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COMPACTNESS AND REGULARITY VIA MAXIMAL OPEN AND MINIMAL CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce and study the notion of maximal open cover which in turn leads us to define and study m-compact spaces. We prove that there always exists a maximal open cover in an infinite T_1 topological space. We also obtain some results on minimal c-regular and minimal c-normal spaces. We prove that a Hausdorff m-compact topological space is minimal c-normal.

1. Introduction

We simply write X to denote a topological space $(X, \mathscr{P})$. By a proper open set (resp., closed set) of a topological space X, we mean an open set $G \neq \emptyset, X$ (resp., $E \neq \emptyset, X$). We also write $|A|$ to denote the cardinality of the subset A of X.

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Nakaoka and Oda [7, 5] introduced and studied the concept of minimal open sets in topological spaces. The natural dualization of minimal open sets pave the way to introduce and study the concept of maximal open sets in topological spaces [6]. As the consequences of maximal and minimal open sets, Nakaoka and Oda [5] also introduced and studied the notions of maximal and minimal closed sets in topological spaces. The investigative aspects of maximal and minimal open and closed sets are still vivid and still yielding new and interesting concepts e.g. mean open sets [10]. In contrary to use of maximal and minimal open sets in separation, covering properties e.g. [2, 3, 8]; mean open sets [1] pertain to the study of cut-point spaces [4].

2. Some known definitions and theorems

Firstly, we recall some known definitions and theorems to make the article self-sufficient as much as possible and to use them in the sequel.

Definition 1 (Nakaoka and Oda [5, 7]). *A proper open set U of X is said to be a minimal open set if G is an open set of X contained in U, then $G = \emptyset$ or $G = U$.*

Definition 2 (Nakaoka and Oda [5, 6]). *A proper open set U of X is said to be a maximal open set if U contained in an open set G of X, then $G = U$ or $G = X$.*

Definition 3 (Nakaoka and Oda [5]). *A proper closed set E of X is said to be a minimal closed set if $\emptyset$ and E are only closed sets of X contained in E.*

Definition 4 (Nakaoka and Oda [5]). *A proper closed set E of X is said to be a maximal closed set if X and E are only closed sets of X containing E.*

Definition 5 (Mukharjee and Bagchi [10]). *An open set M of X is said to be a mean open set of X if there exist two distinct proper open sets $U, V (\neq M)$ of X such that $U \subset M \subset V$.*

Definition 6 (Mukharjee and Bagchi [10]). *A closed set E of X is said to be a mean closed set if there exist two distinct proper closed sets $D, F (\neq E)$ of X such that $D \subset E \subset F$.*

Lemma 1 (Bagchi and Mukharjee [1]). *Every nonempty open set G of a T_1-connected topological space X is infinite. In particular, if $G \neq X$, then G is not a minimal open set in X.*

Theorem 2 (Bagchi and Mukharjee [1])**.** *Let X be a T_1-connected topological space. A proper open set G of X is a mean open set iff $G \neq X - \{x\}$ for any $x \in X$.*

3. Maximal open covers and m-compact spaces

We now introduce and study the concepts of maximal open covers. Thereafter, we introduce the concept of m-compact spaces using maximal open covers and study it.

We remind that a cover $\mathscr{A}$ of X is called a refinement of the cover $\mathscr{B}$ of X if for each $A \in \mathscr{A}$, there exists a $B \in \mathscr{B}$ such that $A \subset B$.

Definition 7. *Let $\mathscr{A}$ and $\mathscr{B}$ be two covers of a topological space X. $\mathscr{A}$ is said to be an s-refinement of $\mathscr{B}$ if for each $A \in \mathscr{A}$ there is a $B \in \mathscr{B}$ such that $A \subsetneq B$. An s-refinement $\mathscr{A}$ of $\mathscr{B}$ is said to be an open s-refinement of $\mathscr{B}$ if all members of $\mathscr{A}$ and $\mathscr{B}$ are open.*

Note that if $\mathscr{B} = \{X\}$ and $A \neq X$ for each $A \in \mathscr{A}$, then $\mathscr{A}$ is an s-refinement of $\mathscr{B}$. If $\mathscr{A}$ is an s-refinement of $\mathscr{B}$ then $\mathscr{A}$ is a refinement of $\mathscr{B}$. Also we see that no element of an s-refinement of a cover of X is maximal open.

Definition 8. *An open cover $\mathscr{A}$ of a topological space X is said to be a maximal open cover of X if $\mathscr{A}$ is not an s-refinement of any other open cover of X.*

Lemma 3. *An open cover containing a maximal open set is maximal.*

Proof. Easy and hence omitted. ∎

Theorem 4 (Existence of maximal open covers)**.** *There exists a maximal open cover in an infinite T_1 topological space.*

Proof. Let X be an infinite T_1 topological space. Then for each $x \in X$, $X - \{x\}$ is a maximal open set in X.

Let $a \in X$. We choose a finite subset $A = \{x_i \in X \mid x_i \neq a, i \in \{1, 2, \ldots, n\}, n \in \mathbb{N}\}$ of X. Since X is T_1, A is closed in X. Then $\{X - \{a\}, X - A\}$ is an open cover of X containing a maximal open set $X - \{a\}$. Hence by Lemma 3, $\{X - \{a\}, X - A\}$ is a maximal open cover of X. ∎

Theorem 5. *Let $\mathscr{G}$ be an open cover of an infinite T_1 topological space X. Then $\mathscr{G}$ is a maximal open cover of X if and only if $\mathscr{G}$ contains a maximal open set.*

Proof. Let $\mathscr{G} = \{G_\alpha \mid \alpha \in \Delta\}$ be a maximal open cover of X such that no $G_\alpha, \alpha \in \Delta$ is maximal open. By Theorem 2, G_α is not also

minimal open for each $\alpha \in \Delta$. It means that each $G_\alpha, \alpha \in \Delta$ is mean open. So for each $\alpha \in \Delta$, there exists a proper open set H_α such that $G_\alpha \subsetneq H_\alpha$. Let $\mathscr{H} = \{H_\alpha \mid G_\alpha \subsetneq H_\alpha, G_\alpha \in \mathscr{G}\}$. Clearly, $\mathscr{H}$ is a cover of X. Thus we see that $\mathscr{G}$ is an s-refinement of $\mathscr{H}$, a contradiction to the fact that $\mathscr{G}$ is a maximal open cover of X. Hence $\mathscr{G}$ has a maximal open set as one among its members.
The converse part follows by Lemma 3. ∎

Definition 9. *A topological space X is said to be m-compact if each maximal open cover of X has a finite open s-refinement.*

Theorem 6. *Every infinite T_1 connected topological space is m-compact.*

Proof. Let $\mathscr{G}$ be a maximal open cover of an infinite T_1 connected topological space X. By Theorem 5, $\mathscr{G}$ contains a maximal open set G. By Theorem 2, we may write $G = X - \{x\}$ for some $x \in X$. There is an $H \in \mathscr{G}$ such that $x \in H$. By Lemma 1, for $x, y \in H$ with $x \neq y$ we may have open sets $H_1 = X - \{x, y\}$, $H_2 = H - \{x\}$, $H_3 = H - \{y\}$ of X. Then $\{H_1, H_2, H_3\}$ is an s-refinement of $\mathscr{G}$. ∎

Example 1. *Let R denote the set of all real numbers and $a \in R$. Then R is a compact topological space with respect to the topology $\mathscr{T} = \{\emptyset, R, (-\infty, a), [a, \infty)\}$. But $(R, \mathscr{T})$ is not m-compact.*

Remark 1. *According to Theorem 6, the real number space R with the usual topology is m-compact but it is known to us that this topological space is not compact. Therefore using the Theorem 6 together with Example 1, we conclude that the notions compactness and m-compactness are independent.*

Definition 10. *Let X and Y be two topological spaces. A function $f : X \to Y$ is said to be m-continuous if $f^{-1}(U)$ is a maximal open set in X for each proper open set U in Y.*

Theorem 7. *Let X be a m-compact topological space and $f : X \to Y$ be a bijective m-continuous function. Then Y is m-compact.*

Proof. Let $\mathscr{G}^{(Y)}$ be a maximal open cover of Y. Then $\mathscr{G}^{(X)} = \{f^{-1}(G) \mid G \in \mathscr{G}^{(Y)}\}$ is a maximal open cover of X. By m-compactness of X, $\mathscr{G}^{(X)}$ has a finite s-refinement $\mathscr{G}_1^{(X)} = \{f^{-1}(G_k) \mid G_k \in \mathscr{G}^{(Y)}, k \in \{1, 2, \ldots, n\}\}$. It implies that $\mathscr{G}_1^{(Y)} = \{f(f^{-1}(G_k)) \mid G_k \in \mathscr{G}^{(Y)}, k \in \{1, 2, \ldots, n\}\} = \{G_k \mid G_k \in \mathscr{G}^{(Y)}, k \in \{1, 2, \ldots, n\}\}$ covers Y. For each $k \in \{1, 2, \ldots, n\}$, there exists $G \in \mathscr{G}^{(Y)}$ such that $f^{-1}(G_k) \subsetneq$

$f^{-1}(G)$ which implies $G_k \subsetneq G$. Therefore $\mathscr{G}_1^{(Y)}$ is a finite s-refinement of $\mathscr{G}^{(Y)}$. ∎

Definition 11. *A point x of a topological space X is said to be m-complete accumulation point of a subset K of X if $|G \cap K| = |K|$ for each maximal open set G containing x.*

Theorem 8. *Each infinite subset of a m-compact space has an m-complete accumulation point.*

Proof. Let K be an infinite subset of a m-compact topological space X. Suppose for each $x \in X$, there is a maximal open set V_x containing x and satisfying $|V_x \cap K| < |K|$. Since $\{V_x \mid x \in X\}$ is an open cover of X consists of maximal open sets, by Lemma 3, $\{V_x \mid x \in X\}$ is a maximal open cover of X. So there is a finite s-refinement $\{V_{x_i} \mid x_i \in X, i \in \{1, 2, \ldots, n\}\}$ of $\{V_x \mid x \in X\}$. $|K| = |\bigcup_{i=1}^{n}(V_{x_i} \cap K)| < |K|$, a contradiction. ∎

4. Minimal c-regular and c-normal spaces

Definition 12 (Benchalli et al. [2][1]). *A topological space X is said to be minimal c-regular if for each $x \in X$ and each minimal closed set F with $x \notin F$, there exist disjoint open sets U, V such that $x \in U$ and $F \subset V$.*

Theorem 9. *In a topological space X, the following are equivalent:*

(i) X is minimal c-regular.

(ii) Given a point $x \in X$ and a maximal open set U containing x there is an open set V such that $x \in V \subset Cl(V) \subset U$.

(iii) For a point $x \in X$ and a minimal closed set F with $x \notin F$, there exists an open set U containing x such that $Cl(U) \cap F = \emptyset$.

Proof. (i)⇒(ii): Follows by Theorem 3.5 [2].
(ii)⇒(iii) and (iii)⇒(i): Easy to follow. ∎

Definition 13 (Benchalli et al. [3]). *A topological space X is said to be minimal c-normal if for each pair of distinct minimal closed sets E, F there exist disjoint open sets U, V such that $E \subset U$ and $F \subset V$.*

Theorem 10. *In a topological space X, the following are equivalent:*

[1]As contents of the present paper are developed, the first author noticed that the notion of strong regularity [8] is same as the notion of minimal o-regularity [2] due to Benchalli et al. and this mistake to proper citation of the works happen inadvertently by him.

(i) X is minimal c-normal.
(ii) For each minimal closed set A and each maximal open set U with $A \subset U$ there is an open set V such that $A \subset V \subset Cl(V) \subset U$.
(iii) For each pair of distinct minimal closed sets A, B, there exists disjoint open sets U, V such that $A \subset U, Cl(U) \cap B = \emptyset$ and $B \subset V, Cl(V) \cap A = \emptyset$.
(iv) For each pair of distinct minimal closed sets A, B, there exists a pair of disjoint open sets U, V such that $A \subset U$, $B \subset V$ and $Cl(U) \cap Cl(V) = \emptyset$.

Proof. (i)⇒(ii): Follows by Theorem 2.5 [3].
(ii)⇒(iii): $X - B$ is a maximal open set such that $A \subset X - B$. By (ii), there exists an open set U such that $A \subset U \subset Cl(U) \subset X - B$. $Cl(U) \subset X - B$ implies that $Cl(U) \cap B = \emptyset$. Putting $V = X - Cl(U)$, we get $B \subset V \subset X - U \subset X - A$. Since $X - U$ is closed, $B \subset Cl(V) \subset X - U \subset X - A$. $Cl(V) \subset X - A$ implies that $Cl(V) \cap A = \emptyset$. It is obvious that $U \cap V = \emptyset$.
(iii)⇒(iv): By (iii), we have two disjoint open sets U, V such that $A \subset U, Cl(U) \cap B = \emptyset$ and $B \subset V, Cl(V) \cap A = \emptyset$. $Cl(U) \cap B = \emptyset$ and $Cl(V) \cap A = \emptyset$ together imply that $Cl(U) \cap Cl(V) = \emptyset$.
(iv)⇒(i): Easy to follow and hence omitted. ∎

Theorem 11. *Every Hausdorff m-compact space is minimal c-regular.*

Proof. Let X be a Hausdorff m-compact space. Suppose F is a minimal closed set in X and $p \in X$ such that $p \notin F$. Since X is Hausdorff, for each $q \in F$, there exist disjoint open sets U_q, V_q such that $p \in U_q, q \in V_q$. Let $\mathscr{G} = \{V_q \mid q \in F\} \cup \{X - F\}$. Then $\mathscr{G}$ is a maximal open cover of X by Lemma 3. By m-compactness of X, there is a finite s-refinement $\mathscr{H}$ of $\mathscr{G}$. Let $U = \bigcup\{H \in \mathscr{H} \mid H \cap F \neq \emptyset\}$. So U is an open set which contains F. Let $H_1, H_2, \ldots, H_n$ be the only members of $\mathscr{H}$ such that $H_k \cap F \neq \emptyset, k \in \{1, 2, \ldots, n\}$. For each $k \in \{1, 2, \ldots, n\}$, there exists $q_k \in F$ such that $H_k \subsetneq V_{q_k}, k \in \{1, 2, \ldots, n\}$. We put $V = \bigcap_{k=1}^{n} U_{q_k}$. Then $p \in V$. It it is easy to show that $U \cap V = \emptyset$. ∎

Corollary 12. *A Hausdorff m-compact space is minimal c-normal.*

Proof. Let A, B be distinct minimal closed sets in a Hausdorff m-compact space X. By Theorem 11, X is minimal c-regular. So for each $a \in A$, there exist open sets U_a and V_a such that $a \in U_a, B \subset V_a$ and $U_a \cap V_a = \emptyset$. The collection $\mathscr{G} = \{U_a \mid a \in A\} \cup \{X - A\}$ is a maximal open cover of X by Lemma 3. Now proceeding like the proof of Theorem 11, we obtain two open sets U and V such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$. ∎

Lemma 13. *If Y is a closed (resp. open) subset of a topological space X, then minimal closed (resp. minimal open) sets in the subspace Y of X are minimal closed (resp. minimal open) sets in X.*

Proof. Let Y be a closed subset of a topological space X and A be a minimal closed set in Y. There exists a closed set E in X such that $A = E \cap Y$ which implies that A is also closed in X. If possible, suppose we have a closed set F in X such that $F \subset A$. Then $F \cap Y$ is closed in Y such that $F \cap Y \subset F \subset A$. Since A is minimal closed in Y, we have $F \cap Y = A$ or $F \cap Y = \emptyset$. $F \cap Y = A$ implies that $F \cap Y = A = F$. It now need to show that if $F \cap Y = \emptyset$, then $F = \emptyset$. We see that $F \subset A \subset Y$ as A is a subset of Y. So we have $F \cap Y = F \neq \emptyset$ if $F \neq \emptyset$. Hence we get $F = \emptyset$.

Proceeding similarly, it can be proved that if Y is a open subset of a topological space X, then minimal open sets in the subspace Y of X are minimal open sets in X. ∎

Definition 14. *A subspace Y of a topological space X is said to be minimally closed (resp. minimally open) invariant if minimal closed (resp. minimal open) sets of Y are also minimal closed (resp. minimal open) sets of X.*

Theorem 14. *Minimally closed invariant subspaces of minimal c-normal spaces are minimal c-normal.*

Proof. Let Y be a minimally closed invariant subspace of a minimal c-normal space X. Let A, B be two distinct minimal closed sets in Y. Therefore A, B are minimal closed sets in X. Since X is minimal c-normal, the exist disjoint open sets U, V in X such that $A \subset U$ and $B \subset V$. Obviously, $(Y \cap U) \cap (Y \cap V) = \emptyset$. We see that $Y \cap U$ and $Y \cap V$ are disjoint open sets in Y such that $A \subset Y \cap U$ and $B \subset Y \cap V$. ∎

Corollary 15. *Each closed subspace of a minimal c-normal space is minimal c-normal.*

Proof. Using Lemma 13, we have to proceed like that of Theorem 14. ∎

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