_1^+(\mathrm{Int}(\mathrm{Cl}(K)))$ and $U \subset F_2^+(\mathrm{Int}(\mathrm{Cl}(H)))$. Thus, we obtain that F_1 and F_2 are upper almost ω-continuous multifunctions. The proof of the lower almost ω continuity of F_1 and F_2 is similar to the above. $\square$

Lemma 3.45. [1] *Let A and B be subsets of a topological space (X, τ). Then*

(1) *If $A \in \omega O(X)$ and $B \in \tau$, then $A \cap B \in \omega O(B)$;*
(2) *If $A \in \omega O(B)$ and $B \in \tau_\omega$, then $A \in \tau_\omega$.*

Lemma 3.46. *If $F : (X, \tau) \to (Y, \sigma)$ is an upper almost ω-continuous (lower almost ω-continuous) multifunction and $U \in \tau$, then $F_{|U} : (U, \tau_U) \Rightarrow (Y, \sigma)$ is upper almost ω-continuous (lower almost ω-continuous).*

Proof. Suppose that V is an open subset of Y. Let $x \in U$ and let $x \in (F_{|U})^-(V)$. Since F is lower almost ω-continuous multifunction, there exists an ω-open set G such that $x \in G \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$. By Lemma 3.45, we obtain that $x \in G \cap U \in \omega O(U)$ and $G \cap U \subset (F_{|U})^-(\mathrm{Int}(\mathrm{Cl}(V)))$. Hence $F_{|U}$ is lower almost ω-continuous. The proof of the upper almost ω-continuity of $F_{|U}$ is similar to the above. $\square$

Theorem 3.47. *Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of a space (X, τ). Then a multifunction $F : (X, \tau) \to (Y, \sigma)$ is upper almost ω-continuous (resp. lower almost ω-continuous) if and only if the restriction $F_{|U_\alpha} : (U_\alpha, \tau_\alpha) \Rightarrow (Y, \sigma)$ is upper almost ω-continuous (resp. lower almost ω-continuous) for each $\alpha \in \Lambda$.*

Proof. We prove only the case for F upper almost ω-continuous, the proof for F lower almost ω-continuous being analogous. Let $\alpha \in \Lambda$ and V be any open set of Y. Since F is upper almost ω-continuous, $F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ is ω-open in X. By Lemma 3.45, $(F_{|U_\alpha})^+(\mathrm{Int}(\mathrm{Cl}(V))) = F^+(\mathrm{Int}(\mathrm{Cl}(V))) \cap U_\alpha$ is ω-open in U_α and hence $F_{|U_\alpha}$ is upper almost ω-continuous. Conversely, let V be any open set of Y. Since $F_{|U_\alpha}$ is upper almost ω-continuous for each $\alpha \in \Lambda$, $(F_{|U_\alpha})^+(\mathrm{Int}(\mathrm{Cl}(V))) = F^+(\mathrm{Int}(\mathrm{Cl}(V))) \cap U_\alpha$ is ω-open in U_λ. By Lemma 3.45, $(F_{|U_\alpha})^+(\mathrm{Int}(\mathrm{Cl}(V)))$ is ω-open in X for each $\alpha \in \Lambda$. We obtain that $F^+(\mathrm{Int}(\mathrm{Cl}(V))) = \bigcup\limits_{\alpha\in\Lambda} (F_{|U_\alpha})^+(\mathrm{Int}(\mathrm{Cl}(V)))$ is ω-open in X. Hence F is upper almost ω-continuous. $\square$

Recall that a multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be punctually connected if for each $x \in X$, $F(x)$ is connected.

Definition 3.48. A topological space (X, τ) is called ω-connected [2] provided that X is not the union of two nonempty disjoint ω-open sets.

Theorem 3.49. *Let F be a multifunction from an ω-connected topological space (X,τ) onto a topological space (Y,σ) such that F is punctually connected. If F is an upper almost ω-continuous multifunction, then Y is a connected space.*

Proof. The proof follows from Theorem 9.1 of [15]. □

Recall that a multifunction $F:(X,\tau)\to(Y,\sigma)$ is said to be punctually closed if for each $x\in X, F(x)$ is closed.

Theorem 3.50. *Let F be an upper almost ω-continuous punctually closed multifunction and G be an upper almost continuous punctually closed multifunction from a space (X,τ) to a normal space (Y,σ). Then the set $K=\{x\in X : F(x)\cap G(x)\neq\emptyset\}$ is ω-closed in X.*

Proof. Let $x\in X\backslash K$. Then $F(x)\cap G(x)=\emptyset$. Since F and G are punctually closed multifunctions and Y is a normal space, there exists disjoint open sets U and V containing $F(x)$ and $G(x)$, respectively. Since F and G are upper almost ω-continuous and upper almost continuous, respectively the sets $F^{+}(\mathrm{Int}(\mathrm{Cl}(U)))$ and $G^{+}(\mathrm{Int}(\mathrm{Cl}(V)))$ are ω-open and open sets, respectively containing x. Let $H=F^{+}(\mathrm{Int}(\mathrm{Cl}(U)))\cap G^{+}(\mathrm{Int}(\mathrm{Cl}(V)))$. Then H is an ω-open set containing x and $H\cap K=\emptyset$. Hence, K is ω closed in X. □

Definition 3.51. A topological space (X,τ) is said to be ω-T_2 [2], if for each pair of distinct points x and y in X, there exist disjoint ω-open sets U and V in X such that $x\in U$ and $y\in V$.

Theorem 3.52. *Let $F:(X,\tau)\to(Y,\sigma)$ be an upper almost ω-continuous multifunction and punctually closed from a topological space (X,τ) to a normal topological space (Y,σ) and let $F(x)\cap F(y)=\emptyset$ for each distinct pair $x,y\in X$. Then X is an ω-T_2 space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x)\cap F(y)=\emptyset$. Since (Y,σ) is a normal space, it follows that there exist disjoints open sets U and V containing $F(x)$ and $F(y)$, respectively. Thus $F^{+}(\mathrm{Int}(\mathrm{Cl}(U)))$ and $F^{+}(\mathrm{Int}(\mathrm{Cl}(V)))$ are disjoint ω-open sets containing x and y, respectively. Thus, it is obtained that (X,τ) is ω-T_2. □

Definition 3.53. [2] The ω-frontier of a subset A of a space (X,τ), denoted by $\omega Fr(A)$, is defined by $\omega Fr(A)=\omega\,\mathrm{Cl}(A)\cap\omega\,\mathrm{Cl}(X\backslash A)=\omega\,\mathrm{Cl}(A)\backslash\omega\,\mathrm{Int}(A)$.

Theorem 3.54. *The set all points of X at which a multifunction $F:(X,\tau)\to(Y,\sigma)$ is not upper almost ω-continuous (lower almost*

ω-continuous) is identical with the union of the ω-frontier of the upper (lower) inverse images of regular open sets containing (meeting) $F(x)$.

Proof. The proof follows from Theorem 3.11 of [14].
In case F is lower almost ω-continuous, the proof is similar. $\square$

In the following $(D, >)$ is a directed set, (F_λ) is a net of multifunction $F_\lambda : (X, \tau) \to (Y, \sigma)$ for every $\lambda \in D$ and F is a multifunction from X into Y .

Definition 3.55. Let $(F_\lambda)_{\lambda \in D}$ be a net of multifunctions from X to Y. A multifunction $F^\star : (X, \tau) \to (Y, \sigma)$ is defined as follows: for each $x \in X$, $F^\star(x) = \{y \in Y\colon$ for each open neighborhood V of y and each $\mu \in D$, there exists $\lambda \in D$ such that $\lambda > \mu$ and $V \cap F_\lambda(x) \neq \emptyset\}$ is called the upper topological limit of the net $(F_\lambda)_{\lambda \in D}$ [4].

Definition 3.56. A net $(F_\lambda)_{\lambda \in D}$ is said to be equally upper almost ω-continuous at $x_0 \in X$, if for every open set V containing $F_\lambda(x_0)$, there exists an ω-open set U containing x_0 such that $F_\lambda(U) \subset \operatorname{Int}(\operatorname{Cl}(V_\lambda))$ for all $\lambda \in D$.

Theorem 3.57. *Let $(F_\lambda)_{\lambda \in D}$ be a net of multifunctions from a topological space (X, τ) into a compact space (Y, σ). If the following are satisfied:*

(1) *$\cup\{F_\mu(x) : \mu > \lambda\}$ is closed in Y for each $\lambda \in D$ and each $x \in X$;*
(2) *$(F_\lambda)_{\lambda \in D}$ is equally upper almost ω-continuous on X, then $F^\star$ is upper almost ω-continuous on X, then $F^\star$ is upper almost ω-continuous on X.*

Proof. We have $F^\star(x) = \cap\{(\cup\{F_\mu(x) : \mu > \lambda\}) : \lambda \in D\}$. Since the net $(\cup\{F_\mu(x) : \mu > \lambda\})_{\lambda \in D}$ is a family of closed sets having the finite intersection property and Y is compact, $F^\star(x) \neq \emptyset$ for each $x \in X$. Now, let $x_0 \in X$ and let V be a proper open subset of Y such that $F^\star(x_0) \subset V$ .Since $F^\star(x_0) \cap (Y \backslash V) = \emptyset$, $F^\star(x_0) \neq \emptyset$ and $Y \backslash V \neq \emptyset$, $\cap\{(\cup\{F_\mu(x_0) : \mu > \lambda\}) : \lambda \in D\} \cap (Y \backslash V) = \emptyset$ and hence $\cap\{(\cup\{F_\mu(x_0) \cap (Y \backslash V) : \mu > \lambda\}) : \lambda \in D\} = \emptyset$. Since Y is compact and the family $\{(\cup\{F_\mu(x_0) \cap (Y \backslash V) : \mu > \lambda\}) : \lambda \in D\}$ is a family of closed sets with the empty intersection, there exists $\lambda \in D$ such that $F_\mu(x_0) \cap (Y \backslash V) = \emptyset$ for each $\mu \in D$ with $\mu > \lambda$. Since the net $(F_\lambda)_{\lambda \in D}$ is equally upper almost ω-continuous on X, there exists an ω-open set U containing x_0 such that $F_\mu(U) \subset \operatorname{Int}(\operatorname{Cl}(V))$ for each $\mu > \lambda$, that is, $F_\mu(x) \cap (Y \backslash \operatorname{Int}(\operatorname{Cl}(V))) = \emptyset$ for each $x \in U$. Then we have $\cup\{F_\mu(x) \cap (Y \backslash \operatorname{Int}(\operatorname{Cl}(V))) : \mu > \lambda\} = \emptyset$ and hence

$\cap\{\cup\{F_\mu(x) : \mu > \lambda\} : \lambda \in D\} \cap (Y \setminus \mathrm{Int}(\mathrm{Cl}(V))) = \emptyset$. This implies that $F^*(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. If $V = Y$, then it is clear that for each ω-open set U containing x_0 we have $F^\star(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Hence $F^\star$ is upper almost ω-continuous at x_0. Since x_0 is arbitrary, the proof completes. □

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