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SUPERMINIMIZERS FOR ENERGY INTEGRALS IN ORLICZ-SOBOLEV SPACES ON METRIC SPACES

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Abstract. We extend the basic part of the study of superminimizers for Dirichlet energy integrals on metric spaces, initiated in a seminal paper by J. Kinnunen and O. Martio (2002) and thoroughly undertaken in the monograph of A. Björn and J. Björn (2011), to a case where the role of Newtonian spaces is played by more general Orlicz-Sobolev spaces. We prove a comparison principle for obstacle problems in this generalized setting, then we give some characterizations of superminimizers and methods of constructing new superminimizers from existing ones. Finally, we establish a two-way connection between the solutions of obstacle problems and the superminimizers associated to an energy integral.

Dedicated to Professor Valeriu Popa on the Occasion of His 80th Birthday

1. Introduction

Laplace equation is the prototype for linear elliptic partial differential equations of second order and the properties of its solutions, harmonic functions, are the object of study of potential theory.

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Dirichlet's principle for Laplace equation establishes a strong connection between PDE's and calcululs of variations. Consider the energy integral $I(v) = \int\limits_{\Omega} |\nabla v(x)|^2 \, dx$, where $\Omega \subset \mathbb{R}^n$ is a nonempty open set and $v \in C^2(\Omega)$. Then $u \in C^2(\Omega)$ is harmonic in Ω if and only if u is a stationary point of the functional I, in the sense that $\frac{d}{dt} I(u + t\varphi)\big|_{t=0} = 0$ for all $\varphi \in C_0^\infty(\Omega)$. In the special case where Ω is a bounded domain with a C^1-boundary, a function $u \in C^2\left(\overline{\Omega}\right)$ is harmonic in Ω if and only if u is a minimizer of I, with respect to the functions having the same boundary values: $I(u) \leq I(w)$ whenever $w \in C^2\left(\overline{\Omega}\right)$ satisfies $w|_{\partial\Omega} = u|_{\partial\Omega}$.

Nonlinear potential theory extends the study of harmonic functions to solutions of partial differential equations that are nonlinear, possibly degenerate, elliptic or parabolic.

A first step in generalizing Laplace equation by a nonlinear elliptic equation in divergence form has been taken by introducing the $p-$Laplace equation

$$-\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = 0.$$

Solutions of the $p-$Laplace equation in domains in $\mathbb{R}^n$, called $p-$harmonic functions, coincide with the minimizers of the $p-$Dirichlet energy integral $I_p(u) = \int\limits_{\Omega} |\nabla u(x)|^p \, dx$. Moreover, supersolutions of the $p-$Laplace equation coincide with the superminimizers of the $p-$Dirichlet energy integral.

A classical obstacle problem (in calculus of variations) requires to find the equilibrium position of an elastic membrane whose boundary is held fixed, and which is constrained to lie above a given obstacle. This problem is related to the study of minimal surfaces and of the capacity of a set in potential theory. The obstacle problem with the obstacle identical $-\infty$ is the corresponding Dirichlet problem. Obstacle problems have applications in continuous media mechanics (the study of fluid filtration in porous media, elasto-plasticity), optimal control (applied e.g. to the study of the electrochemical machining problem), financial mathematics and many others [24].

[14] was the first monograph dealing with a potential theory for a class of second order quasilinear elliptic equations that are "measurable perturbations" of $p-$Laplace equation and whose solutions are in a weighted Sobolev space of first order, with exponent p. An important property of these solutions is that they are quasiminimizers of a weighted $p-$Dirichlet integral.

During the last two decades, potential theory has been developed in the setting of doubling metric measure spaces supporting a $p-$Poincaré inequality. Nonlinear potential theory on metric measure spaces unifies the theory of variational integrals related to nonlinear elliptic PDE's, studied in various settings: in weighted Euclidean spaces with Muckenhoupt weights, on Riemannian manifolds with nonnegative Ricci curvature, on Carnot groups, on graphs, etc.

In the metric setting, the role of the length of the gradient is taken by the notion of upper gradient, that was introduced by Heinonen and Koskela [13]. The study of $p-$harmonic functions as solutions of a partial differential equation is replaced by their study study as minimizers of the $p-$Dirichlet integral, which belong to the Newtonian space $N_{loc}^{1,p}(X)$. Superminimizers and solutions of the obstacle problem for the $p-$Dirichlet integral play an important role in this study, as shown in [16]. Quasiminimizers, that in first place have been used as tools in studying regularity of minimizers of variational integrals, are interesting in their own right, their potential theory on metric spaces being developed in [17].

The monograph [3], considered a metric space enriched version of [14], is a self-contained exposition on nonlinear potential theory of $p-$harmonic functions on metric measure spaces supporting a $p-$Poincaré inequality.

We extend the basic part of the study of superminimizers for Dirichlet energy integrals on metric spaces, initiated in a seminal paper by J. Kinnunen and O. Martio [16], developed in papers as [2] and thoroughly undertaken in the monograph of A. Björn and J. Björn [3], to a case where the role of Newtonian spaces is played by more general Orlicz-Sobolev spaces. We prove a comparison principle for obstacle problems in this generalized setting, then we give some characterizations of superminimizers and methods of constructing new superminimizers from existing ones. Finally, we establish a two-way connection between the solutions of obstacle problems and the superminimizers associated to an energy integral.

2. Preliminaries. Weak upper gradients and Orlicz-Sobolev spaces

Let (X, d, μ) be a metric measure space. It is assumed that μ is a Borel regular measure, finite and positive on balls. We will denote by $B(x, r)$ the open ball centered at $x \in X$ of radius $r > 0$. By a curve we mean a continuous mapping from a compact interval into X. We will denote the image of a curve $\gamma : [a,b] \to X$ by $|\gamma|$.

The measure μ is *doubling* if there exists a constant $C_\mu \geq 1$ such that

$$\mu(B(x,2r)) \leq C_\mu \mu(B(x,r))$$

whenever $x \in X$ and $r > 0$. Doubling measures play an important role in harmonic analysis, through the notion of space of homogeneous type.

We recall the notion of Orlicz space on a measure space [23]. The notions of Young function and $N-$function are well-known. In what follows, $\Psi : [0,\infty) \to [0,\infty]$ will always be a Young function.

We deal with the growth rates given by $\Delta_2-$and ∇_2-conditions for Young functions. Ψ is said to satisfy a Δ_2-condition (and is called *doubling*) if there is a constant $C_\Psi > 0$ such that $\Psi(2t) \leq C_\Psi \Psi(t)$ for every $t \in [0,\infty)$. On the other hand, $\Psi : [0,\infty) \to [0,\infty)$ is said to satisfy a ∇_2-condition if there is a constant $C > 1$ such that $\Psi(t) \leq \frac{1}{2C}\Psi(Ct)$ for all $t \in [0,\infty)$.

Let $(X, \mathcal{A}, \mu)$ be a measure space with μ a complete and $\sigma-$finite measure. The Orlicz space associated to Ψ is

$$L^\Psi(X) = \left\{ u : X \to \overline{\mathbb{R}} \text{ measurable: } \int_X \Psi(\lambda |u|) d\mu < \infty \text{ for some } \lambda > 0 \right\}.$$

The Orlicz space is a Banach space with the *Luxemburg norm* defined by

$$\|u\|_{L^\Psi(X)} = \inf \left\{ \lambda > 0 : \int_X \Psi\left(\frac{|u|}{\lambda}\right) d\mu \leq 1 \right\}.$$

If a Young function Ψ satisfies both a Δ_2-condition and a ∇_2-condition, then the space $L^\Phi(\Omega)$ is reflexive [6].

The notion on modulus of a curve family, with respect to an Orlicz space, was introduced in [27] as a generalization of the $p-$modulus. Denote by $\Gamma(X)$ the family of all rectifiable non-constant curves $\gamma : [a,b] \to X$. Let $\Gamma \subset \Gamma(X)$. Then $\mathcal{A}(\Gamma)$, the set of admissible function for Γ, is defined as follows: $\rho \in \mathcal{A}(\Gamma)$ if $\rho : X \to [0,\infty]$ is a Borel function such that $\int_\gamma \rho \, ds \geq 1$. The $\Psi-$modulus $Mod_\Psi(\Gamma)$ of Γ is defined by

$$Mod_\Psi(\Gamma) = \inf_{\rho \in \mathcal{A}(\Gamma)} \|\rho\|_{L^\Psi(X)}.$$

The $\Psi-$modulus has the properties of an outer measure on $\Gamma(X)$. It is said that a property of curves holds for $\Psi-$almost every curve if the $\Psi-$modulus of the set of curves in $\Gamma(X)$ for which the property does not hold is zero [27].

In first order calculus on metric spaces, the notion of upper gradient, introduced by Heinonen and Koskela in [13], is a substitute for the length of the gradient of a smooth function. A related notion is that of upper gradient along a curve. All non-constant rectifiable curves $\gamma : [a,b] \to X$ are assumed to be parameterized by arc length. Denoting the length of γ by l_γ, we will consider curves parameterized by arc length written as $\gamma : [0, l_\gamma] \to X$.

A Borel measurable function $g : X \to [0, \infty]$ is an *upper gradient* of an extended real-valued function u on X if

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_\gamma g \, ds. \tag{2.1}$$

Here the following convention is used: if at least one of $u(\gamma(0))$ and $u(\gamma(l_\gamma))$ is not finite, then $\int_\gamma g \, ds = \infty$ ([3, page 9]).

Definition 1. *A Borel measurable function $g : X \to [0,\infty]$ is an upper gradient of an extended real-valued function u on X, along a non-constant rectifiable curve $\gamma : [0, l_\gamma] \to X$, if for every subcurve $\gamma_1 : [0, l_{\gamma_1}] \to X$ of γ we have*

$$|u(\gamma_1(0)) - u(\gamma_1(l_{\gamma_1}))| \leq \int_{\gamma_1} g \, ds.$$

The upper gradients are sensitive, being not preserved neither under changes almost everywhere, nor under limits. These drawbacks are eliminated by using the more general notion of $p-$weak upper gradient, introduced by Koskela and MacManus [19]. We can define the notion of $\Psi-$weak upper gradient associated to a Young function Ψ, slightly modifying the definition from [27], by omitting the Borel measurability requirement, as it was first suggested in [4]. As a consequence of the Borel regularity of the measure μ, every measurable function $f : X \to \overline{\mathbb{R}}$ agrees a.e. with a Borel function [3, Proposition 1.2]. Note that for $\Psi(t) = t^p$ the Borel measurable $\Psi-$weak upper gradients coincide with $p-$weak upper gradients.

Definition 2. *A measurable function $g : X \to [0,\infty]$ is a $\Psi-$weak upper gradient of an extended real-valued function u on X if (2.1) holds for $\Psi-$almost every non-constant rectifiable curve $\gamma : [0, l_\gamma] \to X$.*

The fact that $\int_\gamma g\ ds$ is defined for $\Psi-$almost every curve $\gamma \in \Gamma(X)$ follows as shown in [3, Lemma 1.43] for the case $\Psi(t) = t^p$, as we will explain below.

Obviously, every upper gradient is a $\Psi-$weak upper gradient. The converse is not true, but every $\Psi-$weak upper gradient is an upper gradient along $\Psi-$almost every curve. Moreover, every $\Psi-$weak upper gradient can be approximated in Orlicz norm by upper gradients, with arbitrarily small error.

Remark 1. *If Γ_0 and Γ are curve families such that each $\gamma \in \Gamma$ has a subcurve $\gamma_0 \in \Gamma_0$, then $Mod_\Psi(\Gamma) \leq Mod_\Psi(\Gamma_0)$* [27, page 22].

Remark 2. *As in* [3, Lemma 1.40], *it follows that for each $\Psi-$weak upper gradient g of u the family of curves $\gamma \in \Gamma(X)$ for which g is not an* upper gradient along *γ has zero $\Psi-$modulus.*

For a set $E \subset X$, denote $\Gamma_E = \{\gamma \in \Gamma(X) \text{ with } |\gamma| \cap E \neq \emptyset\}$ and $\Gamma_E^+ = \{\gamma \in \Gamma(X) \text{ with } \Lambda_1(\gamma^{-1}(E)) \neq 0\}$, where Λ_1 is the outer Lebesgue measure on $\mathbb{R}$.

As in [3, Lemma 1.42], it is easy to prove that $\mu(E) = 0$ implies $Mod_\Psi\left(\Gamma_E^+\right) = 0$.

Lemma 1. *Let $g : X \to [0,\infty]$ be a measurable function and let $\widetilde{g} : X \to [0,\infty]$ be a Borel function such that $\widetilde{g} = g$ a.e. Then, for $\Psi-$almost every curve $\gamma \in \Gamma(X)$, we have $\int_\gamma g\ ds = \int_\gamma \widetilde{g}\ ds$, in particular $\int_\gamma g\ ds$ is well defined.*

Proof. As $\widetilde{g}$ is a Borel function, $\int_\gamma \widetilde{g}\ ds$ is defined for every curve $\gamma \in \Gamma(X)$.

Let $E = \{x \in X : g(x) \neq \widetilde{g}(x)\}$. If $\gamma \in \Gamma(X) \setminus \Gamma_E^+$, then $\int_\gamma g\ ds = \int_\gamma \widetilde{g}\ ds$. But $Mod_\Psi\left(\Gamma_E^+\right) = 0$, since we have $\mu(E) = 0$. ∎

Corollary 1. *Each nonnegative function that agrees a.e. with a $\Psi-$weak upper gradient of u is also a $\Psi-$weak upper gradient of u. In particular, a function possesing a measurable $\Psi-$weak upper gradient also has a Borel $\Psi-$weak upper gradient.*

The first part of the above corollary was proved for Borel measurable functions in [27, Lemma 4.4].

The following extension of [3, Lemma 2.6] can be proved as in the case $L^{\Psi}(X) = L^{p}(X)$, using only basic properties of $\Psi-$weak upper gradients and of $\Psi-$modulus, as well as the property from Remark 2.

Lemma 2. *If $g_1, g_2 \in L^{\Psi}(X)$ are $\Psi-$weak upper gradients of a measurable function u in X, then $g = \min\{g_1, g_2\}$ is also a $\Psi-$weak upper gradient of u.*

The problem of approximation in L^{Ψ} norm of a $\Psi-$weak upper gradient by upper gradients was first solved in [19, Lemma 2.4] for $\Psi(t) = t^{p}$, then in [27, Lemma 4.3] for an arbitrary Young function Ψ. Note that is not necessary to assumme that the $\Psi-$weak upper gradient belongs to $L^{\Psi}(X)$. From [27, Lemma 4.3] and Lemma 1 we obtain

Lemma 3. *Let $g : X \to [0, \infty]$ be a $\Psi-$weak upper gradient of A function u. Then there is a decreasing sequence $(g_i)_{i\geq 1}$ of upper gradients of u such that $\|g_i - g\|_{L^{\Psi}(X)} \to 0$ as $i \to \infty$.*

Now we can recall the definition of the Orlicz-Sobolev space $N^{1,\Psi}(X)$ introduced in [27]. First, the family of all functions $u \in L^{\Psi}(X)$ having a $\Psi-$weak upper gradient $g \in L^{\Psi}(X)$ is denoted by $\widetilde{N}^{1,\Psi}(X)$.

$\widetilde{N}^{1,\Psi}(X)$ is a vector space. One defines a seminorm on $\widetilde{N}^{1,\Psi}(X)$ by setting

$$\|u\|_{1,\Psi} = \|u\|_{L^{\Psi}(X)} + \inf_{g} \|g\|_{L^{\Psi}(X)},$$

where the infimum is taken over all $\Psi-$weak upper gradients $g \in L^{\Psi}(X)$ of u.

Remark 3. *Lemma 1 shows that the definition of $\widetilde{N}^{1,\Psi}(X)$ is not affected if we use measurable, not necessarily Borel, $\Psi-$weak upper gradients. In addition, by Lemma 3 the seminorm $\|u\|_{1,\Psi}$ is not changed if we take the infimum only over the upper gradients $g \in L^{\Psi}(X)$ of u.*

As in the case of $L^{p}-$spaces, the seminormed space $\widetilde{N}^{1,\Psi}(X)$ is turned into a normed space via an equivalence relation. It is said that functions $u, v \in \widetilde{N}^{1,\Psi}(X)$ are equivalent, $u \sim v$, if $\|u - v\|_{1,\Psi} = 0$. Denote the equivalence class of u by $\widehat{u}$.

One defines the normed space $\left(N^{1,\Psi}(X), \|\cdot\|_{N^{1,\Psi}(X)}\right)$ with $N^{1,\Psi}(X) = \widetilde{N}^{1,\Psi}(X)/\sim$ and $\|\widehat{u}\|_{N^{1,\Psi}(X)} = \|u\|_{1,\Psi}$ for every $u \in \widetilde{N}^{1,\Psi}(X)$. To ease the notation, we write $u \in N^{1,\Psi}(X)$ instead of

$\widehat{u} \in N^{1,\Psi}(X)$, understanding that we choose an arbitrary representative of a given equivalence class belonging to $N^{1,\Psi}(X)$. The normed space $\left(N^{1,\Psi}(X), \|\cdot\|_{N^{1,\Psi}(X)}\right)$ is the *Orlicz-Sobolev space* associated to the Young function Ψ.

In the above definitions we may replace X by a nonempty open subset $\Omega \subset X$. It is said that a function $u : X \to \overline{\mathbb{R}}$ belongs to the *local Orlicz-Sobolev space* $N^{1,\Psi}_{loc}(X)$ if $u \in N^{1,\Psi}(B)$ for each ball $B \subset X$ [27, Definition 6.3].

Some function spaces related to $N^{1,\Psi}(X)$ are very useful.

The Dirichlet space $D^{\Psi}(X)$ can be defined as the space of measurable functions $u : X \to \overline{\mathbb{R}}$ having an upper gradient in $L^{\Psi}(X)$ (equivalently, by Lemma 1, having a $\Psi-$weak upper gradient in $L^{\Psi}(X)$). We see that $\widetilde{N}^{1,\Psi}(X) = L^{\Psi}(X) \cap D^{\Psi}(X)$.

The space $ACC_{\Psi}(X)$ of functions that are absolutely continuous on $\Psi-$almost every curve in X is defined in [27, Definition 4.5], as follows: $u \in ACC_{\Psi}(X)$ if $u : X \to \mathbb{R}$ and $u \circ \gamma$ is absolutely continuous on $[0, l_{\gamma}]$ for $\Psi-$almost every curve $\gamma \in \Gamma(X)$.

It is shown in [27, Lemma 4.6] that every $u : X \to \mathbb{R}$ with $u \in D^{\Psi}(X)$ also belongs to $ACC_{\Psi}(X)$. Note that we may drop the restriction of finiteness of u. The existence of an upper gradient $g \in L^{\Psi}(X)$ of u implies that the family $\Gamma_{\infty} \subset \Gamma(X)$, of curves γ with $\int_{\gamma} g ds = \infty$, has zero $\Psi-$modulus. According to Remark 2, the family Γ_0 of curves $\gamma \in \Gamma(X)$ for which g is not an *upper gradient along* γ also has zero $\Psi-$modulus. If $\gamma \in \Gamma(X) \setminus (\Gamma_{\infty} \cup \Gamma_0)$, it follows that $u \circ \gamma$ is absolutely continuous on $[0, l_{\gamma}]$, see the proof of [3, Theorem 1.56]. Then $D^{\Psi}(X) \subset ACC_{\Psi}(X)$.

In particular, we get $\widetilde{N}^{1,\Psi}(X) \subset ACC_{\Psi}(X)$ [27, Corollary 6.4].

A Sobolev capacity with respect to the space $N^{1,\Psi}(X)$ is the $\Psi-$*capacity* introduced in [27, Definitions 6.1 and 6.2], defined by $Cap_{\Psi}(E) = \inf\{\|u\|_{N^{1,\Psi}(X)} : u \in N^{1,\Psi}(X) \text{ and } u \geq 1 \text{ on } E\}$ for each set $E \subset X$. The $\Psi-$capacity is an outer measure. [27, Proposition 7.3]. We have $Cap_{\Psi}(E) = 0$ if and only if $\mu(E) = 0$ and $Mod_{\Psi}(\Gamma_E) = 0$ [27, Proposition 7.4].

It is said that a property regarding points in X holds $\Psi-$*quasieverywhere* ($\Psi-$q.e.) if the set of points for which the property does not hold has zero $\Psi-$capacity.

Remark 4. *The $\Psi-$capacity represents the correct gauge for distinguishing between two functions representing classes in $N^{1,\Psi}(X)$: if $u \in N^{1,\Psi}(X)$ and $v : X \to \overline{\mathbb{R}}$, then $v = u$ $\Psi-$q.e. if and only if $v \in N^{1,\Psi}(X)$ and $\|u - v\|_{N^{1,\Psi}(X)} = 0$, see* [3, Proposition 1.61], [21, Proposition 4].

A proof similar to that of [3, Proposition 1.59] shows that two functions belonging to $ACC_\Psi(X)$, that agree a.e., actually agree $\Psi-$q.e., see also [27, Corollary 6.12]. It follows that every representative of a class in $N^{1,\Psi}(X)$ is finite $\Psi-$q.e., see [3, Corollary 1.70].

In order to compare boundary values of Orlicz-Sobolev functions we need *Orlicz-Sobolev spaces with zero boundary values* on a set $E \subset X$. Denote by $\widetilde{N}_0^{1,\Psi}(E)$ be the collection of functions $u : E \to \overline{\mathbb{R}}$ for which there exists $\overline{u} \in \widetilde{N}^{1,\Psi}(X)$ such that $\overline{u} = u$ a.e. on E and $\overline{u}(x) = 0$ $\Psi-$q.e in $X \backslash E$. If $u, v \in \widetilde{N}_0^{1,\Psi}(E)$ define $u \simeq v$ if $u = v$ a.e. on E. Then $\simeq$ is an equivalence relation. We consider the quotient space $N_0^{1,\Psi}(E) = \widetilde{N}_0^{1,\Psi}(E)/\simeq$. A norm on $N_0^{1,\Psi}(E)$ is unambiguously defined by $\|u\|_{N_0^{1,\Psi}(E)} := \|\overline{u}\|_{N^{1,\Psi}(X)}$. By Remark 4, we have

$$N_0^{1,\Psi}(E) = \left\{ u|_E : u \in N^{1,\Psi}(X) \text{ and } u = 0 \text{ in } X \backslash E \right\}.$$

Denote by $Lip_c(E)$ the set of Lipschitz functions with compact support in E. If $\Omega \subset X$ is open, then $Lip_c(\Omega) \subset N_0^{1,\Psi}(\Omega)$.

The following lemma generalizes [3, Lemma 2.37].

Lemma 4. *Assume that $E \subset X$ is measurable. If $u \in N^{1,\Psi}(E)$ and $v, w \in N_0^{1,\Psi}(E)$ satisfy the inequalities $v \le u \le w$ $\Psi-$q.e. in E, then $u \in N_0^{1,\Psi}(E)$.*

Proof. We denote the extension by zero of a function $f : E \to \overline{\mathbb{R}}$ to X also by f. The extensions by zero of v, w belong to $N^{1,\Psi}(X)$.

We define the difference

$$d(x) = \begin{cases} u(x) - v(x), \text{ if } x \in E \text{ and } v(x) < +\infty \\ 0, \text{ if } x \in E \text{ and } v(x) = +\infty \text{ or if } x \in X \setminus E \end{cases}.$$

Similarly, we define D replacing u by w. Note that $D \in N_0^{1,\Psi}(E)$.

Then we have $u = v + d$ and $w = v + D$ on X.

Since $v \le u \le w$ $\Psi-$q.e. in E, it follows that $0 \le d \le D$. It suffices to prove that $d \in N_0^{1,\Psi}(E)$.

The above discussion show that it suffices to prove the lemma in the case where $v = 0$. After redefining functions on sets of zero

$\Psi-$capacity, we may assume that $0 \leq u \leq w$ everywhere in E. We prove that the extension by zero of u belongs to $N^{1,\Psi}(X)$.

Fix $g_u \in L^\Psi(E)$ an upper gradient of u in E and $g_w \in L^\Psi(X)$ an upper gradient of w in X.

Define

$$g = \begin{cases} g_u + g_w \text{ in } E \\ g_w \text{ in } X \setminus E \end{cases}.$$

Clearly, $g \in L^\Psi(X)$. We prove that g is an upper gradient of u in X, hence $u \in N^{1,\Psi}(X)$.

Let $\gamma : [0, l] \to X$ be a curve of length $0 < l < \infty$, parameterized by arc length. Denote $|\gamma| = \gamma([0, l(\gamma)])$.

Case 1. If $|\gamma| \subset E$, then $|\widetilde{u}(\gamma(0)) - \widetilde{u}(\gamma(l))| = |u(\gamma(0)) - u(\gamma(l))| \leq \int_\gamma g_u ds \leq \int_\gamma g ds.$

Case 2. If $\{\gamma(0), \gamma(l)\} \subset X \setminus E$, then $|\widetilde{u}(\gamma(0)) - \widetilde{u}(\gamma(l))| = 0 \leq \int_\gamma g ds.$

Case 3. It remains to study the case when $|\gamma| \cap (X \setminus E) \neq \emptyset$ and $\{\gamma(0), \gamma(l)\} \cap E \neq \emptyset$.

a) If only one of the points $\gamma(0)$, $\gamma(l)$ belongs to E, we may assume that $\gamma(0) \in E$, $\gamma(l) \in X \setminus E$, since the case $\gamma(l) \in E$, $\gamma(0) \in X \setminus E$ is similar. Then

$$\begin{aligned} |\widetilde{u}(\gamma(0)) - \widetilde{u}(\gamma(l))| &= |u(\gamma(0))| = u(\gamma(0)) \\ &\leq w(\gamma(0)) = |w(\gamma(0)) - w(\gamma(l))| \\ &\leq \int_\gamma g_w ds \leq \int_\gamma g ds. \end{aligned}$$

b) If $\gamma(0), \gamma(l) \in E$, we consider $0 < t < l$ such that $\gamma(t) \in X \setminus E$. The above argument shows that $|\widetilde{u}(\gamma(0)) - \widetilde{u}(\gamma(t))| \leq \int_{\gamma|_{[0,t]}} g ds$ and that $|\widetilde{u}(\gamma(t)) - \widetilde{u}(\gamma(l))| \leq \int_{\gamma|_{[t,l]}} g ds$. Then

$$|\widetilde{u}(\gamma(0)) - \widetilde{u}(\gamma(l))| \leq \int_{\gamma|_{[0,t]}} g ds + \int_{\gamma|_{[t,l]}} g ds = \int_\gamma g ds.$$

∎

Obviously, upper gradients, as well as $\Psi-$weak upper gradients, are not unique (the sum between such a gradient and an arbitrary nonnegative Borel function being a gradient of the same type). In order to substitute, in the problem of minimizing energy integral, the length of the gradient $|\nabla u|$ by an appropriate $\Psi-$weak upper gradient of u, we need the notion of *minimal* $\Psi-$*weak upper gradient.*

Definition 3. *Let $u \in D^{\Psi}(X)$. A $\Psi-$weak upper gradient $g_u \in L^{\Psi}(X)$ of u is a* minimal $\Psi-$weak upper gradient *if $g_u \leq g$ a.e. in X for every $\Psi-$weak upper gradient $g \in L^{\Psi}(X)$ of u.*

Note that $g_u \leq g$ a.e. implies $\int_X \Psi(g_u)\,d\mu \leq \int_X \Psi(g)\,d\mu$ and $\|g_u\|_{L^{\Psi}(X)} \leq \|g\|_{L^{\Psi}(X)}$.

Tuominen [27] proved the following existence result for minimal weak upper gradients of Orlicz-Sobolev functions.

Proposition 1. [27, Corollary 6.9] *Assume that Ψ is a doubling Young function (i.e., satisfies the Δ_2-condition). Then for each $u \in N^{1,\Psi}(X)$ there exists a $\Psi-$weak upper gradient $g_u \in L^{\Psi}(X)$ of u such that $\int_X \Psi(g_u)\,d\mu = \inf_g \int_X \Psi(g)\,d\mu$ and $\|g_u\|_{L^{\Psi}(X)} = \inf_g \|g\|_{L^{\Psi}(X)}$, where the infimum is taken over all $\Psi-$weak upper gradients of u. Moreover, for every $\Psi-$weak upper gradient $g \in L^{\Psi}(X)$ of u we have $g_u \leq g$ a.e. in X.*

Remark 5. *An analysis of the proofs from* [27, 6.1], *including the proof of* [27, Corollary 6.9], *shows that the assumption $u \in L^{\Psi}(X)$ is not used. Therefore the results regarding minimal $\Psi-$weak upper gradients in* [27, 6.1] *hold for $u \in D^{\Psi}(X)$.*

Note that a $\Psi-$weak upper gradient $g \in L^{\Psi}(X)$ of $u \in D^{\Psi}(X)$ is determined up to sets of zero measure.

Now we can prove a glueing lemma for (measurable) $\Psi-$weak upper gradients, extending [3, Lemma 2.19] from the case where $L^{\Psi}(X) = L^{p}(X)$, with a similar proof. The glueing lemma for $\Psi-$weak upper gradients that are Borel functions was stated (without proof) in [27, Lemma 4.11].

Lemma 5. *Let $u \in ACC_{\Psi}(X)$ and let $v, w : X \to \mathbb{R}$ with $g, h \in L^{\Psi}(X)$ being $\Psi-$weak upper gradients of v and w, respectively . Assume that there exists a a measurable set $F \subset X$ such that $u|_F = v$ and $u|_{X\setminus F} = w$. Then the function $\rho = g\chi_F + h\chi_{X\setminus F}$ is a $\Psi-$weak upper gradient of u. Moreover, if Ψ is doubling and $g = g_v$ and $h = g_w$ are minimal $\Psi-$weak upper gradients of v and w, respectively, then ρ is a minimal $\Psi-$weak upper gradient of u.*

Proof. Let $g_1 = g + h\chi_{X\setminus F}$ and $g_2 = g\chi_F + h$. Note that $g_1, g_2 \in L^{\Psi}(X)$ and $\rho = \min\{g_1, g_2\}$. It suffices to prove that g_1 is a $\Psi-$weak upper gradient of u. By symmetry, we see that g_2 is also a $\Psi-$weak upper gradient of u. Then using Lemma 2 it follows that ρ is a $\Psi-$weak upper gradient of u.

By Remark 2 and Lemma 1, there exists a curve family $\Gamma_0 \subset \Gamma(X)$ of zero $\Psi-$modulus such that for every $\gamma \in \Gamma(X) \setminus \Gamma_0$, $\gamma : [0, l] \to X$ the following properties hold:

(1) $\gamma^{-1}(F)$ is measurable;

(2) u, v and w are absolutely continuous on γ;

(3) g and h are upper gradients along γ of v and w, respectively.

Case 1. If $|\gamma| \subset X \setminus F$, then

$$|u(\gamma(0)) - u(\gamma(l))| = |w(\gamma(0)) - w(\gamma(l))| \leq \int_{\gamma} h ds \leq \int_{\gamma} g_1 ds,$$

hence

$$|u(\gamma(0)) - u(\gamma(l))| \leq \int_{\gamma} g_1 ds.$$

Case 2. Now assume that $|\gamma| \cap F \neq \emptyset$. Let $\alpha = \inf\{t \in [0, l] : \gamma(t) \in F\}$ and $\beta = \sup\{t \in [0, l] : \gamma(t) \in F\}$.

If $\alpha > 0$, then $|u(\gamma(0)) - u(\gamma(\alpha))| = \lim_{t \nearrow \alpha} |w(\gamma(0)) - w(\gamma(t))| \leq \lim_{t \nearrow \alpha} \int_{\gamma|_{[0,t]}} h ds = \int_{\gamma|_{[0,\alpha]}} h ds \leq \int_{\gamma|_{[0,\alpha]}} g_1 ds$, hence

$$|u(\gamma(0)) - u(\gamma(\alpha))| \leq \int_{\gamma|_{[0,\alpha]}} g_1 ds.$$

For $\alpha = 0$ the above inequality is trivial.

Similarly, $|u(\gamma(\beta)) - u(\gamma(l))| \leq \int_{\gamma|_{[\beta,l]}} g_1 ds$.

Using the continuity of u and v along γ we get

$$|u(\gamma(\alpha)) - u(\gamma(\beta))| = |v(\gamma(\alpha)) - v(\gamma(\beta))| \leq \int_{\gamma|_{[\alpha,\beta]}} g ds \leq \int_{\gamma|_{[\alpha,\beta]}} g_1 ds.$$

By the triangle inequality,

$$|u(\gamma(0)) - u(\gamma(l))| \leq \int_{\gamma|_{[0,\alpha]}} g_1 ds + \int_{\gamma|_{[\alpha,\beta]}} g_1 ds + \int_{\gamma|_{[\beta,l]}} g_1 ds = \int_{\gamma} g_1 ds,$$

hence the first part of the proof is complete.

We proved that ρ, that clearly belongs to $L^{\Psi}(X)$, is a $\Psi-$weak upper gradient of u, hence $u \in D^{\Psi}(X)$.

Now assume that Ψ is doubling (i.e., satisfies a Δ_2-condition). By Lemma 1 and Remark 5, there exists a minimal $\Psi-$weak upper gradient g_u of u. Then $g_u \leq \rho$ a.e. on X.

For $g = g_u$ and $h = g_v$, the first part of this lemma shows that

$$\rho_1 := g_v \chi_F + g_w \chi_{X \setminus F}$$

is a $\Psi-$weak upper gradient of u.

Applying the first part of the lemma, but with the roles of u and v interchanged, we obtain that $\rho_2 := g_u \chi_F + g_v \chi_{X \setminus F}$ is a $\Psi-$weak upper gradient of v. Then, by the a.e. minimality of g_v and g_u, we get

$$g_v \leq \rho_2 = g_u \leq \rho = g_v \text{ } a.e. \text{ on } F.$$

Then $g_u = g_v$ a.e. on F, hence $g_u = \rho_1$ a.e. on F.

Similarly, applying the first part of this lemma, with the roles of u and w interchanged, we obtain that $\rho_3 := g_w \chi_F + g_u \chi_{X \setminus F}$ is a $\Psi-$weak upper gradient of w. Then, by the a.e. minimality of g_w and g_u, we get

$$g_w \leq \rho_3 = g_u \leq \rho = g_w \text{ } a.e. \text{ on } X \setminus F.$$

Then $g_u = g_w$ a.e. on $X \setminus F$, hence $g_u = \rho_1$ a.e. on X.

We proved that ρ_1 is a minimal $\Psi-$weak upper gradient of u. ∎

Corollary 2. *Assume that the Young function Ψ is doubling. If $v, w \in D^{\Psi}(X)$, then $g_v = g_w$ a.e. on $E := \{x \in X : v(x) = w(x)\}$.*

Proof. Define $u = v\chi_{X \setminus E} + w\chi_E$. Since $w = v$ on E, we see that $u = v$ on X. In particular, $u \in D^{\Psi}(X) \subset ACC_{\Psi}(X)$.

By the second part of the glueing lemma, Lemma 5, with $F := X \setminus E$, it follows that

$$\rho := g_v \chi_{X \setminus E} + g_w \chi_E$$

is a minimal $\Psi-$weak upper gradient of u, hence of v, on X.

Since g_v is also a $\Psi-$weak upper gradient of v, we have $g_v = \rho$ a.e. on X. This implies $g_v = g_w$ a.e. on E. ∎

Following the steps from the proof of [3, Lemma 2.23] and using Remark 2 and Lemma 5 we get the next lemma. Note that the restriction of a minimal weak upper gradient is not always minimal [3, page 51].

Lemma 6. *Let $\Omega \subset X$ be a nonempty open set. If $u \in D^{\Psi}(X)$ and g_u is a minimal $\Psi-$weak upper gradient of u in X, then the restriction $g_u|_{\Omega}$ of g_u to Ω is a minimal $\Psi-$weak upper gradient of u in Ω.*

Using Lemma 6, Proposition 1 can be extended from $N^{1,\Psi}(X)$ to $D^{\Psi}_{loc}(X)$, by the following result that generalizes [3, Theorem 2.25], showing that every function in $D^{\Psi}_{loc}(X)$ has a minimal $\Psi-$weak upper gradient, determined up to a set of measure zero.

Proposition 2. *Assume that the Young function Ψ is doubling. Then for each $u \in D^{\Psi}_{loc}(X)$ there exists a minimal $\Psi-$weak upper gradient $g_u \in L^{\Psi}_{loc}(X)$ of u, i.e. $g_u \le g$ a.e. for all $\Psi-$weak upper gradients $g \in L^{\Psi}_{loc}(X)$ of u.*

Proposition 3. *Assume that the Young function Ψ is doubling. If $v, w \in D^{\Psi}_{loc}(X)$, then $g_v = g_w$ a.e. on $E := \{x \in X : v(x) = w(x)\}$.*

Proof. Fix $x_0 \in X$. For every $n \ge 1$ it follows, by Lemma 6 and Corollary 2, that there exists a set $E_n \subset E \cap B(x_0, n)$ of zero measure such that $g_v = g_w$ on $(E \cap B(x_0, n)) \setminus E_n$. For every $x \in E \setminus \bigcup\limits_{n=1}^{\infty} E_n$ there exists $k \ge 1$ such that $x \in (E \cap B(x_0, k)) \setminus E_k$, hence $g_v(x) = g_w(x)$. Since $\bigcup\limits_{n=1}^{\infty} E_n$ has zero measure, $g_v = g_w$ a.e. on E. ∎

3. A comparison principle for the solutions of obstacle problems

We recall the definition the obstacle problem in Orlicz-Sobolev spaces and an existence result for this problem.

Let Ψ be a doubling Young function and let $\Omega \subset X$ be a bounded nonempty open set. For $u \in N^{1,\Psi}(\Omega)$ denote by g_u a minimal $\Psi-$weak upper gradient of u in Ω. The existence of a minimal $\Psi-$weak upper gradient, which is determined up to sets of zero measure, is guaranteed by Proposition 1.

The obstacle problem's requirement is to minimize the energy integral $I(u, \Omega) = \int\limits_{\Omega} \Psi(g_u) d\mu$ among the Orlicz-Sobolev functions $u \in N^{1,\Psi}(\Omega)$ that have given boundary values $\beta \in N^{1,\Psi}(\Omega)$ and lie above an obstacle function $\omega : \Omega \to \overline{\mathbb{R}}$ a.e.

Note that $I(u, \Omega) < \infty$ whenever $u \in N^{1,\Psi}(\Omega)$, since $g_u \in L^{\Psi}(\Omega)$ and the Young function Ψ is doubling.

The set of admissible functions for the obstacle problem is

$$K_{\omega,\beta}(\Omega) = \{v \in N^{1,\Psi}(\Omega) : v - \beta \in N^{1,\Psi}_0(\Omega),\ v \ge \omega \ a.e.\}.$$

If $Cap_{\Psi}(X \setminus \Omega) = 0$, then $N^{1,\Psi}_0(\Omega) = N^{1,\Psi}(\Omega)$, therefore the condition $v - \beta \in N^{1,\Psi}_0(\Omega)$ becomes void. One defines obstacle problems assuming that $\Omega \subset X$ is a bounded nonempty open set such that $Cap_\Psi(X \setminus \Omega) > 0$. Note that $\mu(X \setminus \Omega) > 0$ implies $Cap_\Psi(X \setminus \Omega) > 0$.

Definition 4. *A function $u \in K_{\omega,\beta}(\Omega)$ is said to be a solution to the $K_{\omega,\beta}(\Omega)-$obstacle problem if $\int_\Omega \Psi(g_u)d\mu \leq \int_\Omega \Psi(g_v)d\mu$, for every $v \in K_{\omega,\beta}(\Omega)$.*

It was proved in [22] that the obstacle problem stated above has a unique solution under some assumptions that guarantee that a $(\Psi, \Psi)-$ Poincaré inequality for functions in $N_0^{1,\Psi}(\Omega)$ holds.

Theorem 1. [22, Theorem 4]*Let X be a proper metric space, equipped with a doubling measure, supporting a weak $(1, \Phi)-$Poincaré inequality for some strictly increasing Young function Φ. Assume that Ψ is a doubling $N-$ function and that $\Psi \circ \Phi^{-1}$ is an $N-$function satisfying a ∇_2-condition. Let $\Omega \subset X$ be a bounded nonempty open set such that $diam(\Omega) < diam(X)/3$. If $K_{\omega,\beta}(\Omega)$ is nonempty, then there exists a solution to the $K_{\omega,\beta}(\Omega)-$obstacle problem. Moreover, if Ψ is also strictly convex, then the solution to this obstacle problem is unique.*

The uniqueness of the solution means that, whenever u_1, u_2 are solutions of the $K_{\omega,\beta}(\Omega)-$obstacle problem, the functions u_1, u_2 generate the same equivalence class of $N^{1,\Psi}(\Omega)$, in particular $u_1 = u_2$ $\Psi-$q.e. in Ω.

Theorem 1 gives a partial extension of Theorem 3.2 proved by Kinnunen and Martio in [16], in the case $\Psi(t) = t^p$. The word "partial" refers to the fact that in Theorem 1 the more general assumption $\mu(X \setminus \Omega) > 0$ used in [16] is replaced by the assumption $diam(\Omega) < diam(X)/3$, needed in the proof of a $(\Psi, \Psi)-$Poincaré inequality for functions in $N_0^{1,\Psi}(\Omega)$ [22, Theorem 2]. Note that in [16] it was assumed that the metric measure space supports a weak $(1, q)-$Poincaré inequality for some $1 < q < p$, but later it turned out that this follows from the apparently weaker assumption that the space supports a weak $(1, p)-$Poincaré inequality for some $p > 1$ [15].

A variant of the above obstacle problem was studied in [3]. Replacing the condition $v \geq \omega$ a.e. in the definition of the admissible set $K_{\omega,\beta}(\Omega)$ by the stronger condition $v \geq \omega$ $\Psi-$q.e., the admissible set turns into

$$\widetilde{K}_{\omega,\beta}(\Omega) = \{v \in N^{1,\Psi}(\Omega) : v - \beta \in N_0^{1,\Psi}(\Omega),\, v \geq \omega\ \Psi - \text{q.e.}\}.$$

A function $u \in \widetilde{K}_{\omega,\beta}(\Omega)$ is said to be a solution to the $\widetilde{K}_{\omega,\beta}(\Omega)-$obstacle problem if $\int_\Omega \Psi(g_u)d\mu \leq \int_\Omega \Psi(g_v)d\mu$ for every $v \in \widetilde{K}_{\omega,\beta}(\Omega)$.

Note that $\widetilde{K}_{\omega,\beta}(\Omega) \subset K_{\omega,\beta}(\Omega)$, since zero capacity sets have zero measure. If $u \in \widetilde{K}_{\omega,\beta}(\Omega)$ is a solution to the $K_{\omega,\beta}(\Omega)-$obstacle problem, then u is also a solution to the $\widetilde{K}_{\omega,\beta}(\Omega)-$obstacle problem.

An analysis of the proof of Theorem 1 shows that its assumptions actually ensure the existence of a solution of the $\widetilde{K}_{\omega,\beta}(\Omega)-$obstacle problem, if $\widetilde{K}_{\omega,\beta}(\Omega)$ is non-empty, as we will describe in the following.

Let $I := \inf\left\{I(v,\Omega) : v \in \widetilde{K}_{\omega,\beta}(\Omega))\right\}$. Consider a minimizing sequence $(u_i)_{i\geq 1}$ in $\widetilde{K}_{\omega,\beta}(\Omega)$, i.e. $I = \lim\limits_{i\to\infty} I(g_i,\Omega)$, where $g_i := g_{u_i}$ for every $i \geq 1$. Using the $(\Psi,\Psi)-$Poincaré inequality as in the proof of [22, Theorem 2], we may assume, passing to some subsequences, that $(u_i)_{i\geq 1}$ and $(g_i)_{i\geq 1}$ are weakly convergent in $L^\Psi(X)$, to u and g, respectively, where $g \geq 0$. By a consequence of Mazur's lemma, [27, Theorem 4.17], there are sequences $(\overline{u_j})_{j\geq 1}$ and $(\overline{g_j})_{j\geq 1}$ of convex combinations

$$\overline{u_j} = \sum_{k=j}^{n_j} \lambda_{kj} u_k,\ \overline{g_j} = \sum_{k=j}^{n_j} \lambda_{kj} g_k,$$

such that $\overline{u_j} \to u$ and $\overline{g_j} \to g$ in $L^\Psi(X)$ as $j \to \infty$. It is clear that $\overline{u_j} \in \widetilde{K}_{\omega,\beta}(\Omega)$ and $\overline{g_j}$ is a $\Psi-$weak upper gradient of $\overline{u_j}$, for each $j \geq 1$. Moreover, the convexity of Ψ and the minimality of I imply $\lim\limits_{j\to\infty} \int_\Omega \Psi\left(\overline{g_j}\right) d\mu = I$. In Orlicz spaces norm convergence implies $\Psi-$mean convergence, hence $I = \int_\Omega \Psi(g)\, d\mu$.

Let $\widetilde{u} = \limsup\limits_{j\to\infty} \overline{u_j}$ and $\widehat{u} = \liminf\limits_{j\to\infty} \overline{u_j}$. As in the proof of [3, Proposition 2.3], it follows that g is a $\Psi-$weak upper gradient of $\widetilde{u}$ and of $\widehat{u}$. Then $\widetilde{u}$, $\widehat{u} \in \widetilde{N}^{1,\Psi}(\Omega)$. On the other hand, since there exists a subsequence of $(\overline{u_j})_{j\geq 1}$ that converges a.e. to u, we see that $\widetilde{u} = u = \widehat{u}$ a.e. Having $\widetilde{u}$, $\widehat{u} \in \widetilde{N}^{1,\Psi}(\Omega)$ and $\widetilde{u} = \widehat{u}$ a.e., it follows that $\widetilde{u}$ and $\widehat{u}$ represent the same equivalence class in $N^{1,\Psi}(\Omega)$, hence $\widetilde{u} = \widehat{u}$ $\Psi-$q.e.

Considering a subsequence $(\overline{u_{j_m}})_{m\geq 1}$ such that $\widetilde{u} = \lim\limits_{m\to\infty} \overline{u_{j_m}}$, we see that $\widetilde{u} \geq \omega$ $\Psi-$q.e. As in the proof of [22, Theorem 2], we obtain $\widetilde{u} - \beta \in N_0^{1,\Psi}(\Omega)$. Then $\widetilde{u} \in \widetilde{K}_{\omega,\beta}(\Omega)$.

Then $I \leq \int\limits_\Omega \Psi(g_{\widetilde{u}})\, d\mu \leq \int\limits_\Omega \Psi(g)\, d\mu = I$. We proved that $\widetilde{u}$ is a solution of the $\widetilde{K}_{\omega,\beta}(\Omega)-$obstacle problem. Similarly, $\widehat{u}$ is a solution of the $\widetilde{K}_{\omega,\beta}(\Omega)-$obstacle problem.

The uniqueness in $N^{1,\Psi}(\Omega)$ of the solution for Ψ strictly convex follows as in the proof of [22, Theorem 2].

In the particular case when the obstacle ω belongs to $N^{1,\Psi}(X)$, the set $\widetilde{K}_{\omega,\beta}(\Omega)$ is nonempty if and only if $(\omega-\beta)_+ \in N_0^{1,\Psi}(\Omega)$. For $\Psi(t) = t^p$ with $1 < p < \infty$ this was proved in [3, Proposition 7.4].

Proposition 4. *Assume that Ω is a bounded nonempty open set in the metric measure space X, such that $Cap_\Psi(X \setminus \Omega) > 0$. Let $\beta, \omega \in N^{1,\Psi}(\Omega)$. Then $\widetilde{K}_{\omega,\beta}(\Omega)$ is nonempty if and only if $(\omega-\beta)_+ \in N_0^{1,\Psi}(\Omega)$.*

Proof. Note that $(\omega-\beta)_+ = \max\{\omega,\beta\} - \beta$.

Necessity. Let $u \in \widetilde{K}_{\omega,\beta}(\Omega)$. Then $0 \leq (\omega-\beta)_+ \leq (u-\beta)_+$ $\Psi-$q.e. This inequality and $(u-\beta)_+ \in N_0^{1,\Psi}(\Omega)$ imply $(\omega-\beta)_+ \in N_0^{1,\Psi}(\Omega)$, according to Lemma 4.

Sufficiency. Define $v := \max\{\omega,\beta\}$. Since $\beta,\omega \in N^{1,\Psi}(\Omega)$, we have $v \in N^{1,\Psi}(\Omega)$. Clearly, $v - \beta = (\omega-\beta)_+ \in N_0^{1,\Psi}(\Omega)$ and $v \geq \omega$ on Ω. Then $v \in \widetilde{K}_{\omega,\beta}(\Omega)$. ∎

Assume that $1 < p < \infty$, that X is a doubling metric measure space supporting a $(1,p)-$Poincaré inequality and $\Omega \subset X$ is a bounded nonempty open set such that $Cap_p(X \setminus \Omega) > 0$. Let $\beta \in N^{1,p}(\Omega)$ and $\omega : \Omega \to \overline{\mathbb{R}}$. Under these assumptions, it was proved in [3, Theorem 7.3] that $\widetilde{K}_{\omega,\beta}(\Omega)$ is nonempty if and only if the Choquet integral $\int\limits_\Omega (\omega-\beta)_+^p \, d\, cap_p(\cdot,\Omega)$ is finite.

We do not know if there is an analogue for the above Adams' criterion in the setting of Orlicz-Sobolev spaces on metric measure spaces.

We prove a comparison principle for the solutions of a $K_{\omega,\beta}(\Omega)-$obstacle problem, assuming that every such problem has an unique solution in $N^{1,\Psi}(\Omega)$. This holds under the assumptions of Theorem 1.

Proposition 5. *Assume that every $K_{\omega,\beta}(\Omega)-$obstacle problem has an unique solution in $N^{1,\Psi}(\Omega)$, whenever $K_{\omega,\beta}(\Omega)$ is nonempty.*

For $j=1,2$, let $\omega_j:\Omega\to\overline{\mathbb{R}}$ and $\beta_j\in N^{1,\Psi}(\Omega)$ be such that $K_{\omega_j,\beta_j}(\Omega)$ is nonempty and let u_j be a solution of the $K_{\omega_j,\beta_j}(\Omega)-$obstacle problem. If $\omega_1\le\omega_2$ a.e. in Ω and $(\beta_1-\beta_2)_+\in N_0^{1,\Psi}(\Omega)$, then $u_1\le u_2$ $\Psi-$q.e. in Ω.

Proof. For each $f\in N^{1,\Psi}(\Omega)$ we will denote by g_f a minimal $\Psi-$weak upper gradient of f.

Let $u=\min\{u_1,u_2\}$. Then $u\in N^{1,\Psi}(\Omega)$. We will prove that $u=u_1$ $\Psi-$q.e. in Ω.

First we show that $u\in K_{\omega_1,\beta_1}(\Omega)$. Clearly, $u\ge\omega_1$ a.e. in Ω.

Define $h:=(u_1-\beta_1)-(u_2-\beta_2)$. Note that $h\in N_0^{1,\Psi}(\Omega)$. We have

$$\begin{aligned} u-\beta_1 &= \min\{u_1-\beta_1,(u_2-\beta_2)+(\beta_2-\beta_1)\}\\ &= (u_2-\beta_2)+\min\{h,\beta_2-\beta_1\}, \end{aligned}$$

hence $u-\beta_1\in N_0^{1,\Psi}(\Omega)$. It follows that $u\in K_{\omega_1,\beta_1}(\Omega)$.

Similarly, for $v=\max\{u_1,u_2\}\in N^{1,\Psi}(\Omega)$ we show that $v\in K_{\omega_2,\beta_2}(\Omega)$. Obviously, $v\ge\omega_2$ a.e. in Ω. We have

$$\begin{aligned} v-\beta_2 &= \max\{u_1-\beta_1,(u_2-\beta_2)+(\beta_2-\beta_1)\}\\ &= (u_2-\beta_2)+\max\{h,\beta_2-\beta_1\}, \end{aligned}$$

hence $v-\beta_2\in N_0^{1,\Psi}(\Omega)$. It follows that $v\in K_{\omega_2,\beta_2}(\Omega)$.

Next we compare $\int\limits_\Omega\Psi(g_u)d\mu$ and $\int\limits_\Omega\Psi(g_{u_1})d\mu$.

Let $A:=\{x\in\Omega:u_1(x)>u_2(x)\}$. The set A is measurable. Since $u(x)=u_2(x)$ for $x\in A$ and $u(x)=u_1(x)$ for $x\in\Omega\setminus A$, the function $\widetilde{g}_u=g_{u_2}\chi_A+g_{u_1}\chi_{\Omega\setminus A}$ *is a* $\Psi-$weak upper gradient of u. Similarly, since $v(x)=u_1(x)$ for $x\in A$ and $v(x)=u_2(x)$ for $x\in\Omega\setminus A$, the function $\widetilde{g}_v=g_{u_1}\chi_A+g_{u_2}\chi_{\Omega\setminus A}$ *is a* $\Psi-$weak upper gradient of v, by Lemma 5.

Since u_2 is a solution of the K_{ω_2,β_2}-obstacle problem and $v\in K_{\omega_2,\beta_2}(\Omega)$, we have

$$\int\limits_\Omega\Psi(g_{u_2})d\mu\le\int\limits_\Omega\Psi(g_v)d\mu\le\int\limits_\Omega\Psi(\widetilde{g}_v)d\mu=\int\limits_A\Psi(g_{u_1})d\mu+\int\limits_{\Omega\setminus A}\Psi(g_{u_2})d\mu.$$

The above inequalities imply

$$\int\limits_A\Psi(g_{u_2})d\mu\le\int\limits_A\Psi(g_{u_1})d\mu,$$

hence

$$\int_{\Omega} \Psi(g_u)d\mu \leq \int_{\Omega} \Psi(\widetilde{g}_u)d\mu = \tag{3.1}$$
$$\int_{A} \Psi(g_{u_2})d\mu + \int_{\Omega\setminus A} \Psi(g_{u_1})d\mu \leq \int_{\Omega} \Psi(g_{u_1})d\mu.$$

Since u_1 is a solution of the $K_{\omega_1,\beta_1}(\Omega)$-obstacle problem and $u \in K_{\omega_1,\beta_1}(\Omega)$, inequality (3.1) shows that u is also a solution of the $K_{\omega_1,\beta_1}(\Omega)$-obstacle problem.

By the uniqueness property for the solution of an obstacle problem, it follows that $u = u_1$ $\Psi-$q.e. in Ω. ∎

Remark 6. *Replacing in the statement and in the proof of Proposition 5 inequalities a.e. by inequalities $\Psi-$q.e. we obtain a comparison principle for $\widetilde{K}_{\omega,\beta}(\Omega)-$obstacle problem, extending* [3, Lemma 7.6].

4. Superminimizers

In the following we assume that Ψ is a doubling Young function, in order to ensure the existence of minimal $\Psi-$weak upper gradient $g_u \in L^{\Psi}_{loc}(\Omega)$ for each functions $u \in N^{1,\Psi}_{loc}(\Omega)$, see Proposition 2.

Given a doubling Young function Ψ and a nonempty open set $\Omega \subset X$, we want to find functions $u \in N^{1,\Psi}_{loc}(\Omega)$ that locally minimize the energy integral $I(v, D) = \int\limits_{D} \Psi(g_v)\, d\mu$ for all nonempty open sets $D \subset\subset \Omega$ among all functions $v \in u + N^{1,\Psi}_0(\Omega)$ or at least among all functions $v \in u + Lip_c(\Omega)$.

The case $\Psi(t) = t^p$ was thoroughly studied in several papers, such as [8], [26], [18], [16], [2], see the monograph [3, Chapter 7].

If $u \in N^{1,\Psi}_{loc}(\Omega)$ and $v \in u + N^{1,\Psi}_0(\Omega)$, then $g_v \in L^{\Psi}(D)$ for every nonempty open set $D \subset\subset \Omega$, hence $I(v, D) < \infty$ by the doubling property of the Young function Ψ.

Let $Q \geq 1$ be a number.

Definition 5. *A function $u \in N^{1,\Psi}_{loc}(\Omega)$ is a $Q-$*quasiminimizer *in Ω if for all $\varphi \in Lip_c(\Omega)$ we have*

$$\int_{\varphi\neq 0} \Psi(g_u)\, d\mu \leq Q \int_{\varphi\neq 0} \Psi(g_{u+\varphi})\, d\mu. \tag{4.1}$$

Definition 6. *A function* $u \in N^{1,\Psi}_{loc}(\Omega)$ *is a* $Q-$quasisuperminimizer *(a* $Q-$quasisubminimizer*) in* Ω *if the inequality (4.1) holds for all* $\varphi \in Lip_c(\Omega)$ *with* $\varphi \geq 0$ *(respectively, with* $\varphi \leq 0$*).*

Definition 7. *A function* $u \in N^{1,\Psi}_{loc}(\Omega)$ *is*

1) a $Q-$*quasiminimizer* in a strong sense *in* Ω *if the inequality (4.1) holds for all* $\varphi \in N^{1,\Psi}_0(\Omega)$*;*

2) a $Q-$*quasisuperminimizer (a* $Q-$*quasisubminimizer)* in a strong sense *in* Ω *if the inequality holds (4.1) holds for all* $\varphi \in N^{1,\Psi}_0(\Omega)$ *with* $\varphi \geq 0$ *(respectively, with* $\varphi \leq 0$*).*

When we take $Q = 1$ in the above definitions, the prefix "$Q-$*quasi*" will be omitted, so that we talk about *minimimizers, superminimizers, subminimizers.*

We will show below that the class of $Q-$quasiminimizers is the intersection between the class of $Q-$quasisuperminimizers and the class of $Q-$quasisubminimizers. Note that u is a $Q-$quasisubminimizer (in a strong sense) if and only if $(-u)$ is a $Q-$quasisuperminimizer (in a strong sense). Therefore, it suffices to study $Q-$quasisuperminimizers.

Remark 7. *Note that we obtain an equivalent definition of* $Q-$*quasisuper-minimizers (in a strong sense) by replacing the condition "*$\varphi \geq 0$*" by "*$\varphi \geq 0$ *a.e.", see* [3, Remark 7.10]. *Indeed, if a function* $\varphi \in N^{1,\Psi}(\Omega)$ *is assumed to be nonnegative a.e., then* $\max\{\varphi, 0\} \in N^{1,\Psi}(\Omega)$ *by the lattice property of Orlicz-Sobolev spaces and* $\varphi = \max\{\varphi, 0\}$ *a.e., therefore* $\varphi = \max\{\varphi, 0\} \geq 0$ $\Psi-$*q.e. in* Ω*. Moreover, replacing* φ *by a nonnegative representative of its equivalence class in* $N^{1,\Psi}(\Omega)$ *the integrals involved in (4.1) do not change.*

Lemma 7. *Let* $u \in N^{1,\Psi}_{loc}(\Omega)$*. Then* u *is a* $Q-$*quasiminimizer (in a strong sense) if and only if* u *is both a* $Q-$*quasisuperminimizer and a* $Q-$*quasisubminimizer (in a strong sense).*

Proof. The necessity is obvious.

Sufficiency: Let $\varphi \in Lip_c(\Omega)$ ($\varphi \in N^{1,\Psi}_0(\Omega)$). Then $\varphi_+, \varphi_- \in Lip_c(\Omega)$ (respectively, $\varphi_+, \varphi_- \in N^{1,\Psi}_0(\Omega)$).

As u is a $Q-$quasisuperminimizer (in a strong sense),
$$\int_{\varphi_+ \neq 0} \Psi\left(g_u\right) d\mu \leq Q \int_{\varphi_+ \neq 0} \Psi\left(g_{u+\varphi_+}\right) d\mu = Q \int_{\varphi > 0} \Psi\left(g_{u+\varphi}\right) d\mu.$$
As u is a $Q-$quasisubminimizer (in a strong sense),
$$\int_{\varphi_- \neq 0} \Psi\left(g_u\right) d\mu \leq Q \int_{\varphi_- \neq 0} \Psi\left(g_{u-\varphi_-}\right) d\mu = Q \int_{\varphi < 0} \Psi\left(g_{u+\varphi}\right) d\mu..$$
Adding the above inequalities we get (4.1). ∎

In the case $\Psi(t) = t^p$, $1 < p < \infty$, several characterizations of $Q-$quasisuperminimizers have been proved in [2, Proposition 3.2], see also [3, Proposition 7.9] for the special case $Q = 1$. One of these characterizations proves the equivalence between the above definition of $Q-$quasisuperminimizers involving (4.1) and the definition of $Q-$quasisuperminimizers introduced in [17] (called here *$Q-$quasisuper-minimizers in a strong sense*).

We extend below some characterizations of superminimizers.

Proposition 6. *Let $u \in N^{1,\Psi}_{loc}(\Omega)$. Then u is a minimizer (superminimizer, subminimizer) in Ω if and only if*

$$\int_{supp\varphi} \Psi(g_u)\, d\mu \leq \int_{supp\varphi} \Psi(g_{u+\varphi})\, d\mu \tag{4.2}$$

for all $\varphi \in Lip_c(\Omega)$ (that in addition are nonnegative, respectively nonpositive).

Proof. Let $\varphi \in Lip_c(\Omega)$. Denote $A = \{x \in \mathrm{supp}\varphi : \varphi(x) = 0\}$. Since $\mathrm{supp}\varphi$ is compact, the integrals in (4.2) are finite.

For all $v \in N^{1,\Psi}_{loc}(\Omega)$ we have

$$\int_{\mathrm{supp}\varphi} \Psi(g_v)\, d\mu = \int_{\varphi\neq 0} \Psi(g_v)\, d\mu + \int_A \Psi(g_v)\, d\mu.$$

By Proposition 3, $g_u = g_{u+\varphi}$ a.e. in A, hence $\int_A \Psi(g_u)\, d\mu = \int_A \Psi(g_{u+\varphi})\, d\mu < \infty$. It follows that

$$\begin{aligned}\int_{\mathrm{supp}\varphi} \Psi(g_u)\, d\mu &\leq \int_{\mathrm{supp}\varphi} \Psi(g_{u+\varphi})\, d\mu \text{ if and only if}\\ \int_{\varphi\neq 0} \Psi(g_u)\, d\mu &\leq \int_{\varphi\neq 0} \Psi(g_{u+\varphi})\, d\mu.\end{aligned}$$

∎

Now compare the definition of minimizers (superminimizers, subminimizers) used here with the extension of the definition introduced by Kinnunen and Martio [16] in the case $\Psi(t) = t^p$.

Proposition 7. *Let $u \in N^{1,\Psi}_{loc}(\Omega)$. Then u is a minimizer (superminimizer, subminimizer) in Ω if and only if*

$$\int_D \Psi(g_u)\, d\mu \le \int_D \Psi(g_{u+\varphi})\, d\mu \tag{4.3}$$

holds for all open nonempty sets $D \subset\subset \Omega$ and all $\varphi \in Lip_c(D)$ (that in addition are nonnegative, respectively nonpositive).

Proof. It suffices to study the case of superminimizers.

Sufficiency: Let $\varphi \in Lip_c(\Omega)$. It suffices to assume that $\varphi \ge 0$, but φ is not identically zero. Let $D = \{x \in \Omega : \varphi(x) \ne 0\}$. Then D is open, nonempty and $D \subset\subset \Omega$ and (4.3) turns into (4.1).

Necessity: Fix an open nonempty set $D \subset\subset \Omega$ and a nonnegative function $\varphi \in Lip_c(D)$. Then we also have $\varphi \in Lip_c(\Omega)$. Moreover, $\{x \in \Omega : \varphi(x) \ne 0\} \subset D$, hence $D = \{x \in \Omega : \varphi(x) \ne 0\} \cup A$, where $A := \{x \in D : \varphi(x) = 0\}$.

Since u is a superminimizer in Ω, we have $\int\limits_{\varphi\ne 0} \Psi(g_u)\, d\mu \le \int\limits_{\varphi\ne 0} \Psi(g_{u+\varphi})\, d\mu$.

As $g_u = g_{u+\varphi}$ a.e. in A, we have $\int\limits_A \Psi(g_u)\, d\mu = \int\limits_A \Psi(g_{u+\varphi})\, d\mu =: I(u, A)$. Adding $I(u, A)$ to both members of the above inequality we get (4.3). ∎

If Ω is bounded and X is proper it is enough to test the minimizing property from the definition introduced by Kinnunen and Martio using (4.3) only for $D = \Omega$, as it is shown below.

Proposition 8. *Let $u \in N^{1,\Psi}_{loc}(\Omega)$. Assume that*

$$\int_D \Psi(g_u)\, d\mu \le \int_D \Psi(g_{u+\varphi})\, d\mu \tag{4.4}$$

for all open nonempty sets $D \subset\subset \Omega$ and all $\varphi \in Lip_c(D)$ (that in addition are nonnegative, respectively nonpositive). If Ω is bounded and X is proper, then (4.4) also holds for $D = \Omega$ and $\varphi \in Lip_c(\Omega)$.

Proof. Let $\varphi \in Lip_c(\Omega)$. It suffices to assume that $\varphi \ge 0$, but φ is not identically zero.

For $n \ge 1$, define $D_n := \left\{x \in \Omega : d(x, \partial\Omega) > \frac{1}{n}\right\}$. The set D_n is nonempty for n large enough. By the continuity of the function $x \mapsto d(x, \partial\Omega)$, the set D_n is always open. Since $\overline{D_n} \subset$

$\left\{x \in \overline{\Omega} : d\left(x, \partial\Omega\right) \geq \frac{1}{n}\right\} \subset \Omega$ and we have Ω bounded and X proper, it follows that $\overline{D_n}$ is compact, hence $D_n \subset\subset \Omega$.

Note that for large n we have $\text{supp}\varphi \subset D_n$, hence $\varphi \in Lip_c\left(D_n\right)$.

By Proposition 3, $g_u = g_{u+\varphi}$ a.e. in $\{x \in \Omega : \varphi\left(x\right) = 0\}$. Then we have

$$\int\limits_{D_n} \Psi\left(g_u\right) d\mu = \int\limits_{D_n \cap \{\varphi \neq 0\}} \Psi\left(g_u\right) d\mu + \int\limits_{D_n \cap \{\varphi = 0\}} \Psi\left(g_u\right) d\mu$$

and

$$\int\limits_{D_n} \Psi\left(g_{u+\varphi}\right) d\mu = \int\limits_{D \cap \{\varphi \neq 0\}} \Psi\left(g_{u+\varphi}\right) d\mu + \int\limits_{D \cap \{\varphi = 0\}} \Psi\left(g_u\right) d\mu.$$

Using (4.4) for $D = D_n$ and the fact that $\int\limits_{D_n \cap \{\varphi = 0\}} \Psi\left(g_u\right) d\mu < \infty$ it follows that

$$\int\limits_{D_n \cap \{\varphi \neq 0\}} \Psi\left(g_u\right) d\mu d\mu \leq \int\limits_{D_n \cap \{\varphi \neq 0\}} \Psi\left(g_u\right) d\mu. \tag{4.5}$$

Since $D_n \subset D_{n+1}$ for all $n \geq 1$ and $\Omega = \bigcup\limits_{n=1}^{\infty} D_n$, letting $n \to \infty$ in (4.5) we get (4.3) for $D = \Omega$. ∎

Since we have $Lip_c\left(\Omega\right) \subset N_0^{1,\Psi}\left(\Omega\right)$, every $Q-$quasiminimizer in a strong sense is also a $Q-$quasiminimizer. The analogous statements hold for $Q-$quasisuperminimizers and $Q-$quasubsiminimizers. It is very important to point out that the converse holds in the case where $\Psi\left(t\right) = t^p$, $1 < p < \infty$, as it is proved in [2, Proposition 3.2], using the density of compactly supported Lipschitz function in Newtonian spaces with zero boundary values under some usual assumptions on the metric measure space X. The proofs of implications $(e) \Rightarrow (a)$ and $(c) \Rightarrow (f)$ in [2, Proposition 3.2], that use this density property are not easy to extend to our case.

We recall some results on the density of compactly supported Lipschitz functions in an Orlicz-Sobolev space with zero boundary value, [20, Theorem 1] and [20, Corollary 2].

Lemma 8. *(*[20, Theorem 1]*) Let X be a proper metric measure space and let Ψ be a doubling $N-$function. If locally Lipschitz functions are dense in $N^{1,\Psi}\left(X\right)$, then $Lip_c\left(\Omega\right)$ is a dense subset of $N_0^{1,\Psi}\left(\Omega\right)$, for every nonempty open set $\Omega \subset X$.*

Remark 8. *The proof of the above result shows that every* nonnegative *function in* $N_0^{1,\Psi}(\Omega)$ *is the limit in* $N^{1,\Psi}(\Omega)$ *of a sequence of* nonnegative *functions belonging to* $Lip_c(\Omega)$.

By [27, Theorem 6.17], if Ψ is a doubling Young function and if the metric measure space X is doubling and supports a $(1,\Psi)-$Poincaré inequality, then Lipschitz functions are dense in $N^{1,\Psi}(X)$, both in norm and in Lusin's sense. The following sufficient conditions for the density of $Lip_c(\Omega)$ in $N_0^{1,\Psi}(\Omega)$ follows.

Corollary 3. *(*[20, Corollary 2]*) Assume that* Ψ *is a doubling* $N-$*function and that the metric measure space* X *is proper, doubling and supports a* $(1,\Psi)-$*Poincaré inequality. Then* $Lip_c(\Omega)$ *is a dense subset of* $N_0^{1,\Psi}(\Omega)$, *for every open set* $\Omega\subset X$.

Proposition 9. *Let* X *be a proper metric measure space and let* Ψ *be a doubling* $N-$*function, such that locally Lipschitz functions are dense in* $N^{1,\Psi}(X)$. *Assume that* $u\in N_{loc}^{1,\Psi}(\Omega)$. *If* u *is a minimizer (superminimizer, subminimizer) in* Ω, *then* u *is a* $Q-$*quasiminimizer (*$Q-$*quasisuperminimizer,* $Q-$*quasisubminimizer) in a strong sense in* Ω, *where* Q *depends only on the doubling constant of* Ψ.

Proof. It suffices to assume that u is a superminimizer.

Let $\varphi\in N_0^{1,\Psi}(\Omega)$ be nonnegative. Let $0<\varepsilon<1$. By Lemma 8 and Remark 8, there exists a nonnegative function $f\in Lip_c(\Omega)$ such that $\|f-\varphi\|_{L^\Psi(\Omega)}<\varepsilon$.

By [11, Lemma 3.8.4], the following inequalities between the $\Psi-$integral and the Luxemburg norm hold:

$\int_\Omega \Psi(|v|)\,d\mu\le\|v\|_{L^\Psi(\Omega)}$ if $\|v\|_{L^\Psi(\Omega)}\le 1$, respectively $\int_\Omega \Psi(|v|)\,d\mu\ge\|v\|_{L^\Psi(\Omega)}$ if $\|v\|_{L^\Psi(\Omega)}\ge 1$. Since $\|f-\varphi\|_{L^\Psi(\Omega)}<\varepsilon<1$, we have $\int_\Omega \Psi(|f-\varphi|)\,d\mu\le\|f-\varphi\|_{L^\Psi(\Omega)}<\varepsilon$.

As u is a superminimizer in a weak sense,

$\int_\Omega \Psi(g_u)\,d\mu\le\int_\Omega \Psi(g_{u+f})\,d\mu$.

Writing $u+f=u+\varphi+(f-\varphi)$, we see that $g_{u+f}\le g_{u+\varphi}+g_{f-\varphi}$ a.e.. Then, using the doubling constant C_Ψ of Ψ and the properties of monotonicity and convexity of Ψ, we get

$$\Psi(g_{u+f})\le\Psi(g_{u+\varphi}+g_{f-\varphi})\le\frac{1}{2}C_\Psi\left(\Psi(g_{u+\varphi})+\Psi(g_{f-\varphi})\right).$$

Then

$$\begin{aligned}\int_\Omega \Psi(g_u)\,d\mu &\le \int_\Omega \Psi(g_{u+f})\,d\mu \\ &\le \frac{1}{2}C_\Psi \int_\Omega \Psi(g_{u+\varphi})\,d\mu + \frac{1}{2}C_\Psi \int_\Omega \Psi\left(g_{f-\varphi}\right)d\mu,\end{aligned}$$

hence $\int_\Omega \Psi(g_u)\,d\mu < \frac{1}{2}C_\Psi \int_\Omega \Psi(g_{u+\varphi})\,d\mu + \frac{1}{2}C_\Psi\varepsilon$.

Letting ε tend to zero, we get

$$\int_\Omega \Psi(g_u)\,d\mu < \frac{1}{2}C_\Psi \int_\Omega \Psi(g_{u+\varphi})\,d\mu.$$

Since $0 \le \varphi \in N_0^{1,\Psi}(\Omega)$ is arbitrary, it follows that u is a $Q-$quasisuper-minimizer, where $Q := \frac{1}{2}C_\Psi$. ∎

We consider the problem of constructing new superminimizers from existing ones.

Lemma 9. *Consider an ascending sequence of open sets* $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega = \bigcup_{j=1}^{\infty} \Omega_j$ *and* $u \in N_{loc}^{1,\Psi}(\Omega)$. *Then u is a superminimizer in Ω if and only if u is a superminimizer in Ω_j for each $j \ge 1$.*

Proof. Necessity: Use $Lip_c(\Omega_j) \subset Lip_c(\Omega)$, where $j \ge 1$.

Sufficiency: Assume that u is a superminimizer in Ω_j for each $j \ge 1$. Then u is defined on $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$.

Let $\varphi \in Lip_c(\Omega)$ be nonnegative. Since $\mathrm{supp}\varphi$ is a compact subset of Ω and $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega = \bigcup_{j=1}^{\infty} \Omega_j$, there exists $i \ge 1$ such that $\mathrm{supp}\varphi \subset \Omega_i$, in particular $\{x \in \Omega : \varphi(x) \ne 0\} \subset \Omega_i$. Since u is a superminimizer in Ω_i we get (4.1).

It follows that u is a superminimizer in Ω. ∎

Theorem 2 (**Pasting superminimizers**). *Let $\Omega_1 \subset \Omega_2$ be open sets. Assume that u_j is a superminimizer in a strong sense in Ω_j, where $j = 1, 2$. Define*

$$u = \begin{cases} \min\{u_1, u_2\} & \text{in } \Omega_1 \\ u_2 & \text{in } \Omega_2 \setminus \Omega_1 \end{cases}.$$

If $u \in N_{loc}^{1,\Psi}(\Omega_2)$, then u is a superminimizer in Ω_2.

Proof. Let $\varphi \in Lip_c(\Omega_2)$ be nonnegative. We have to show that

$$\int_G \Psi(g_u)\,d\mu \le \int_G \Psi(g_v)\,d\mu, \tag{4.6}$$

where $G := \{x \in \Omega_2 : \varphi(x) > 0\}$ and $v := u + \varphi$. Note that $G \subset\subset \Omega_2$ is an open set.

Consider $M := \{x \in \Omega_2 : u_1(x) < u_2(x)\}$. Then

$$u = \begin{cases} u_1 \text{ in } \Omega_1 \cap M \\ u_2 \text{ in } (\Omega_2 \setminus \Omega_1) \cup (\Omega_1 \setminus M) \end{cases}.$$

Let $A := \{x \in \Omega_2 : u_2(x) < v(x)\}$ and $B := \{x \in \Omega_2 : u_2(x) < v(x)\}$. The above representation of u shows that

$$G = (B \cap \Omega_1 \cap M) \cup (A \cap ((\Omega_2 \setminus \Omega_1) \cup (\Omega_1 \setminus M))).$$

Note that $A \subset G$, since $u \le u_2$ in Ω_2.

Obviously, $B \cap \Omega_1 \cap M \subset G \cap \Omega_1 \cap M$. For every $x \in G \cap \Omega_1 \cap M$ we have $u(x) = u_1(x) < v(x)$, hence $x \in B$. It follows that $B \cap \Omega_1 \cap M = G \cap \Omega_1 \cap M$.

Denote $E := G \cap \Omega_1 \cap M$. We conclude that $G = E \cup A$.

Since $u = u_1\chi_E + u_2\chi_{G\setminus E}$, we have $g_u = g_{u_1}\chi_E + g_{u_2}\chi_{G\setminus E}$ a.e. in G, by Lemma 5. Noticing that $G \setminus E = A \setminus E$, it follows that

$$\int_G \Psi(g_u)\,d\mu = \int_E \Psi(g_{u_1})\,d\mu + \int_{A\setminus E} \Psi(g_{u_2})\,d\mu. \tag{4.7}$$

The following are equivalent: (a) $x \in E$; (b) $x \in \Omega_1$ and $u_1(x) < \min\{u_2(x), v(x)\}$; (c) $x \in \Omega_2$ and $u(x) < \min\{u_2(x), v(x)\}$.

Denote $m(x) := \min\{u_2(x), v(x)\}$.

Then $E = \{x \in \Omega_2 : u(x) < m(x)\}$.

As $E \subset G \subset\subset \Omega_2$, we have $(m-u) \in N^{1,\Psi}(E)$. But $(m-u)_+ = m - u_1$ on E and $m - u = 0$ in $\Omega_2 \setminus E$, hence $(m-u)_+ \in N_0^{1,\Psi}(E)$. Since u_1 is a superminimizer in a strong sense and $m = u_1 + (m-u)_+$ in E, it follows that

$$\int_E \Psi(g_{u_1})\,d\mu \le \int_E \Psi(g_m)\,d\mu. \tag{4.8}$$

But $m = u_2\chi_A + v\chi_{\Omega_2\setminus A}$, therefore $g_m = g_{u_2}\chi_A + g_v\chi_{\Omega_2\setminus A}$ a.e. in Ω_2. Then $g_m = g_{u_2}\chi_{E\cap A} + g_v\chi_{E\setminus A}$ a.e. in E, hence

$$\int_E \Psi(g_m)\,d\mu = \int_{E\cap A} \Psi(g_{u_2})\,d\mu + \int_{E\setminus A} \Psi(g_v)\,d\mu. \tag{4.9}$$

Using (4.7), (4.8) and (4.9), we obtain

$$\int_G \Psi(g_u)\,d\mu \le \int_{E\setminus A} \Psi(g_v)\,d\mu + \int_{E\cap A} \Psi(g_{u_2})\,d\mu + \int_{A\setminus E} \Psi(g_{u_2})\,d\mu,$$

i.e.

$$\int_G \Psi(g_u)\,d\mu \le \int_{E\setminus A} \Psi(g_v)\,d\mu + \int_A \Psi(g_{u_2})\,d\mu.$$

As $A \subset G \subset\subset \Omega_2$, we have $(v-u_2) \in N^{1,\Psi}(A)$. But $(v-u_2)_+ = v-u_2$ in A and $(v-u_2)_+ = 0$ in $\Omega_2 \setminus A$, therefore $(v-u_2)_+ \in N_0^{1,\Psi}(A)$. Since u_2 is a superminimizer in a strong sense and $v = u_2 + (v-u_2)_+$ in A, we have

$$\int_A \Psi(g_{u_2})\,d\mu \le \int_A \Psi(g_v)\,d\mu.$$

The latter two inequalities imply

$$\int_G \Psi(g_u)\,d\mu \le \int_{E\setminus A} \Psi(g_v)\,d\mu + \int_A \Psi(g_v)\,d\mu.$$

This is the required inequality (4.6), since A and $E \setminus A$ are disjoint and $A \cup (E \setminus A) = A \cup E = G$. ∎

Corollary 4. *If u_1 and u_2 are superminimizers in a strong sense in Ω, then $u = \min\{u_1, u_2\}$ is a superminimizer in Ω.*

Proof. Apply the above theorem with $\Omega_1 = \Omega_2 = \Omega$. ∎

5. Connections between superminimizers and obstacle problems

In the following we assume that Ψ is a doubling Young function and $\Omega \subset X$ is a nonempty open set.

Proposition 10. *Let $\Omega \subset X$ be a bounded nonempty open set. Assume that $\beta \in N^{1,\Psi}(\Omega)$ and $\omega : \Omega \to \overline{\mathbb{R}}$. If u is a solution of the $K_{\omega,\beta}(\Omega)-$obstacle problem or of the $\widetilde{K}_{\omega,\beta}(\Omega)-$obstacle problem, then u is a superminimizer in a strong sense.*

Proof. Let u be a solution of the $K_{\omega,\beta}(\Omega)-$obstacle problem (or of the $\widetilde{K}_{\omega,\beta}(\Omega)-$obstacle problem). Let $0 \le \varphi \in N_0^{1,\Psi}(\Omega)$. Then we have

$u+\varphi \in K_{\omega,\beta}(\Omega)$ (respectively, $u+\varphi \in \widetilde{K}_{\omega,\beta}(\Omega)$), hence

$$\int_{\Omega} \Psi\left(g_u\right) d\mu \leq \int_{\Omega} \Psi\left(g_{u+\varphi}\right) d\mu.$$

Let $A=\{x \in \Omega : \varphi(x)=0\}$. By Corollary 2, $g_u = g_{u+\varphi}$ a.e. in A. Substracting $\int_A \Psi\left(g_u\right) d\mu = \int_A \Psi\left(g_{u+\varphi}\right) d\mu < \infty$ from both sides of the above inequality, it follows that

$$\int_{\varphi\neq 0} \Psi\left(g_u\right) d\mu \leq \int_{\varphi\neq 0} \Psi\left(g_{u+\varphi}\right) d\mu.$$

∎

Proposition 11. *Let u be a superminimizer in a strong sense in Ω. Assume that $D \subset \Omega$ is a bounded nonempty open set with $Cap_{\Psi}\left(X \setminus D\right) > 0$ and $u \in N^{1,\Psi}(D)$. Then u is a solution of the $\widetilde{K}_{u,u}(D)-$obstacle problem.*

Proof. Obviously, $u \in \widetilde{K}_{u,u}(D)$. Let $v \in \widetilde{K}_{u,u}(D)$. We have to prove that $\int_D \Psi(g_u)d\mu \leq \int_D \Psi(g_v)d\mu$.

Denote $\varphi = v-u$. Then $\varphi \in N_0^{1,\Psi}(D)$ and $\varphi \geq 0$ $\Psi-$q.e.

Let $E=\{x \in D : \varphi(x) < 0\}$. By our assumption, $Cap_{\Psi}\left(E\right)=0$.

Define α on D by $\alpha(x)=\varphi(x)$ if $x \in D\setminus E$ and $\alpha(x)=0$ if $x \in E$. Then $0 \leq \alpha \in N_0^{1,\Psi}(D)$ and α represents the same equivalence class as φ in $N^{1,\Psi}(D)$.

Since u is a superminimizer in a strong sense in Ω, $\int_{\alpha\neq 0} \Psi\left(g_u\right) d\mu \leq \int_{\alpha\neq 0} \Psi\left(g_{u+\alpha}\right) d\mu$, hence $\int_D \Psi\left(g_u\right) d\mu \leq \int_D \Psi\left(g_{u+\alpha}\right) d\mu$.

We have $g_{u+\alpha} = g_v$ a.e. in $D \setminus E$, hence $g_{u+\alpha} = g_v$ a.e. in D. Then $\int_D \Psi(g_v)d\mu = \int_D \Psi\left(g_{u+\alpha}\right) d\mu \geq \int_D \Psi\left(g_u\right) d\mu$. ∎

Proposition 12. *Assume that either X is unbounded or $\Omega \neq X$. If $u \in N_{loc}^{1,\Psi}(\Omega)$ is a solution of $\widetilde{K}_{u,u}(D)-$obstacle problem for all nonempty open sets $D \subset\subset \Omega$, then u is a superminimizer in Ω.*

Proof. Let $0 \leq \varphi \in Lip_c(\Omega)$. Then $G=\{x \in \Omega : \varphi(x) > 0\}$ is an open set with $G \subset\subset \Omega$.

Since u is a solution of the $\widetilde{K}_{u,u}(G)-$obstacle problem and $0 \le \varphi \in N_0^{1,\Psi}(G)$, we have $\int_G \Psi(g_u)d\mu \le \int_G \Psi(g_{u+\varphi})d\mu$, i.e.

$$\int_{\varphi\neq 0} \Psi(g_u)d\mu \le \int_{\varphi\neq 0} \Psi(g_{u+\varphi})d\mu.$$

This proves that u is a superminimizer in Ω. ∎

In the case $\Psi(t) = t^p$, $1 < p < \infty$, the following equivalence is proved in [3]: assuming that either X is unbounded or $\Omega \neq X$, $u \in N_{loc}^{1,p}(\Omega)$ is a superminimizer in Ω if and only if u is a solution of the $\widetilde{K}_{u,u}(D)-$*obstacle problem for all* $D \subset\subset \Omega$. Recall that in this case the notions of *superminimizer* and *superminimizer in a strong sense* are equivalent.

Proposition 11 and the counterpart of the comparison principle, Proposition 5, discussed in Remark 6 imply the following extension of [3, Corollary 7.17], showing that a solution of an obstacle problem is the smallest superminimizer with the prescribed boundary values, which lies above the obstacle.

Corollary 5. *Assume that $\Omega \subset X$ is a bounded nonempty open set with $Cap_\Psi(X \setminus \Omega) > 0$, $\beta \in N^{1,\Psi}(\Omega)$ and $\omega : \Omega \to \overline{\mathbb{R}}$. If u is a solution of the $\widetilde{K}_{\omega,\beta}(\Omega)-$obstacle problem and if $v \in \widetilde{K}_{\omega,\beta}(\Omega)$ is a superminimizer in a strong sense in Ω, then $u \le v$ $\Psi-$q.e. in Ω.*

Proof. By Proposition 11, v is a solution of the $\widetilde{K}_{v,v}(\Omega)-$obstacle problem. But u is a solution of the $\widetilde{K}_{\omega,\beta}(\Omega)-$obstacle problem and as $\omega \le v$ $\Psi-$q.e. in Ω and $(\beta - v)_+ \in N_0^{1,\Psi}(\Omega)$. Then the counterpart for $\widetilde{K}_{\omega,\beta}(\Omega)-$obstacle problems of the comparison principle, Proposition 5, implies $u \le v$ $\Psi-$q.e. in Ω. ∎

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