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SOME FORMS OF (1, 2)-CONTINUITY IN BITOPOLOGICAL SPACES

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Abstract In this paper, by using mg-closed sets [35] and M-continuity [41] in m-spaces, we obtain the unified definitions and properties for $\tau_1\tau_2$-continuity, (1,2)-semi-continuity, (1,2)-precontinuity, (1,2)-α-continuity, and (1,2)-semi-precontinuity in bitopological spaces.

1. Introduction

Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the researching of generalizations of continuity of functions and multifunctions in topological spaces and bitopological spaces. The notions of quasi-open sets [12], [46] or $\tau_1\tau_2$-open sets [45] in bitopological spaces are introduced and investigated. The notions of quasi-continuity or $\tau_1\tau_2$-continuity, (1, 2)-semi-open sets and (1, 2)-semi-continuity, (1, 2)-preopen sets and (1, 2)-precontinuity, (1, 2)-semi-preopen sets and (1, 2)-semi-precontinuity or (1, 2)-β-continuity are introduced in [12], [46] [18], [31], [19], [4], [44].

In [41] and [42], the present authors introduced and studied the notions of minimal structures, m-spaces and M-continuity.

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The concept of generalized closed sets in topological spaces was introduced by Levine [21]. The notion of g-continuity is introduced and studied in [6], [7], [31] and other papers. Quite recently, Noiri [35] has introduced the notion of mg-closed sets. Recently, the present authors [9], [38] reduced the study of (1,2)*-continuity between bitopological spaces to the study of m-continuity and M-continuity between topological spaces.

In the present paper we reduce the study of (1, 2)-continuity between bitopological spaces to the study of M-continuity between m-spaces. Properties of M-continuity are obtained in [41].

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. We recall some generalized open sets in topological spaces.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be

(1) *α-open* [34] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(2) *semi-open* [20] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(3) *preopen* [26] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,
(4) *β-open* [1] or *semi-preopen* [3] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,

The family of all α-open (resp. semi-open, preopen, β-open) sets in (X, τ) is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\beta(X)$ or $\mathrm{SPO}(X)$).

Definition 2.2. Let (X, τ) be a topological space. A subset A of X is said to be *α-closed* [28] (resp. *semi-closed* [10], *preclosed* [26], *β-closed* [1] or *semi-preclosed* [3]) if the complement of A is α-open (resp. semi-open, preopen, β-open).

Definition 2.3. Let (X, τ) be a topological space and A a subset of X. The intersection of all α-closed (resp. semi-closed, preclosed, β-closed) sets of X containing A is called the *α-closure* [28] (resp. *semi-closure* [10], *preclosure* [13], *β-closure* [2] or *semi-preclosure* [3]) of A and is denoted by $\alpha\mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, ${}_\beta\mathrm{Cl}(A)$ or $\mathrm{spCl}(A)$)).

Definition 2.4. Let (X, τ) be a topological space and A a subset of X. The union of all α-open (resp. semi-open, preopen, β-open) sets of X contained in A is called the *α-interior* [28] (resp. *semi-interior* [10], *preinterior* [13], *β-interior* [2] or *semi-preinterior* [3]) of A and is denoted by $\alpha\mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, ${}_\beta\mathrm{Int}(A)$ or $\mathrm{spInt}(A)$)).

Throughout the present paper, (X, τ) and (Y, σ) always denote topological spaces and (X, τ_1, τ_2) and (Y, σ_1, σ_2) always denote bitopological spaces.

3. Minimal structures and m-continuity

Definition 3.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a *minimal structure* (briefly *m-structure*) on X [41], [42] if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty set X with an m-structure m_X on X and call it an m-space. Each member of m_X is said to be *m_X-open* (briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (briefly *m-closed*).

Remark 3.1. Let (X, τ) be a topological space. The families $\alpha(X)$, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, and $\beta(X)$ are all m-structures on X.

Definition 3.2. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m-closure* of A and the *m-interior* of A are defined in [25] as follows:
(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,
(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 3.2. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$), then we have
(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_{\beta}\mathrm{Cl}(A)$ or $\mathrm{spCl}(A)$),
(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_{\beta}\mathrm{Int}(A)$ or $\mathrm{spInt}(A)$).

Lemma 3.1. (Maki et al. [25]). *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*
(1) $\mathrm{mCl}(X - A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X - A) = X - \mathrm{mCl}(A)$,
(2) If $(X - A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,
(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,
(4) If $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,
(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Lemma 3.2. (Popa and Noiri [41]). *Let (X, m_X) be an m-space and A a subset of X. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 3.3. An m-structure m_X on a nonempty set X is said to have *property* $\mathcal{B}$ [25] if the union of any family of subsets belong to m_X belongs to m_X.

Remark 3.3. If (X, τ) is a topological space, then SO(X), PO(X), $\alpha(X)$, and $\beta(X)$ have property $\mathcal{B}$,

Lemma 3.3. (Popa and Noiri [43]). *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:*
(1) $A \in m_X$ if and only if $\mathrm{mInt}(A) = A$,
(2) A is m_X-closed if and only if $\mathrm{mCl}(A) = A$,
(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is m_X-closed.*

Definition 3.4. A subset A of a bitopological space (X, τ_1, τ_2) is said to be *quasi-open* [12], [46] or $\tau_1\tau_2$*-open* [17] if $A = B \cup C$, where $B \in \tau_1$ and $C \in \tau_2$.

The complement of a $\tau_1\tau_2$-open set is said to be $\tau_1\tau_2$*-closed*. The intersection of all $\tau_1\tau_2$*-closed* sets containing a subset A of X is called the $\tau_1\tau_2$*-closure* of A and is denoted by $\tau_1\tau_2\mathrm{Cl}(A)$. The union of all $\tau_1\tau_2$-open sets contained in A is called the $\tau_1\tau_2$*-interior* of A and is denoted by $\tau_1\tau_2\mathrm{Int}(A)$. The family of all $\tau_1\tau_2$-open sets of (X, τ_1, τ_2) is denoted by (1,2)O(X).

Definition 3.5. Let (X, τ_1, τ_2) be a bitopological space. A subset A of X is said to be
(1) *(1,2)-semi-open* [17] if $A \subset \tau_1\tau_2\mathrm{Cl}(\tau_1\mathrm{Int}(A))$,
(2) *(1,2)-preopen* [17] if $A \subset \tau_1\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(A))$,
(3) *(1,2)-α-open* [17] if $A \subset \tau_1\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(\tau_1\mathrm{Int}(A)))$,
(4) *(1,2)-semi-preopen* [19], [44] if $A \subset \tau_1\tau_2\mathrm{Cl}(\tau_1\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(A)))$.

The complement of a (1,2)-semi-open (resp. (1,2)-preopen, (1,2)-α-open, (1,2)-semi-preopen) set is said to be (1,2)-semi-closed (resp. (1,2)-preclosed, (1,2)-α-closed, (1,2)-semi-preclosed). The collection of all (1,2)-semi-open (resp. (1,2)-preopen, (1,2)-α-open, (1,2)-semi-preopen) sets of (X, τ_1, τ_2) is denoted by (1,2)SO(X) (resp. (1,2)PO(X), (1,2)$\alpha(X)$, (1,2)SPO(X)).

Remark 3.4. Let (X, τ_1, τ_2) be a bitopological space.
(1) The families (1,2)O(X), (1,2)SO(X), (1,2)PO(X), (1,2)$\alpha(X)$, and (1,2)SPO(X)) are all minimal structures on X having property $\mathcal{B}$.
(2) We denote by (1,2)mO(X) minimal structures on X determined

by τ_1 and τ_2. Then, if (1,2)mO(X) = (1,2)O(X) (resp. (1,2)SO(X), (1,2)PO(X), (1,2)α(X), and (1,2)SPO(X)), then we have
(1,2)mCl(A) = $\tau_1\tau_2$Cl(A) (resp. (1,2)sCl(A), (1,2)pCl(A), (1,2)αCl(A), (1,2)spCl(A)),
(1,2)mInt(A) = $\tau_1\tau_2$Int(A) (resp. (1,2)sInt(A), (1,2)pInt(A), (1,2)αInt(A), (1,2)spInt(A)),
By (1) and Lemma 3.3, we have the following properties:

(i) A is $\tau_1\tau_2$-closed (resp. (1,2)-semi-closed, (1,2)-preclosed, (1,2)-α-closed, (1,2)-semi-preclosed) if and only if $A = \tau_1\tau_2$Cl(A) (resp. $A =$ (1,2)sCl(A), $A =$ (1,2)pCl(A), $A =$ (1,2)αCl(A), $A =$ (1,2)spCl(A)),

(ii) A is $\tau_1\tau_2$-open (resp. (1,2)-semi-open, (1,2)-preopen, (1,2)-α-open, (1,2)-semi-preopen) if and only if $A = \tau_1\tau_2$Int(A) (resp. $A =$ (1,2)sInt(A), $A =$ (1,2)pInt(A), $A =$ (1,2)αInt(A), $A =$ (1,2)spInt(A)).

(3) By using Lemma 3.1, we obtain the relations between (1,2)mCl(A) and (1,2)mInt(A).

(4) By Lemma 3.2, we obtain the result of Lemma 8(iii) of [4].

4. *mg*-CLOSED SETS IN BITOPOLOGICAL SPACES

Definition 4.1. Let (X, τ) be a topological space. A subset A of X is said to be

(1) *g-closed* [21] if Cl(A) $\subset U$ whenever $A \subset U$ and $U \in \tau$,

(2) *gα-closed* [24] if αCl(A) $\subset U$ whenever $A \subset U$ and $U \in \alpha(X)$,

(3) *sg-closed* [5] if sCl(A) $\subset U$ whenever $A \subset U$ and $U \in$ SO(X),

(4) *pg-closed* [36] if pCl(A) $\subset U$ whenever $A \subset U$ and $U \in$ PO(X),

(5) *spg-closed* [35] if spCl(A) $\subset U$ whenever $A \subset U$ and $U \in$ SPO(X),

Definition 4.2. Let (X, m_X) be an m-space. A subset A of X is said to be *mg-closed* [35] if mCl(A) $\subset U$ whenever $A \subset U$ and $U \in m_X$.

The complement of an mg-closed set is said to be *mg-open*. The collection of mg-open sets in (X, m_X) is denoted by mGO(X). The mGO(X) is a minimal structure on X.

Remark 4.1. Let (X, τ) be a topological space and m_X an m-structure on X. If $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, SPO(X)), then, an mg-closed set is a g-closed (resp. sg-closed, pg-closed, $g\alpha$-closed, spg-closed) set.

Definition 4.3. Let (X, τ_1, τ_2) be a bitopological space. A subset A of X is said to be

(1) *(1,2)-g-closed* [19] if $\tau_1\tau_2\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in (1,2)\mathrm{O}(X)$,

(2) *(1,2)-sg-closed* [19] if $(1,2)\mathrm{sCl}(A) \subset U$ whenever $A \subset U$ and $U \in (1,2)\mathrm{SO}(X)$,

(3) *(1,2)-pg-closed* [19] if $(1,2)\mathrm{pCl}(A) \subset U$ whenever $A \subset U$ and $U \in (1,2)\mathrm{PO}(X)$,

(4) *(1,2)-αg-closed* [19] if $(1,2)\alpha\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in (1,2)\alpha(X)$,

(5) *(1,2)-spg-closed* [19], [33] if $(1,2)\mathrm{spCl}(A) \subset U$ whenever $A \subset U$ and $U \in (1,2)\mathrm{SPO}(X)$,

Definition 4.4. Let (X, τ_1, τ_2) be a bitopological space and (1,2)mO(X) a minimal structure on X determined by τ_1 and τ_2. A subset A of X is said to be *(1,2)-mg-closed* if A is mg-closed in (X, (1,2)mO(X)).

A subset A of X is said to be *(1,2)-mg-open* if $X - A$ is (1,2)-mg-closed. The collection of all (1,2)-g-open (resp. (1,2)-sg-open, (1,2)-pg-open, (1,2)-αg-open, (1,2)-spg-open) sets of X is denoted by (1,2)GO(X) (resp. (1,2)SGO(X), (1,2)PGO(X), (1,2)αGO(X), (1,2)SPGO(X)).

Remark 4.2. Let (X, τ_1, τ_2) be a bitopological space.

(1) Let (1,2)mO(X) = (1,2)O(X) (resp. (1,2)SO(X), (1,2)PO(X), (1,2)α(X), (1,2)SPO(X)), then an mg-closed set is (1,2)-g-closed (resp. (1,2)-sg-closed, (1,2)-pg-closed, (1,2)-αg-closed, (1,2)-spg-closed).

(2) (1,2)GO(X), (1,2)SGO(X), (1,2)PGO(X), (1,2)αGO(X) and (1,2)SPGO(X) are minimal structures on X. But they do not always satisfy property $\mathcal{B}$ (see Example 22 of [45]).

Lemma 4.1. (Noiri [35]). *Let (X, m_X) be an m-space. For subsets A and B of X, the following properties hold:*

(1) If A is m-closed, then A is mg-closed,

(2) If m_X has property $\mathcal{B}$ and A is mg-closed and m-open, then A is m-closed,

(3) If A is mg-closed and $A \subset B \subset \mathrm{mCl}(A)$, then B is mg-closed.

Theorem 4.1. *Let (X, τ_1, τ_2) be a bitopological space and (1,2)mO(X) a minimal structure on X determined by τ_1 and τ_2. For subsets A and B of X, the following properties hold:*

(1) If A is (1,2)-m-closed, then A is (1,2)-mg-closed,

(2) If (1,2)mO(X) has property $\mathcal{B}$ and A is (1,2)-mg-closed and (1,2)-m-open, then A is (1,2)-m-closed,

(3) If A is (1,2)-mg-closed and $A \subset B \subset (1,2)\mathrm{mCl}(A)$*, then B is (1,2)-mg-closed.*

Proof. The proof follows from Definition 4.4 and Lemma 4.1.

Corollary 4.1. *Let* (X, τ_1, τ_2) *be a bitopological space. For subsets A, B of X, the following properties hold:*

(1) If A is (1,2)-sp-closed, then A is (1,2)-spg-closed,

(2) If A is (1,2)-spg-closed and (1,2)-sp-open, then A is (1,2)-sp-closed,

(3) If A is (1,2)-spg-closed and $A \subset B \subset (1,2)\mathrm{spCl}(A)$*, then B is (1,2)-spg-closed.*

Lemma 4.2. (Noiri [35]). *Let* (X, m_X) *be an m-space. Then for each* $x \in X$*, either* $\{x\}$ *is m-closed or* $\{x\}$ *is mg-open.*

Theorem 4.2. *Let* (X, τ_1, τ_2) *be a bitopological space and (1,2)mO(X) a minimal structure on X determined by* τ_1 *and* τ_2*. Then for each* $x \in X$*, either* $\{x\}$ *is (1,2)-m-closed or* $\{x\}$ *is (1,2)-mg-open.*

Proof. The proof follows from Definition 4.4 and Lemma 4.2.

Corollary 4.2. *Let* (X, τ_1, τ_2) *be a bitopological space. Then for each* $x \in X$*, either* $\{x\}$ *is (1,2)-sp-closed or* $\{x\}$ *is (1,2)-spg-open.*

Lemma 4.3. (Noiri [35]). *Let* (X, m_X) *be an m-space. A subset A of X is mg-open if and only if* $F \subset \mathrm{mInt}(A)$ *whenever* $F \subset A$ *and F is m-closed.*

Theorem 4.3. *Let* (X, τ_1, τ_2) *be a bitopological space and (1,2)mO(X) a minimal structure on X determined by* τ_1 *and* τ_2*. A subset A of X is (1,2)-mg-open if and only if* $F \subset (1,2)\mathrm{mInt}(A)$ *whenever* $F \subset A$ *and F is (1,2)-m-closed.*

Proof. The proof follows from Definition 4.4 and Lemma 4.3.

Corollary 4.3. *A subset A of a bitopological space* (X, τ_1, τ_2) *is (1,2)-spg-open if and only if* $F \subset (1,2)\mathrm{spInt}(A)$ *whenever* $F \subset A$ *and F is (1,2)-sp-closed.*

Lemma 4.4. (Noiri [35]). *Let* (X, m_X) *be an m-space. For subsets A and B of X, the following properties hold:*

(1) If A is m-open, then A is mg-open,

(2) If m_X *has property* $\mathcal{B}$ *and A is mg-open and m-closed, then A is m-open,*

(3) If A is mg-open and $\mathrm{mInt}(A) \subset B \subset A$*, then B is mg-open.*

Theorem 4.4. *Let* (X, τ_1, τ_2) *be a bitopological space and (1,2)mO(X) a minimal structure on X determined by* τ_1 *and* τ_2. *For subsets A and B of X, the following properties hold:*

(1) If A is (1,2)-m-open, then A is (1,2)-mg-open,

(2) If (1,2)mO(X) has property $\mathcal{B}$ *and A is (1,2)-mg-open and (1,2)-m-closed, then A is (1,2)-m-open,*

(3) If A is (1,2)-mg-open and $(1,2)\mathrm{mInt}(A) \subset B \subset A$, *then B is (1,2)-mg-open.*

Proof. The proof follows from Definition 4.4 and Lemma 4.4.

Corollary 4.4. *Let* (X, τ_1, τ_2) *be a bitopological space. For subsets A and B of X, the following properties hold:*

(1) If A is (1,2)-semi-preopen, then A is (1,2)-spg-open,

(2) If A is (1,2)-spg-open and (1,2)-semi-preclosed, then A is (1,2)-semi-preopen,

(3) If A is (1,2)-spg-open and $(1,2)\mathrm{spInt}(A) \subset B \subset A$, *then B is (1,2)-spg-open.*

Lemma 4.5. (Noiri [35]). *Let* (X, m_X) *be an m-space, where* m_X *has property* $\mathcal{B}$. *Then, for a subset A of X, the following properties are equivalent.*

(1) A is mg-closed;

(2) $\mathrm{mCl}(A) - A$ *does not contain any nonempty m-closed set;*

(3) $\mathrm{mCl}(A) - A$ *is mg-open.*

Theorem 4.5. *Let* (X, τ_1, τ_2) *be a bitopological space and (1,2)mO(X) a minimal structure on X determined by* τ_1 *and* τ_2 *having property* $\mathcal{B}$. *Then, for a subset A of X, the following properties are equivalent.*

(1) A is (1,2)-mg-closed;

(2) $(1,2)\mathrm{mCl}(A) - A$ *does not contain any nonempty (1,2)-m-closed set;*

(3) $(1,2)\mathrm{mCl}(A) - A$ *is (1,2)-mg-open.*

Proof. The proof follows from Definition 4.4 and Lemma 4.5.

Corollary 4.5. *Let* (X, τ_1, τ_2) *be a bitopological space. Then, for a subset A of X, the following properties are equivalent.*

(1) A is (1,2)-spg-closed;

(2) $(1,2)\mathrm{spCl}(A) - A$ *does not contain any nonempty (1,2)-semi-preclosed set;*

(3) $(1,2)\mathrm{spCl}(A) - A$ *is (1,2)-spg-open.*

Remark 4.3. Corollary 4.5(2) is an improvement of Theorem 4.8 of [19].

Lemma 4.6. (Noiri [35]). *Let (X, m_X) be an m-space. A subset A of X is mg-closed if and only if* $\mathrm{mCl}(A) \cap F = \emptyset$ *whenever* $A \cap F = \emptyset$ *and* F *is m-closed.*

Theorem 4.6. *Let (X, τ_1, τ_2) be a bitopological space and (1,2)mO(X) a minimal structure on X determined by τ_1 and τ_2. Then, a subset A of X is (1,2)-mg-closed if and only if* $(1,2)\mathrm{mCl}(A) \cap F = \emptyset$ *whenever* $A \cap F = \emptyset$ *and F is (1,2)-m-closed.*

Proof. The proof follows from Definition 4.4 and Lemma 4.6.

Corollary 4.6. *Let (X, τ_1, τ_2) be a bitopological space. Then, A is (1,2)-spg-closed if and only if* $(1,2)\mathrm{spCl}(A) \cap F = \emptyset$ *whenever* $A \cap F = \emptyset$ *and F is (1,2)-semi-preclosed.*

Lemma 4.7. (Noiri [35]). *Let (X, m_X) be an m-space, where m_X have property $\mathcal{B}$. A subset A of X is mg-closed if and only if* $\mathrm{mCl}(\{x\}) \cap A \neq \emptyset$ *for each* $x \in \mathrm{mCl}(A)$.

Theorem 4.7. *Let (X, τ_1, τ_2) be a bitopological space and (1,2)mO(X) a minimal structure on X with property $\mathcal{B}$ determined by τ_1 and τ_2. Then, a subset A of X is (1,2)-mg-closed if and only if* $(1,2)\mathrm{mCl}(\{x\}) \cap A \neq \emptyset$ *for each* $x \in (1,2)\mathrm{mCl}(A)$.

Proof. The proof follows from Definition 4.4 and Lemma 4.7.

Corollary 4.7. *Let (X, τ_1, τ_2) be a bitopological space. Then, a subset A of X is (1,2)-spg-closed if and only if* $(1,2)\mathrm{spCl}(\{x\}) \cap A \neq \emptyset$ *for each* $x \in (1,2)\mathrm{spCl}(A)$.

Definition 4.5. A subset A of an m-space (X, m_X) is said to be *locally m-closed* [35] if $A = U \cap F$, where $U \in m_X$ and F is m-closed.

Remark 4.4. Let (X, τ) be a topological space and $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{SPO}(X)$), then a locally m-closed set is locally closed [14] (resp. semi-locally closed [47], locally pre-closed [35], α-locally closed [15], β-locally closed [16]).

Lemma 4.8. *Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. For a subset A of X, the following properties are equivalent:*

(1) A is locally m-closed;
(2) $A = U \cap \mathrm{mCl}(A)$ *for some* $U \in m_X$;
(3) $\mathrm{mCl}(A) - A$ *is m-closed;*
(4) $A \cup (X - \mathrm{mCl}(A)) \in m_X$;
(5) $A \subset \mathrm{mInt}[A \cup (X - \mathrm{mCl}(A))]$.

Definition 4.6. Let (X, τ_1, τ_2) be a bitopological space and (1,2)mO(X) a minimal structure on X determined by τ_1 and τ_2. A subset A of X is said to be *locally (1,2)-m-closed* if A is locally m-closed in $(X, (1,2)\mathrm{mO}(X))$.

Hence A is locally (1,2)-m-closed if $A = U \cap F$, where $U \in (1,2)\mathrm{mO}(X)$ and F is (1,2)-m-closed.

Remark 4.5. Let (X, τ_1, τ_2) be a bitopological space and (1,2)mO(X) = (1,2)O(X) (resp. (1,2)SO(X), (1,2)PO(X), (1,2)$\alpha(X)$, (1,2)SPO(X)). If a subset A of X is locally (1,2)-m-closed, then A is locally (1,2)-closed (resp. locally (1,2)-semi-closed, locally (1,2)-preclosed, locally (1,2)-α-closed, locally (1,2)-semi-preclosed).

Theorem 4.8. *Let (X, τ_1, τ_2) be a bitopological space and (1,2)mO(X) a minimal structure on X with property $\mathcal{B}$ determined by τ_1 and τ_2. Then, for a subset A of X, the following properties are equivalent:*

(1) A is locally (1,2)-m-closed;
(2) $A = U \cap (1,2)\mathrm{mCl}(A)$ *for some* $U \in (1,2)\mathrm{mO}(X)$*;*
(3) $(1,2)\mathrm{mCl}(A) - A$ *is (1,2)-m-closed;*
(4) $A \cup [X - (1,2)\mathrm{mCl}(A)] \in (1,2)\mathrm{mO}(X)$*;*
(5) $A \subset (1,2)\mathrm{mInt}[A \cup (X - (1,2)\mathrm{mCl}(A))]$.

Proof. The proof follows from Definition 4.4 and Lemma 4.8.

Corollary 4.8. *Let (X, τ_1, τ_2) be a bitopological space. Then, for a subset A of X, the following properties are equivalent:*

(1) A is locally (1,2)-semi-preclosed;
(2) $A = U \cap (1,2)\mathrm{spCl}(A)$ *for some* $U \in (1,2)\mathrm{SPO}(X)$*;*
(3) $(1,2)\mathrm{spCl}(A) - A$ *is (1,2)-semi-preclosed;*
(4) $A \cup [X - (1,2)\mathrm{spCl}(A)] \in (1,2)\mathrm{SPO}(X)$*;*
(5) $A \subset (1,2)\mathrm{spInt}[A \cup (X - (1,2)\mathrm{spCl}(A))]$.

Lemma 4.9. (Noiri [35]). *Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. Then a subset A of X is m-closed if and only if A is mg-closed and locally m-closed.*

Theorem 4.9. *Let (X, τ_1, τ_2) be a bitopological space and (1,2)mO(X) a minimal structure on X with property $\mathcal{B}$ determined by τ_1 and τ_2. Then, a subset A of X is (1,2)-m-closed if and only if A is (1,2)-mg-closed and locally (1,2)-m-closed.*

Proof. The proof follows from Definition 4.4 and Lemma 4.9

Corollary 4.9. *Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. Then,* *(1) A is $\tau_1\tau_2$-closed if and only if it is (1,2)-g-closed and locally (1,2)-closed,*

(2) A is (1,2)-semi-closed if and only if it is (1,2)-sg-closed and locally (1,2)-semi-closed,

(3) A is (1,2)-preclosed if and only if it is (1,2)-pg-closed and locally (1,2)-preclosed,

(4) A is (1,2)-α-closed if and only if it is (1,2)-αg-closed and locally (1,2)-α-closed,

(5) A is (1,2)-semi-preclosed if and only if it is (1,2)-spg-closed and locally (1,2)-semi-preclosed.

5. M-continuity in topological spaces

Definition 5.1. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *M-continuous* at a point $x \in X$ [41] if for each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *M-continuous* if it has this property at each point $x \in X$.

Remark 5.1. Let (X, τ) be a topological space.

(1) If $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, SPO(X)) and $m_Y = \sigma$ and f is M-continuous, then $f : (X, \tau) \to (Y, \sigma)$ is continuous (resp. semi-continuous [20], precontinuous [26], α-continuous [28], semi-precontinuous [3] or β-continuous [1]).

(2) If $m_X =$ SO(X) (resp. PO(X), $\alpha(X)$, SPO(X) or $\beta(Y)$) and $m_Y =$ SO(Y) (resp. PO(Y), $\alpha(Y)$, SPO(Y) or $\beta(Y)$) and f is M-continuous, then $f : (X, \tau) \to (Y, \sigma)$ is irresolute [11] (resp. pre-irresolute [27], α-irresolute [23], β-irresolute [29]).

(3) If $m_X = \tau$ and $m_Y =$ SO(Y) (resp. $\alpha(Y)$, $\beta(Y)$) and $f : (X, \tau) \to (Y, \sigma)$ is M-continuous, then f is s-continuous [8] (resp. strongly α-irresolute [22], strongly β-irresolute [32]).

(4) If $m_X =$ SO(X) and $m_Y = \alpha(Y)$ and $f : (X, \tau) \to (Y, \sigma)$ is M-continuous, then f is strongly semi-continuous [40].

Theorem 5.1. (Noiri and Popa [39]). *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) f is M-continuous at $x \in X$;

(2) $x \in \mathrm{mInt}(f^{-1}(V))$ for every $V \in m_Y$ containing f(x);

(3) $x \in f^{-1}(\mathrm{mCl}(f(A)))$ for every subset A of X with $x \in \mathrm{mCl}(A)$;

(4) $x \in f^{-1}(\mathrm{mCl}(B))$ for every subset B of Y with $x \in \mathrm{mCl}(f^{-1}(B))$;

(5) $x \in \mathrm{mInt}(f^{-1}(B))$ for every subset B of Y with $x \in$

$f^{-1}(\mathrm{mInt}(B))$;

(6) $x \in f^{-1}(K)$ *for every* m_Y*-closed set* K *of* Y *such that* $x \in \mathrm{mCl}(f^{-1}(K))$.

For a function $f : (X, m_X) \to (Y, m_Y)$, we define $D_M(f)$ as follows:

$$D_M(f) = \{x \in X : f \text{ is not } M\text{-continuous at } x\}.$$

Theorem 5.2. (Noiri and Popa [39]). *For a function* $f : (X, m_X) \to (Y, m_Y)$*, the following properties hold:*

$$\begin{aligned} D_M(f) &= \textstyle\bigcup_{G\in m_Y}\{f^{-1}(G) - \mathrm{mInt}(f^{-1}(G))\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\{f^{-1}(\mathrm{mInt}(B)) - \mathrm{mInt}(f^{-1}(B))\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\{\mathrm{mCl}(f^{-1}(B)) - f^{-1}(\mathrm{mCl}(B))\} \\ &= \textstyle\bigcup_{A\in\mathcal{P}(X)}\{\mathrm{mCl}(A) - f^{-1}(\mathrm{mCl}(f(A)))\} \\ &= \textstyle\bigcup_{K\in\mathcal{F}}\{\mathrm{mCl}(f^{-1}(K)) - f^{-1}(K)\}, \end{aligned}$$

where $\mathcal{F}$ *is the family of* m_Y*-closed sets of* Y.

Lemma 5.1. (Popa and Noiri [41]). *For a function* $f : (X, m_X) \to (Y, m_Y)$*, the following properties are equivalent:*

(1) f *is M-continuous;*

(2) $f^{-1}(V) = \mathrm{mInt}(f^{-1}(V))$ *for every* $V \in m_Y$;

(3) $f^{-1}(K) = \mathrm{mCl}(f^{-1}(K))$ *for every m-closed set* K *of* Y

(4) $f(\mathrm{mCl}(A)) \subset \mathrm{mCl}(f(A))$ *for every subset* A *of* X;

(5) $\mathrm{mCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{mCl}(B))$ *for every subset* B *of* Y;

(6) $f^{-1}(\mathrm{mInt}(B)) \subset \mathrm{mInt}(f^{-1}(B))$ *for every subset* B *of* Y.

Corollary 5.1. (Popa and Noiri [41]). *For a function* $f : (X, m_X) \to (Y, m_Y)$*, where* m_X *has property* $\mathcal{B}$*, the following properties are equivalent:*

(1) f *is M-continuous;*

(2) $f^{-1}(V)$ *is m-open for every* $V \in m_Y$;

(3) $f^{-1}(K)$ *is m-closed in* X *for every m-closed set* K *of* Y.

Definition 5.2. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *M^*-continuous* [30] if $f^{-1}(V)$ is m-open in X for each m-open set V of Y.

Remark 5.2. (1) If $f : (X, m_X) \to (Y, m_Y)$ is M^*-continuous, then it is M-continuous. By Example 3.4 of [30], every M-continuous function is not always M^*-continuous.

(2) If m_X has property $\mathcal{B}$, then M-continuity and M^*-continuity are equivalent.

Definition 5.3. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *mg-continuous* if $f : (X, \mathrm{mGO}(X)) \to (Y, m_Y)$ is M^*-continuous.

Hence $f : (X, m_X) \to (Y, m_Y)$ is mg-continuous if $f^{-1}(K)$ is mg-closed in X for each m-closed set of Y.

Definition 5.4. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *locally mc-continuous* if $f^{-1}(K)$ is locally m-closed in X for each m-closed set K of Y.

Theorem 5.3. *A function $f : (X, m_X) \to (Y, m_Y)$, where m_X has property $\mathcal{B}$, is M-continuous if and only if f is mg-continuous and locally mc-continuous.*

Proof. The proof follows from Definitions 5.3 and 5.4 and Lemma 4.9.

Definition 5.5. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *contra M^*-continuous* if $f^{-1}(K)$ is m-closed in X for each m-open set K of Y.

Theorem 5.4. *If a function $f : (X, m_X) \to (Y, m_Y)$, where m_X has property $\mathcal{B}$, is mg-continuous and contra M^*-continuous, then f is M-continuous.*

Proof. Let V be any m-open set of Y. Since f is mg-continuous, $f^{-1}(V)$ is mg-open. Since f is contra M^*-continuous, $f^{-1}(V)$ is m-closed. By Lemma 4.4, $f^{-1}(V)$ is m-open. Then, by Corollary 5.1 f is M-continuous.

6. M-continuity in bitopological spaces

Definition 6.1. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *$\tau_1\tau_2$-continuous* [18] or *quasi-continuous* [46] (resp. *(1,2)-semi-continuous, (1,2)-precontinuous, (1,2)-α-continuous, (1,2)-semi-precontinuous* or *(1,2)-β-continuous*) if $f^{-1}(V)$ is $\tau_1\tau_2$-open (resp. (1,2)-semi-open, (1,2)-preopen, (1,2)-α-open, (1,2)-semi-preopen) in X for every $\sigma_1\sigma_2$-open set V of Y.

Definition 6.2. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(1,2)-semi-irresolute* [37] (resp. *(1,2)-preirresolute* [18], [19], *(1,2)-α-irresolute* [18], *(1,2)-β-irresolute* [18]) if $f^{-1}(V)$ is (1,2)-semi-open (resp. (1,2)-preopen, (1,2)-α-open, (1,2)-β-open) in X for each (1,2)-semi-open (resp. (1,2)-preopen, (1,2)-α-open, (1,2)-β-open) set V of Y.

Definition 6.3. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(1,2)-M-continuous* (resp. *(1,2)-M^*-continuous)* if f :

$(X, (1,2)\text{mO}(X)) \to (Y, (1,2)\text{mO}(Y))$ is M-continuous (resp. M^*-continuous), where (1,2)mO(X) (resp. (1,2)mO(Y)) is a minimal structure on X (resp. Y) determined by τ_1 and τ_2 (resp. σ_1 and σ_2),

Remark 6.1. Let (X, τ_1, τ_2) be a bitopological space.

(1) Since (1,2)O(X) (resp. (1,2)SO(X), (1,2)PO(X), (1,2)α(X), (1,2)SPO(X)) has property $\mathcal{B}$, if (1,2)mO(X) = (1,2)O(X) (resp. (1,2)SO(X), (1,2)PO(X), (1,2)α(X), (1,2)SPO(X)) and (1,2)mO(Y) = (1,2)O(Y), then by Definition 6.3 we obtain Definition 6.1.

(2) If (1,2)mO(X) = (1,2)SO(X) (resp. (1,2)PO(X), (1,2)α(X), (1,2)SPO(X)) and (1,2)mO(Y) = (1,2)SO(Y) (resp. (1,2)PO(Y), (1,2)α(Y), (1,2)SPO(Y)), then by Definition 6.3 we obtain Definition 6.2.

By Lemma 5.1 and Corollary 5.1, we obtain the following theorem and corollary.

Theorem 6.1. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopolgical spaces and (1,2)mO(X) and (1,2)mO(Y) m-spaces as in Definition 6.3. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (1,2)-M-continuous;

(2) $f^{-1}(V) = (1,2)\text{mInt}(f^{-1}(V))$ for every (1,2)-m-open set V of Y;

(3) $f^{-1}(K) = (1,2)\text{mCl}(f^{-1}(K))$ for every (1,2)-m-closed set K of Y;

(4) $(1,2)\text{mCl}(f^{-1}(B)) \subset f^{-1}((1,2)\text{mCl}(B))$ for every subset B of Y;

(5) $f((1,2)\text{mCl}(A)) \subset (1,2)\text{mCl}(f(A))$ for every subset A of X;

(6) $f^{-1}((1,2)\text{mInt}(B)) \subset (1,2)\text{mInt}(f^{-1}(B))$ for every subset B of Y;

Corollary 6.1. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopolgical spaces and (1,2)mO(X) and (1,2)mO(Y) m-spaces as in Definition 6.3, where (1,2)mO(X) has property $\mathcal{B}$, then the following properties are equivalent:*

(1) f is (1,2)-M-continuous;

(2) $f^{-1}(V)$ is (1,2)-m-open in X for every (1,2)-m-open set V of Y;

(3) $f^{-1}(K)$ is (1,2)-m-closed in X for every (1,2)-m-closed set K of Y.

Theorem 6.2. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (1,2)-semi-precontinuous;

(2) $f^{-1}(V) \in (1,2)\mathrm{SPO}(X)$ *for each $\sigma_1\sigma_2$-open set of Y;*

(3) $f^{-1}(K)$ *is (1,2)-semi-preclosed for each $\sigma_1\sigma_2$-closed set K of Y;*

(4) $(1,2)\mathrm{spCl}(f^{-1}(B)) \subset f^{-1}(\sigma_1\sigma_2\mathrm{Cl}(B))$ *for every subset B of Y;*

(5) $f((1,2)\mathrm{spCl}(A)) \subset \sigma_1\sigma_2\mathrm{Cl}(f(A))$ *for every subset A of X;*

(6) $f^{-1}(\sigma_1\sigma_2\mathrm{Int}(B)) \subset (1,2)\mathrm{spInt}(f^{-1}(B))$ *for every subset B of Y;*

Theorem 6.3. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (1,2)-semi-preirresolute;

(2) $f^{-1}(V) \in (1,2)\mathrm{SPO}(X)$ *for each (1,2)-semi-preopen set of Y;*

(3) $f^{-1}(K)$ *is (1,2)-semi-preclosed for each (1,2)-semi-preclosed set K of Y;*

(4) $(1,2)\mathrm{spCl}(f^{-1}(B)) \subset f^{-1}((1,2)\mathrm{spCl}(B))$ *for every subset B of Y;*

(5) $f((1,2)\mathrm{spCl}(A)) \subset (1,2)\mathrm{spCl}(f(A))$ *for every subset A of X;*

(6) $f^{-1}((1,2)\mathrm{spInt}(B)) \subset (1,2)\mathrm{spInt}(f^{-1}(B))$ *for every subset B of Y;*

For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we define $D_{(1,2)M}(f)$ as follows:

$$D_{(1,2)M}(f) = \{x \in X : f \text{ is not (1,2)-}M\text{-continuous at } x\}.$$

Theorem 6.4. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopolgical spaces and (1,2)mO(X) and (1,2)mO(Y) m-spaces as in Definition 6.3. Then, for a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties hold:*

$$\begin{aligned} D_{(1,2)M}(f) &= \textstyle\bigcup_{G\in(1,2)O(Y)}\{f^{-1}(G) - (1,2)\mathrm{mInt}(f^{-1}(G))\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\ \{f^{-1}((1,2)\mathrm{Int}(B)) - (1,2)\mathrm{mInt}(f^{-1}(B))\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\ \{(1,2)\mathrm{mCl}(f^{-1}(B)) - f^{-1}((1,2)\mathrm{mCl}(B))\} \\ &= \textstyle\bigcup_{A\in\mathcal{P}(X)}\ \{(1,2)\mathrm{mCl}(A) - f^{-1}((1,2)\mathrm{mCl}(f(A)))\}. \end{aligned}$$

Definition 6.4. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(1,2)-g-continuous* (resp. *(1,2)-sg-continuous, (1,2)-pg-continuous, (1,2)-αg-continuous, (1,2)-spg-continuous*) if the inverse image of a $\sigma_1\sigma_2$-closed (resp. (1,2)-*s*-closed, (1,2)-*p*-closed, (1,2)-α-closed, (1,2)-*sp*-closed) set of Y is (1,2)-*g*-closed (resp. (1,2)-*sg*-closed, (1,2)-*pg*-closed, (1,2)-αg-closed, (1,2)-*spg*-closed) set of X.

Definition 6.5. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(1,2)-mg-continuous* if $f : (X, (1,2)\text{mO}(X)) \to (Y, (1,2)\text{mO}(Y))$ is (1,2)-mg-continuous, where (1,2)mO(X) (resp. (1,2)mO(Y)) is a minimal structure on X (resp. Y) determined by τ_1 and τ_2 (resp. σ_1 and σ_2) as in Definition 6.3,

Hence a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (1,2)-mg-continuous if $f^{-1}(K)$ is (1,2)-mg-closed in X for each (1,2)-m-closed set K of Y.

Remark 6.2. If (1,2)mO(X) = (1,2)O(X) (resp. (1,2)SO(X), (1,2)PO(X), (1,2)$\alpha(X)$, (1,2)SPO(X)) and (1,2)mO(Y) = (1,2)O(Y) (resp. (1,2)SO(Y), (1,2)PO(Y), (1,2)$\alpha(Y)$, (1,2)SPO(Y)), then by Definition 6.5, we obtain Definition 6.4.

Definition 6.6. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *locally mc-continuous* if $f : (X, (1,2)\text{mO}(X)) \to (Y, (1,2)\text{mO}(Y))$ is locally mc-continuous, where (1,2)mO(X) (resp. (1,2)mO(Y)) is a minimal structure on X (resp. Y) determined by τ_1 and τ_2 (resp. σ_1 and σ_2) as in Definition 6.3.

Hence a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is locally (1,2)-mc-continuous if $f^{-1}(K)$ is locally (1,2)-m-closed in X for each (1,2)-m-closed set K of Y.

Remark 6.3. If (1,2)mO(X) = (1,2)O(X) (resp. (1,2)SO(X), (1,2)PO(X), (1,2)$\alpha(X)$, (1,2)SPO(X)) and (1,2)mO(Y) = (1,2)O(Y) (resp. (1,2)SO(Y), (1,2)PO(Y), (1,2)$\alpha(Y)$, (1,2)SPO(Y)) and f is locally (1,2)-m-continuous, then f is locally (1,2)-c-continuous (resp. locally (1,2)-sc-continuous, locally (1,2)-pc-continuous, locally (1,2)-αc-continuous, locally (1,2)-spc-continuous).

By Definitions 5.5 and 5.6 and Theorem 4.9, we obtain the following theorem.

Theorem 6.5. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopolgical spaces, (1,2)mO(X) and (1,2)mO(Y) m-spaces as in Definition 6.3 and (1,2)mO(X) have property $\mathcal{B}$. Then a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (1,2)-M-continuous if and only if f is (1,2)-mg-continuous and locally (1,2)-mc-continuous.*

Corollary 6.2. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopolgical spaces, (1,2)mO(X) and (1,2)mO(Y) m-spaces as in Definition 6.3 and (1,2)mO(X) have property $\mathcal{B}$. Then for a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties hold:*

(1) f is $\tau_1\tau_2$-continuous if and only if f is (1,2)-g-continuous and

locally (1,2)-c-continuous,

(2) f is (1,2)-semi-continuous if and only if f is (1,2)-sg-continuous and locally (1,2)-sc-continuous,

(3) f is (1,2)-precontinuous if and only if f is (1,2)-pg-continuous and locally (1,2)-pc-continuous,

(4) f is (1,2)-α-continuous if and only if f is (1,2)-αg-continuous and locally (1,2)-αc-continuous,

(5) f is (1,2)-sp-continuous if and only if f is (1,2)-spg-continuous and locally (1,2)-spc-continuous.

Definition 6.7. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopolgical spaces, (1,2)mO(X) and (1,2)mO(Y) m-spaces as in Definition 6.3. Then a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *contra (1,2)-M^*-continuous* if $f : (X, (1,2)\text{mO}(X)) \to (Y, (1,2)\text{mO}(Y))$ is contra M^*-continuous.

Hence f is contra (1,2)-M^*-continuous if $f^{-1}(V)$ is (1,2)-m-closed in X for each (1,2)-m-open set V of Y.

Remark 6.4. If (1,2)mO(X) = (1,2)O(X) (resp. (1,2)SO(X), (1,2)PO(X), (1,2)α(X), (1,2)SPO(X)) and (1,2)mO(Y) = (1,2)O(Y) (resp. (1,2)SO(Y), (1,2)PO(Y), (1,2)α(Y), (1,2)SPO(Y)) and f is contra (1,2)-M^*-continuous, then f is contra $\tau_1\tau_2$-continuous (resp. contra (1,2)-semi-continuous, contra (1,2)-precontinuous, contra (1,2)-α-continuous, contra (1,2)-semi-precontinuous).

By Definition 4.7 and Theorem 5.4, we obtain the following theorem.

Theorem 6.6. *If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (1,2)-mg-continuous and contra (1,2)-M^*-continuous, then it is (1,2)-M-continuous.*

Corollary 6.3. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties hold:*

(1) If f is (1,2)-g-continuous and contra (1,2)-continuous, then it is $\tau_1\tau_2$-continuous,

(2) If f is (1,2)-sg-continuous and contra (1,2)-semi-continuous, then it is (1,2)-semi-continuous,

(3) If f is (1,2)-pg-continuous and contra (1,2)-precontinuous, then it is (1,2)-precontinuous,

(4) If f is (1,2)-αg-continuous and contra (1,2)-α-continuous, then it is (1,2)-α-continuous,

(5) If f is (1,2)-spg-continuous and contra (1,2)-semi-precontinuous, then it is (1,2)-semi-precontinuous.

7. Properties of M-continuity in bitopological spaces

We can obtain some properties of (1,2)-continuity by using the results of M-continuity established in [41].

Definition 7.1. An m-space (X, m_X) is said to be mT_2 [41] if for any distinct points x, y of X, there exist $U, V \in m_X$ such that $x \in U, y \in V$, and $U \cap V = \emptyset$.

Definition 7.2. Let (X, τ_1, τ_2) be a bitopological space and (1,2)mO(X) a minimal structure on X determined by τ_1 and τ_2. Then the space (X, τ_1, τ_2) is said to be *(1,2)-mT_2* if $(X, (1,2)\mathrm{mO}(X))$ is mT_2.

Remark 7.1. If (1,2)mO(X) = (1,2)O(X) (resp. (1,2)SO(X), (1,2)PO(X), (1,2)$\alpha(X)$, (1,2)SPO(X)) and (X, τ_1, τ_2) is (1,2)-mT_2, then X is ultra-T_2 (resp. ultra semi-T_2, ultra pre-T_2, ultra α-T_2, ultra semi-pre-T_2 [18].

Lemma 7.1. (Popa and Noiri [41]). *If $f : (X, m_X) \to (Y, m_Y)$ is an M-continuous injection and (Y, m_Y) is mT_2, then (X, m_X) is mT_2.*

Theorem 7.1. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a (1,2)-M-continuous injection and (Y, σ_1, σ_2) is (1,2)-mT_2, then (X, τ_1, τ_2) is (1,2)-mT_2.*

Proof. The proof follows from Definition 7.2 and Lemma 7.1.

Corollary 7.1. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a $\tau_1\tau_2$-continuous (resp. (1,2)-semi-continuous, (1,2)-precontinuous, (1,2)-α-continuous, (1,2)-semi-precontinuous) injection and (Y, σ_1, σ_2) is ultra-T_2, then (X, τ_1, τ_2) is ultra-T_2 (resp. ultra semi-T_2, ultra pre-T_2, ultra α-T_2, ultra semi-pre-T_2).*

Corollary 7.2. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a (1,2)-semi-irresolute (resp. (1,2)-preirresolute, (1,2)-α-irresolute, (1,2)-semi-preirresolute) injection and (Y, σ_1, σ_2) is ultra semi-T_2 (resp. ultra pre-T_2, ultra α-T_2, ultra semi-pre-T_2), then (X, τ_1, τ_2) is ultra semi-T_2 (resp. ultra pre-T_2, ultra α-T_2, ultra semi-pre-T_2).*

Definition 7.3. Let (X, m_X) be an m-space and K a subset of X.

(1) K is said to be *m-compact* [41] if every cover of K by sets of m_X has a finite subcover,

(2) (X, m_X) is said to be *m-compact* [41] if the subset X is m-compact.

Definition 7.4. Let (X, τ_1, τ_2) be a bitopological space and (1,2)mO(X) a minimal structure on X determined by τ_1 and τ_2.

(1) The space (X, τ_1, τ_2) is said to be *(1,2)-m-compact* if $(X, (1,2)\mathrm{mO}(X))$ is m-compact,

(2) A subset K is said to be *(1,2)-m-compact* if K is m-compact in $(X, (1,2)\mathrm{mO}(X))$.

Remark 7.2. Let (X, τ_1, τ_2) be a bitopological space. If (1,2)mO(X) = (1,2)O(X) (resp. (1,2)SO(X), (1,2)PO(X), (1,2)α(X), (1,2)SPO(X)) and (X, τ_1, τ_2) is (1,2)-m-compact, then X is (1,2)-compact (resp. (1,2)-semi-compact, (1,2)-precompact, (1,2)-α-compact, (1,2)-semi-precompact).

Lemma 7.2. (Popa and Noiri [41]). *Let $f : (X, m_X) \to (Y, m_Y)$ be an M-continuous function. then the following properties hold:*

(1) If K is an m-compact set of X, then f(K) is m-compact in Y,

(2) If f is surjective and (X, m_X) is m-compact, then (Y, m_Y) is m-compact.

Theorem 7.2. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function and (1,2)mO(X) be defined as in Definition 6.3. If f is a (1,2)-M-continuous, the following properties hold:*

(1) If K is a (1,2)-m-compact set of (X, τ_1, τ_2), then f(K) is a (1,2)-m-compact set of (Y, σ_1, σ_2),

(2) If f is surjective and (X, τ_1, τ_2) is (1,2)-m-compact, then (Y, σ_1, σ_2) is (1,2)-m-compact.

Proof. The proof follows from Definition 7.4 and Lemma 7.2.

Corollary 7.3. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $\tau_1\tau_2$-continuous (resp. (1,2)-semi-continuous, (1,2)-precontinuous, (1,2)-α-continuous, (1,2)-semi-precontinuous) function. Then the following properties hold:*

(1) If K is a (1,2)-m-compact (resp. (1,2)-semi-compact, (1,2)-precompact, (1,2)-α-compact, (1,2)-semi-precompact) set of X, then f(K) is a (1,2)-compact set of Y,

(2) If f is surjective and X is (1,2)-compact (resp. (1,2)-semi-compact, (1,2)-precompact, (1,2)-α-compact, (1,2)-semi-precompact), then Y is (1,2)-compact.

Corollary 7.4. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a (1,2)-semi-irresolute (resp. (1,2)-preirresolute, (1,2)-α-irresolute, (1,2)-semi-preirresolute) function. Then the following properties hold:*

(1) If K is a (1,2)-semi-compact (resp. (1,2)-precompact, (1,2)-α-compact, (1,2)-semi-precompact) set of X, then f(K) is a (1,2)-semi-compact (resp. (1,2)-precompact, (1,2)-α-compact, (1,2)-semi-precompact) set of Y,

(2) If f is surjective and X is (1,2)-semi-compact (resp. (1,2)-precompact, (1,2)-α-compact, (1,2)-semi-precompact), then Y is (1,2)-semi-compact (resp. (1,2)-precompact, (1,2)-α-compact, (1,2)-semi-precompact).

Definition 7.5. An m-space (X, m_X) is said to be *m-connected* [41] if X cannot be written as the union of two nonempty disjoint m-open sets of X.

Definition 7.6. Let (X, τ_1, τ_2) be a bitopological space and (1,2)mO(X) as in Definition 6.3. Then (X, τ_1, τ_2) is said to be *(1,2)-m-connected* if $(X, (1, 2)\mathrm{mO}(X))$ is m-connected.

Remark 7.3. Let (X, τ_1, τ_2) be a bitopological space. If (1,2)mO(X) = (1,2)O(X) (resp. (1,2)SO(X), (1,2)PO(X), (1,2)α(X), (1,2)SPO(X)) and (X, τ_1, τ_2) is (1,2)-m-connected, then X is (1,2)-connected (resp. (1,2)-semi-connected, (1,2)-preconnected, (1,2)-α-connected, (1,2)-semi-preconnected).

Lemma 7.3. (Popa and Noiri [41]). *Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. If $f : (X, m_X) \to (Y, m_Y)$ is an M-continuous surjection and (X, m_X) is m-connected, then (Y, m_Y) is m-connected.*

Theorem 7.3. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function and (1,2)mO(X) be defined as in Definition 6.3, where (1,2)mO(X) has property $\mathcal{B}$. If f is a (1,2)-M-continuous surjection and (X, τ_1, τ_2) is (1,2)-m-connected, then (Y, σ_1, σ_2) is (1,2)-m-connected.*

Proof. The proof follows from Definition 7.6 and Lemma 7.3.

Corollary 7.5. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $\tau_1\tau_2$-continuous (resp. (1,2)-semi-continuous, (1,2)-precontinuous, (1,2)-α-continuous, (1,2)-semi-precontinuous) surjection and (X, τ_1, τ_2) is (1,2)-connected (resp. (1,2)-semi-connected, (1,2)-preconnected, (1,2)-α-connected, (1,2)-semi-preconnected), then (Y, σ_1, σ_2) is (1,2)-connected.*

Corollary 7.6. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a (1,2)-semi-irresolute (resp. (1,2)-preirresolute, (1,2)-α-irresolute, (1,2)-semi-preirresolute) surjection and (X, τ_1, τ_2) is (1,2)-semi-connected (resp. (1,2)-preconnected, (1,2)-α-connected, (1,2)-semi-preconnected), then*

(Y, σ_1, σ_2) is (1,2)-semi-connected (resp. (1,2)-preconnected, (1,2)-α-connected, (1,2)-semi-preconnected).

Definition 7.7. A function $f : (X, m_X) \to (Y, m_Y)$ is said to have a *strongly m-closed graph* (resp. *m-closed graph*) [41] if for each $(x, y) \in (X \times Y) - \mathrm{G}(f)$, there exist $U \in m_X$ containing x and $V \in m_Y$ containing y such that $[U \times \mathrm{mCl}(V)] \cap \mathrm{G}(f) = \emptyset$ (resp. $[U \times V] \cap \mathrm{G}(f) = \emptyset$).

Definition 7.8. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function and (1,2)mO(X) and (1,2)mO(Y) be defined as in Definition 6.3. Then f is said to have a *strongly m-closed graph* (resp. *m-closed graph*) if $f : (X, (1, 2)\mathrm{mO}(X)) \to (Y, (1, 2)\mathrm{mO}(Y))$ has a strongly m-closed graph (resp. m-closed graph).

Lemma 7.4. (Popa and Noiri [41]). *Let $f : (X, m_X) \to (Y, m_Y)$ be a function. Then the following properties hold:*

(1) If f is M-continuous and (Y, m_Y) is mT_2, then f has a strongly m-closed graph,

(2) If f is a surjective function with a strongly m-closed graph, then (Y, m_Y) is mT_2,

(3) If m_X has property $\mathcal{B}$ and f is an M-continuous injection with an m-closed graph, then (X, m_X) is mT_2.

Theorem 7.4. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function and (1,2)mO(X) be defined as in Definition 6.3. Then the following properties hold:*

(1) If f is (1,2)-M-continuous and (Y, σ_1, σ_2) is (1,2)-mT_2, then f has a strongly (1,2)-m-closed graph,

(2) If f is a surjective function with a strongly (1,2)-m-closed graph, then (Y, σ_1, σ_2) is (1,2)-mT_2,

(3) If (1,2)mO*(X) has property $\mathcal{B}$ and f is a (1,2)-M-continuous injection with a (1,2)-m-closed graph, then (X, τ_1, τ_2) is (1,2)-mT_2.*

Proof. The proof follows from Definitions 7.7 and 7.8 and Lemma 7.4.

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