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ON DUAL TOPOLOGIES FOR FUNCTION SPACES OVER $\mathcal{C}_{\mu,\nu}(Y,Z)$

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Abstract. Dual topologies for function space topologies between generalized topological spaces are defined. The point-open topology, compact-open topology and (μ,ν)-topology on $\mathcal{C}_{\mu,\nu}(Y,Z)$ are shown to be family-open. The notions of splittingness and admissibility for such spaces are introduced. It is proved that a topology on $\mathcal{C}_{\mu,\nu}(Y,Z)$ is splitting (respectively, admissible) if and only if its dual topology is splitting ((respectively, admissible). Similarly, a topology on $\mathcal{O}_Z(Y)$ is splitting ((respectively, admissible) if and only if its dual topology on $\mathcal{C}_{\mu,\nu}(Y,Z)$ is so.

1. Introduction

In the recent years, function space topologies have turned out to be an area of active research [2, 5, 6, 7, 8, 10, 11, 12]. While a unified theory of function spaces and hyperspaces is investigated in [6], dual topologies for function space topologies are discussed in [7]. Some open problems regarding function space topologies are presented in [8]. Usually continuous functions between topological spaces are considered for these studies. However, there are some weaker forms of continuous functions which are no less important than the continuous functions themselves.

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For example, semi-continuous mappings are increasingly being used in the field of stochastic analysis [2, 12], theory of optimization [5, 11], multifunction analysis [10, 12] etc. Due to this, there arises the need to investigate these functions in a systematic manner. In [3], Császár has provided an integrated study of some weaker forms of continuity such as semi-continuity, α-continuity, β-continuity etc. Investigation about their function space topologies was initiated in [9]. There several topologies on the continuous functions between generalized topologies were introduced and studied. Important properties like splittingness, admissibility etc. of these spaces have been studied in that paper. Topologies of spaces of semi-continuous mappings, α-continuous mappings, β-continuous mappings etc. are particular cases of this study. The present paper is a sequel to the studies carried out in [9]. Here, we introduce the concept of dual topologies for such function space topologies. We show that each topology on $\mathcal{C}_{\mu,\nu}(Y, Z)$, the class of continuous functions between two generalized topological spaces Y and Z, generates a dual topology on $\mathcal{O}_Z(Y)$, the family of inverse images of open sets of Z, and vice-versa. The notions of splittingness and admissibility for dual topologies are introduced. Using the concept of dual topology, it is shown that the point-open topology, compact-open topology and (μ, ν)-topology on $\mathcal{C}_{\mu,\nu}(Y, Z)$ are family-open. It is found that a topology on $\mathcal{C}_{\mu,\nu}(Y, Z)$ is splitting (resp. admissible) if and only if its dual topology is splitting (resp. admissible). Similarly, a topology on $\mathcal{O}_Z(Y)$ is splitting (resp. admissible) if and only if its dual topology on $\mathcal{C}_{\mu,\nu}(Y, Z)$ is so.

2. Preliminaries

Á. Császár has defined a generalized topology on a set X in the following way:

Definition 2.1. [3] Let X be a non empty set. A collection $\mathcal{G}$ of subsets of X is called a *generalized topology* (GT, in brief) on X if

(i) $\emptyset \in \mathcal{G}$,

(ii) $\mathcal{G}$ is closed under arbitrary union.

The members of $\mathcal{G}$ are called *generalized open sets* (g-open sets, in brief), their complements are called *generalized closed sets* (g-closed sets, in brief).

Definition 2.2. [3] Let (X, μ) and (Y, ν) be two $GTS's$. Then a map $f : X \to Y$ is said to be *(μ, ν)-continuous* if $f^{-1}(U) \in \mu$ for every $U \in \nu$.

With the help of generalized nets (g-nets, in brief), regular points and saddle points, some equivalent characterizations of (μ,ν)-continuity has been provided in [9, 17]:

Proposition 2.3. *Let (X,μ) and (Y,ν) be two GTS's and $f : X \to Y$ be any mapping. Then the following are equivalent:*
(i) (f is (μ,ν)-continuous;
(ii) inverse image of each generalized closed sets is closed;
(iii) for each regular point x and any neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$;
(iv) for each g-net $\{s_n\}_{n\in\mathcal{D}}$ converging to x, the image g-net $\{f(s_n)\}_{n\in\mathcal{D}}$ converges to $f(x)$;
(v) f is (μ,ν)-continuous at each $x \in X$, that is, for each neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Now onward we shall write (μ,ν)-*continuity* as simply *continuity* unless there is any ambiguity.
Let (X,τ_1) and (Y,τ_2) be two topological spaces. If $\sigma(\tau_1)$, $\alpha(\tau_1)$, $\beta(\tau_1)$ and $\pi(\tau_1)$ represent the families of semi-open [13], α-open [15], β-open[1] and pre-open[14] sets of (X,τ_1) respectively, then $(X,\sigma(\tau_1))$, $(X,\alpha(\tau_1))$, $(X,\beta(\tau_1))$ and $(X,\pi(\tau_1))$ are different generalized topologies on X. Then a function $f : X \to Y$ is (τ_1,τ_2)-semi-continuous if and only if it is $(\sigma(\tau_1),\tau_2)$-continuous. Similar is the case for α-continuous, β-continuous and *pre*-continuous mappings.
Let (Y,μ) and (Z,ν) be two generalized topological spaces and $\mathcal{C}_{\mu,\nu}(Y,Z)$ be the class of all continuous mappings form the GTS's Y to Z. In [9], several topologies have been defined for $\mathcal{C}_{\mu,\nu}(Y,Z)$. Some important amongst them are mentioned below:
The (μ,ν)-*topology* $\tau_{\mu,\nu}$ on $\mathcal{C}_{\mu,\nu}(Y,Z)$ is the topology for which the family of subbasic open sets are of the form

$$(U,V) = \{f \in \mathcal{C}_{\mu,\nu}(Y,Z) \mid f(U) \subseteq V\}$$

where $U \in \mu$ and $V \in \nu$.
Similarly, let (Y,μ) and (Z,ν) be two GTS's with $Z \in \nu$. Then the *point-open topology* $\tau^p_{\mu,\nu}$ on $\mathcal{C}_{\mu,\nu}(Y,Z)$ is the topology for which the family of subbasic open sets are of the form

$$(\{y\},V) = \{f \in \mathcal{C}_{\mu,\nu}(Y,Z) \mid f(y) \in V\}$$

where $y \in Y$ and $V \in \nu$.
Let (Y,μ) and (Z,ν) be two GTS's with $Z \in \nu$. Then the *compact-open topology* $\tau^c_{\mu,\nu}$ on $\mathcal{C}_{\mu,\nu}(Y,Z)$ is the topology for which the family

of subbasic open sets are of the form

$$(C, V) = \{f \in \mathcal{C}_{\mu,\nu}(Y, Z) \mid f(C) \subseteq V\}$$

where C is a compact subset of Y and $V \in \nu$.
For two $GTS's$ (Y, μ) and (Z, ν), one can define a GT on $Y \times Z$ in the following way [4]: a subset of $Y \times Z$ is open in the product topology of $Y \times Z$ if it can be expressed as a union of members of the type $U_i \times V_j$, where $U_i \in \mu$, $V_j \in \nu$. The GT obtained this way is defined to be the product GT of μ and ν on $Y \times Z$ and is denoted by $\mu \times \nu$.

Definition 2.4. Let (Y, μ) and (Z, ν) be two $GTS's$. Let (X, λ) be another GTS. For a function $g : X \times Y \to Z$, we can define a mapping $g^* : X \to \mathcal{C}_{\mu,\nu}(Y, Z)$ by $g^*(x)(y) = g(x, y)$.

The mappings g and g^* related in this way are called *associated maps*.

Definition 2.5. Let (Y, μ) and (Z, ν) be two GTS's. A topology τ on $\mathcal{C}_{\mu,\nu}(Y, Z)$ is called

(i) *admissible* if the evaluation mapping $e : \mathcal{C}_{\mu,\nu}(Y, Z) \times Y \to Z$ defined by $e(f, y) = f(y)$ is continuous;

(ii) *splitting* if for each GTS X, continuity of the map $g : X \times Y \to Z$ implies continuity of the map $g^* : X \to \mathcal{C}_{\mu,\nu}(Y, Z)$, where g^* is the associated map of g.

Below, we provide some important results concerning these topologies [9]:

Theorem 2.6. *Let (Y, μ) and (Z, ν) be two GTS's. Then the following hold:*

(i) (μ, ν)-topology is always admissible on $\mathcal{C}_{\mu,\nu}(Y, Z)$.

(ii) Basic separation axioms T_i, where $i = 0, 1, 2$ on (μ, ν)-topology depends on (Z, ν), that is, $\tau_{\mu,\nu}$ is T_i for $i = 0, 1, 2$, if (Z, ν) is T_i for $i = 0, 1, 2$.

(iii) $\tau^p_{\mu,\nu}$ and $\tau^c_{\mu,\nu}$ on $\mathcal{C}_{\mu,\nu}(Y, Z)$ are always splitting.

(iv) $\tau^p_{\mu,\nu}$ is the coarsest topology on $\mathcal{C}_{\mu,\nu}(Y, Z)$, which is coordinately admissible.

3. Dual Topology Concerning $\mathcal{C}_{\mu,\nu}(Y, Z)$

Now we introduce the concept of a dual topology for topologies on $\mathcal{C}_{\mu,\nu}(Y, Z)$.

Definition 3.1. Let (Y, μ) and (Z, ν) be two $GTS's$. Then we define

$$\mathcal{O}_Z(Y) = \{f^{-1}(U) \mid f \in \mathcal{C}_{\mu,\nu}(Y, Z), U \in \nu\}.$$

Definition 3.2. Let (Y,μ) and (Z,ν) be two $GTS's$ and $\mathcal{C}_{\mu,\nu}(Y,Z)$ be the class of all continuous mappings from Y to Z. Then for subsets $\mathbb{H} \subseteq \mathcal{O}_Z(Y)$, $\mathcal{H} \subseteq \mathcal{C}_{\mu,\nu}(Y,Z)$ and $U \in \nu$, we define
$(\mathbb{H},U) = \{f \in \mathcal{C}_{\mu,\nu}(Y,Z) \mid f^{-1}(U) \in \mathbb{H}\}$
$(\mathcal{H},U) = \{f^{-1}(U) \mid f \in \mathcal{H}\}$

Definition 3.3. Let (Y,μ) and (Z,ν) be two $GTS's$. Let $\mathbb{T}$ be any topology on $\mathcal{O}_Z(Y)$ generated by $\mathcal{C}_{\mu,\nu}(Y,Z)$. Then, we define

$$\mathcal{S}(\mathbb{T}) = \{(\mathbb{H},U) \mid \mathbb{H} \in \mathbb{T}, U \in \nu\}$$

Lemma 3.4. *$\mathcal{S}(\mathbb{T})$ forms a subbasis for a topology on $\mathcal{C}_{\mu,\nu}(Y,Z)$.*

Proof. Let $f \in \mathcal{C}_{\mu,\nu}(Y,Z)$ and $U \in \nu$. Since f is continuous, $f^{-1}(U) \in \mu$. Therefore $f \in (\mathbb{H},U)$ for some $\mathbb{H} \in \mathbb{T}$. This holds for all $f \in \mathcal{C}_{\mu,\nu}(Y,Z)$. Thus $\mathcal{C}_{\mu,\nu}(Y,Z) \subseteq \bigcup(\mathbb{H},U)$. Hence $\mathcal{S}(\mathbb{T})$ forms a subbasis for a topology on $\mathcal{C}_{\mu,\nu}(Y,Z)$. ∎

The topology generated by the subbasis $\mathcal{S}(\mathbb{T})$ is called the *dual* topology to $\mathbb{T}$ and is denoted by $\mathfrak{T}(\mathbb{T})$.
On a similar note, we have

Lemma 3.5. *Let (Y,μ) and (Z,ν) be two $GTS's$ and $\mathfrak{T}$ be any topology on $\mathcal{C}_{\mu,\nu}(Y,Z)$. We define*

$$\mathcal{S}(\mathfrak{T}) = \{(\mathcal{H},U) \mid \mathcal{H} \in \mathfrak{T}, U \in \nu\}.$$

Then $\mathcal{S}(\mathfrak{T})$ forms a subbasis for a topology on $\mathcal{O}_Z(Y)$.

Proof. Let $V \in \mathcal{O}_Z(Y)$. Then there exists a function $f \in \mathcal{C}_{\mu,\nu}(Y,Z)$ and a set $U \in \nu$ such that $V = f^{-1}(U)$. Now, for $\mathcal{H} = \mathcal{C}_{\mu,\nu}(Y,Z) \in \mathfrak{T}$, we have $V \in (\mathcal{H},U)$. Hence $\mathcal{O}_Z(Y) \subseteq \bigcup(\mathcal{H},U)$ and hence $\mathcal{S}(\mathfrak{T})$ forms a subbasis for a topology on $\mathcal{O}_Z(Y)$. ∎

The topology generated by the subbasis $\mathcal{S}(\mathfrak{T})$ is called the *dual* topology to $\mathfrak{T}$ and is denoted by $\mathbb{T}(\mathfrak{T})$.

Definition 3.6. A topology τ on $\mathcal{C}_{\mu,\nu}(Y,Z)$ is called *family-open* if it is dual of a topology on $\mathcal{O}_Z(Y)$.

Now, we show that some known topologies such as point-open topology, compact-open topology and (μ,ν)-topology etc. on $\mathcal{C}_{\mu,\nu}(Y,Z)$ are family-open.
First we show that the point-open and compact-open topologies are family-open.

Lemma 3.7. *Let (Y,μ) and (Z,ν) be two GTS's with $Z \in \nu$. For $y \in Y$, let $\mathcal{O}_Z(y)$ be the subfamiliy of $\mathcal{O}_Z(Y)$ consisting of the open neighbourhoods of y in Y. Then $\{\mathcal{O}_Z(y) \,|\, y \in Y\}$ forms a subbasis for a topology $\mathbb{T}$ on $\mathcal{O}_Z(Y)$.*

Proof. We have to show that $\mathcal{O}_Z(Y) = \bigcup_{y\in Y} \mathcal{O}_Z(y)$. Clearly $\bigcup_{y\in Y} \mathcal{O}_Z(y) \subseteq \mathcal{O}_Z(Y)$. Let $f^{-1}(V) \in \mathcal{O}_Z(Y)$ for some $V \in \nu$. If $f^{-1}(V) \neq \emptyset$, then there exists a $y \in Y$ such that $y \in f^{-1}(V)$ and hence $f^{-1}(V) \in \mathcal{O}_Z(y)$ and we have $\mathcal{O}_Z(Y) \subseteq \bigcup_{y\in Y} \mathcal{O}_Z(y)$. Since $Z \in \nu$, therefore $f^{-1}(Z) \neq \emptyset$ for every $f \in \mathcal{C}_{\mu,\nu}(Y,Z)$. Hence $f^{-1}(Z) \in \{\mathcal{O}_Z(y) \,|\, y \in Y\}$. Therefore $\{\mathcal{O}_Z(y) \,|\, y \in Y\}$ forms a subbasis for a topology on $\mathcal{O}_Z(Y)$. ∎

Now, we show that the point-open topology $\tau^p_{\mu,\nu}$ is family-open.

Theorem 3.8. *Let (Y,μ) and (Z,ν) be two GTS's with $Z \in \nu$. Let $\mathbb{T}_p$ be the topology on $\mathcal{O}_Z(Y)$ generated by the subbasis $\{\mathcal{O}_Z(y) \,|\, y \in Y\}$. Then $\mathfrak{T}(\mathbb{T}_p) \equiv \tau^p_{\mu,\nu}$.*

Proof. Let $U \in \nu$ and $f \in \mathcal{C}_{\mu,\nu}(Y,Z)$. Consider $f \in (\{y\}, U)$, that is $f(y) \in U$. Since f is continuous and $U \in \nu$, therefore there exists an open neighbourhood $V \in \mu$ of y such that $f(V) \subseteq U$. Hence $y \in V \subseteq f^{-1}(U)$. Thus $f \in (\mathcal{O}_Z(y), U)$. Therefore $(\{y\}, U) \subseteq (\mathcal{O}_Z(y), U)$. Similarly, we have $(\mathcal{O}_Z(y), U) \subseteq (\{y\}, U)$. Thus $\mathfrak{T}(\mathbb{T}_p) \equiv \tau^p_{\mu,\nu}$. ∎

Corollary 3.9. $\tau^p_{\mu,\nu}$ is family-open.

Working on the same line as above, we obtain the following results:

Theorem 3.10. *Let (Y,μ) and (Z,ν) be two GTS's.*

(1) [(i)]
(2) If $Z \in \nu$, then $\{\mathcal{O}_Z(K) \,|\, K \subseteq Y,\ K$ is compact $\}$, where $\mathcal{O}_Z(K)$ is the subfamily of $\mathcal{O}_Z(Y)$ consisting of all open subsets of Y containing K, forms a subbasis for a topology $\mathbb{T}_c$ on $\mathcal{O}_Z(Y)$ with $\mathfrak{T}(\mathbb{T}_c) \equiv \tau^c_{\mu,\nu}$.
(3) $\{\mathcal{O}_Z(U) \,|\, U \subseteq Y,\ U$ is open $\}$, where $\mathcal{O}_Z(U)$ is the subfamily of $\mathcal{O}_Z(Y)$ consisting of all open subsets of Y containing U, forms a subbasis for a topology $\mathbb{T}_o$ on $\mathcal{O}_Z(Y)$ with $\mathfrak{T}(\mathbb{T}_o) \equiv \tau_{\mu,\nu}$.

Corollary 3.11. $\tau^c_{\mu,\nu}$ and $\tau_{\mu,\nu}$ on $\mathcal{C}_{\mu,\nu}(Y,Z)$ are family-open.

4. Splittingness, admissibility and Dual Topology

In this section, we introduce the notion of admissibility and splittingness for a topology $\mathbb{T}$ on $\mathcal{O}_Z(Y)$. We also investigate the relationship of admissibility and splittingness of $\mathbb{T}$ and that of its dual.

Definition 4.1. Let (Y,μ) and (Z,ν) be two $GTS's$. Let (X,λ) be another GTS and $g : X \times Y \to Z$ be a continuous map. Then a map $\overline{g} : X \times \nu \to \mathcal{O}_Z(Y)$ is defined by $\overline{g}(x,U) = g_x^{-1}(U)$, for every $x \in X$ and $U \in \nu$, where $g_x : Y \to Z$ is defined by $g_x(y) = g(x,y)$. Equivalently, $\overline{g}(x,U) = [g^*(x)]^{-1}(U) = g_x^{-1}(U)$, where $g^* : X \to \mathcal{C}_{\mu,\nu}(Y,Z)$ is the associated map of g.

Definition 4.2. Let (Y,μ) and (Z,ν) be two $GTS's$. Let (X,λ) be another GTS. A map $M : X \times \nu \to \mathcal{O}_Z(Y)$ is called *continuous with respect to the first variable* if the map $M_U : X \to \mathcal{O}_Z(Y)$ defined by $M_U(x) = M(x,U)$ is continuous for every $x \in X$ and for some fixed $U \in \nu$.

Now, we are in a position to define admissibility and splittingness of the topological space $(\mathcal{O}_Z(Y), \mathbb{T})$.

Definition 4.3. Let (Y,μ) and (Z,ν) be two $GTS's$. Let (X,λ) be another GTS. Then a topology $\mathbb{T}$ on $\mathcal{O}_Z(Y)$ is called

(i) *splitting* if continuity of the map $g : X \times Y \to Z$ implies continuity with respect to the first variable of the map $\overline{g} : X \times \nu \to \mathcal{O}_Z(Y)$.

(ii) *admissible* if for every map $g^* : X \to \mathcal{C}_{\mu,\nu}(Y,Z)$, continuity with respect to the first variable of the map $\overline{g} : X \times \nu \to \mathcal{O}_Z(Y)$ implies continuity of the map $g : X \times Y \to Z$.

Below we provide the proof of our main theorems.

Theorem 4.4. *A topology $\mathbb{T}$ on $\mathcal{O}_Z(Y)$ is splitting if and only if its dual topology $\mathfrak{T}(\mathbb{T})$ on $\mathcal{C}_{\mu,\nu}(Y,Z)$ is splitting.*

Proof. Let the topology $\mathbb{T}$ on $\mathcal{O}_Z(Y)$ be splitting, that is, for every GTS (X,λ), continuity of the map $f : X \times Y \to Z$ implies continuity with respect to the first variable of the map $\overline{f} : X \times \nu \to \mathcal{O}_Z(Y)$. We have to show that its dual topology $\mathfrak{T}(\mathbb{T})$ is splitting, that is, continuity of the map $f : X \times Y \to Z$ implies continuity of its associated map $f^* : X \to \mathcal{C}_{\mu,\nu}(Y,Z)$. It is sufficient to show that the map $f^* : X \to \mathcal{C}_{\mu,\nu}(Y,Z)$ is continuous provided the map $\overline{f} : X \times \nu \to \mathcal{O}_Z(Y)$ is continuous with respect to the first variable and vice-versa.
Let $x \in X$ and $(\mathbb{H}, U)$ be any open neighbourhood of $f^*(x)$, that is $f^*(x) \in (\mathbb{H}, U)$. Thus $[f^*(x)]^{-1}(U) \in \mathbb{H}$ and hence $\overline{f}(x,U) \in \mathbb{H}$. We

have $\overline{f}_U(x) \in \mathbb{H}$. Since $\mathbb{H}$ is a subbasic open set in $\mathcal{O}_Z(Y)$. Also for fixed $U \in \nu$, the map $\overline{f}$ is continuous with respect to the first variable. Therefore, there exists an open neighbourhood V of x such that $\overline{f}_U(V) \subseteq \mathbb{H}$. Consider an element $y \in V$, we have $\overline{f}_U(y) \in \mathbb{H}$, that is, $\overline{f}(y, U) \in \mathbb{H}$ and hence $[f^*(y)]^{-1}(U) \in \mathbb{H}$. Therefore, we have $f^*(y) \in (\mathbb{H}, U)$ for every $y \in V$ which implies $f^*(V) \subseteq (\mathbb{H}, U)$. Hence the map f^* is continuous.

Conversely, let the map $f^* : X \to \mathcal{C}_{\mu,\nu}(Y, Z)$ be continuous. We have to show that the map $\overline{f} : X \times \nu \to \mathcal{O}_Z(Y)$ is continuous with respect to the first variable.
For a fixed $U \in \nu$ and $x \in X$, let $\mathbb{H}$ be an open neighbourhood of $\overline{f}_U(x)$. Then $\overline{f}_U(x) \in \mathbb{H}$. That is $\overline{f}(x, U) \in \mathbb{H}$. Thus we have, $[f^*(x)]^{-1}(U) \in \mathbb{H}$ which implies $f^*(x) \in (\mathbb{H}, U)$. Now f^* is continuous and $(\mathbb{H}, U)$ is a subbasic open set in $\mathcal{C}_{\mu,\nu}(Y, Z)$. Therefore there exists an open neighbourhood V of x such that $f^*(V) \subseteq (\mathbb{H}, U)$. Consider an element $y \in V$, we have $f^*(y) \in (\mathbb{H}, U)$, that is $[f^*(y)]^{-1}(U) \in \mathbb{H}$. We have $\overline{f}(y, U) \in \mathbb{H}$. Therefore $\overline{f}_U(y) \in \mathbb{H}$ for every $y \in V$. Hence $\overline{f}_U(V) \subseteq \mathbb{H}$ and $\overline{f}$ is continuous with respect to the first variable. Hence the result. ∎

Our next result is about the counterpart of the previous result, that is

Theorem 4.5. *A topology $\mathfrak{T}$ on $\mathcal{C}_{\mu,\nu}(Y, Z)$ is splitting if and only if its dual topology $\mathbb{T}(\mathfrak{T})$ on $\mathcal{O}_Z(Y)$ is splitting.*

Proof. Let the dual topology $\mathbb{T}(\mathfrak{T})$ on $\mathcal{O}_Z(Y)$ be splitting. We have to show that the topology $\mathfrak{T}$ is splitting and vice-versa. For this, it is sufficient to show that the map $f^* : X \to \mathcal{C}_{\mu,\nu}(Y, Z)$ is continuous provided the map $\overline{f} : X \times \nu \to \mathcal{O}_Z(Y)$ is continuous with respect to the first variable and vice-versa.
Let $x \in X$ and $\mathcal{H}$ be any open neighbourhood of $f^*(x)$. For any fixed $U \in \nu$, we have $[f^*(x)]^{-1}(U) \in (\mathcal{H}, U)$. Therefore $\overline{f}(x, U) \in (\mathcal{H}, U)$ and we have $\overline{f}_U(x) \in (\mathcal{H}, U)$. Since $(\mathcal{H}, U)$ is a subbasic open set in $\mathcal{O}_Z(Y)$ and for fixed $U \in \nu$, the map $\overline{f}$ is continuous with respect to the first variable. Therefore, there exists an open neighbourhood V of x such that $\overline{f}_U(V) \subseteq (\mathcal{H}, U)$. Now, consider an element $y \in V$, then $\overline{f}_U(y) \in (\mathcal{H}, U)$. Thus $\overline{f}(y, U) \in (\mathcal{H}, U)$ and hence $[f^*(y)]^{-1}(U) \in (\mathcal{H}, U)$. Therefore, we have $f^*(y) \in \mathcal{H}$ for every $y \in V$ which implies $f^*(V) \subseteq \mathcal{H}$. Hence the map f^* is continuous.

Conversely, let the map $f^* : X \to \mathcal{C}_{\mu,\nu}(Y, Z)$ be continuous. We have to show that the map $\overline{f} : X \times \nu \to \mathcal{O}_Z(Y)$ is continuous with

respect to the first variable.
For fixed $U \in \nu$ and $x \in X$, consider $\overline{f}_U(x) \in (\mathcal{H}, U)$, where $\mathcal{H} \in \mathfrak{T}$. Thus we have, $\overline{f}(x,U) \in (\mathcal{H}, U)$ which implies $[f^*(x)]^{-1}(U) \in (\mathcal{H}, U)$. Hence $f^*(x) \in \mathcal{H}$. Now f^* is continuous and $\mathcal{H}$ is a subbasic open neighbourhood of $f^*(x)$ in $\mathcal{C}_{\mu,\nu}(Y,Z)$. Therefore there exists an open neighbourhood V of x such that $f^*(V) \subseteq \mathcal{H}$. Consider, an element $y \in V$, we have $f^*(y) \in \mathcal{H}$, that is $[f^*(y)]^{-1}(U) \in (\mathcal{H}, U)$. Hence $\overline{f}(y,U) \in (\mathcal{H}, U)$, therefore, for every $y \in V$, we have $\overline{f}_U(y) \in (\mathcal{H}, U)$. Hence $\overline{f}_U(V) \subseteq (\mathcal{H}, U)$ and $\overline{f}$ is continuous with respect to the first variable. ∎

Our next set of theorems provide the relationship between the topology on $\mathcal{C}_{\mu,\nu}(Y,Z)$ and its dual in the light of admissibility and vice-versa.

Theorem 4.6. *A topology $\mathbb{T}$ on $\mathcal{O}_Z(Y)$ is admissible if and only if its dual topology $\mathfrak{T}(\mathbb{T})$ over $\mathcal{C}_{\mu,\nu}(Y,Z)$ is admissible.*

Proof. Let the topology $\mathbb{T}$ on $\mathcal{O}_Z(Y)$ be admissible, that is, for every GTS (X, λ) and for every map $g^* : X \to \mathcal{C}_{\mu,\nu}(Y,Z)$, continuity of the map $\overline{g} : X \times \nu \to \mathcal{O}_Z(Y)$ with respect the first variable implies continuity of the map $g : X \times Y \to Z$. We have to show that the topology $\mathfrak{T}(\mathbb{T})$ is admissible, that is, continuity of the map $g^* : X \to \mathcal{C}_{\mu,\nu}(Y,Z)$ implies continuity of its associated map $g : X \times Y \to Z$. Thus it is sufficient to prove that $\overline{g} : X \times \nu \to \mathcal{O}_Z(Y)$ is continuous with respect to the first variable provided the map $g^* : X \to \mathcal{C}_{\mu,\nu}(Y,Z)$ is continuous.
Let us consider, for fixed $U \in \nu$ and $x \in X$, $\mathbb{H}$ be a subbasic open neighbourhood of $\overline{g}(x,U)$. That is, $\overline{g}(x,U) \in \mathbb{H}$. Hence $\overline{g}_U(x) \in \mathbb{H}$ which implies $[g^*(x)]^{-1}(U) \in \mathbb{H}$. Thus $g^*(x) \in (\mathbb{H}, U)$. Since the map g^* is given to be continuous and $(\mathbb{H}, U)$ is a subbsaic open neighbourhood of $g^*(x)$, therefore there exists an open neighbourhood V of x such that $g^*(V) \subseteq (\mathbb{H}, U)$. Now, for an element $y \in V$, we have $g^*(y) \in (\mathbb{H}, U)$, that is $[g^*(y)]^{-1}(U) \in \mathbb{H}$. Thus $\overline{g}_U(y) \in \mathbb{H}$ for all $y \in V$. Hence, $\overline{g}_U(V) \subseteq \mathbb{H}$. Therefore the map $\overline{g}$ is continuous with respect to the first variable. Hence the topology $\mathfrak{T}(\mathbb{T})$ is admissible.

Conversely, let $\mathfrak{T}(\mathbb{T})$ be admissible. We have to show that the topology $\mathbb{T}$ on $\mathcal{C}_{\mu,\nu}(Y,Z)$ is admissible. For this, it is sufficient to show that continuity with respect to the first variable of the map $\overline{g} : X \times \nu \to \mathcal{O}_Z(Y)$ implies continuity of the map $g^* : X \to \mathcal{C}_{\mu,\nu}(Y,Z)$. Let $x \in X$ and $(\mathbb{H}, U)$ be a subbasic open neighbourhood of $g^*(x)$, that is $g^*(x) \in (\mathbb{H}, U)$. Thus $[g^*(x)]^{-1}(U) \in \mathbb{H}$. Hence $\overline{g}_U(x) \in \mathbb{H}$. Since the map $\overline{g}$ is given to be continuous with respect to the first variable and $\mathbb{H}$ is a subbaisc open neighbourhood of $\overline{g}_U(x)$, thus there exists an open neighbourhood V of x such that $\overline{g}_U(V) \subseteq \mathbb{H}$. Hence for $y \in V$, we have $\overline{g}_U(y) \in \mathbb{H}$, which implies $\overline{g}(y, U) \in \mathbb{H}$. Hence $[g^*(y)]^{-1}(U) \in \mathbb{H}$. Therefore $g^*(y) \in (\mathbb{H}, U)$ for all $y \in V$. Thus, we have $g^*(V) \subseteq (\mathbb{H}, U)$. Hence the topology $\mathbb{T}$ is admissible. ∎

Theorem 4.7. *A topology $\mathfrak{T}$ on $\mathcal{C}_{\mu,\nu}(Y, Z)$ is admissible if and only if its dual topology $\mathbb{T}(\mathfrak{T})$ on $\mathcal{O}_Z(Y)$ is admissible.*

Proof. Similar to the proof of the previous theorem. ∎

Conclusion. In this paper, we have introduced and investigated the dual topologies for the function spaces over generalized topological spaces. We have also investigated the family-open topologies on $\mathcal{C}_{\mu,\nu}(Y, Z)$. Properties such as splittingness, admissibility etc hold good in such a topology if and only if they hold good in its dual. Further investigation may be carried out to find the effect of the greatest splitting topology on the dual topology.

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