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A GENERAL FIXED POINT THEOREM FOR TWO PAIRS OF MAPPINGS SATISFYING A ϕ - IMPLICIT RELATION IN 0 - COMPLETE PARTIAL METRIC SPACES

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Abstract. In this paper a general fixed point theorem for two pairs of mappings satisfying a ϕ - implicit relation is proved. As application we obtain a fixed point theorem for a sequence of mappings in 0 - complete partial metric spaces, different by results from [24].

1. Introduction

In 1994, Matthews [16] introduced the concept of partial metric spaces as a part of the study of denotional semantics of dataflow networks and proved the Banach contraction principle in such spaces.

Recently, in [2], [5], [7], [13], [14], some fixed point theorems under various contractive conditions are proved.

Romaguera [21] introduced the notions of 0 - Cauchy sequence and 0 - complete partial metric spaces, proving some characterizations of partial metric spaces in terms of completeness and 0 - completeness.

Some fixed point theorems for mappings in 0 - complete partial metric spaces are proved in [4], [17], [8], [10], [23] and in other papers.

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In 1994, Pant [18] introduced the notion of R - weakly commutativity, which is equivalent to commutativity in coincidence points. In [12] is introduced the notion of weakly compatible mappings.

In [13] some fixed point theorems for weakly compatible mappings in partial metric spaces are proved.

Some classical fixed point theorems and common fixed point theorems in metric spaces have been unified considering a general condition by an implicit relation in [19], [20].

Some fixed point theorems for mappings satisfying implicit relations in partial metric spaces are proved in [9] - [11], [23].

In [6], Altun and Türkoglu introduced a new type of implicit relations satisfying a ϕ - maps.

The purpose of this paper is to prove a general fixed point theorem for two pairs of mappings satisfying a new type of ϕ - implicit relation in 0 - complete partial metric spaces including and a Hardy - Rogers type theorem.

As application, we prove a fixed point theorem for a sequence of mappings in 0 - complete partial metric spaces different by the result from [24].

2. Preliminaries

Definition 2.1 ([16]). Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}_+$ is said to be a partial metric on X if for every $x, y, z \in X$, the following conditions hold:
$(P_1) : p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$,
$(P_2) : p(x,x) \leq p(x,y)$,
$(P_3) : p(x,y) = p(y,x)$,
$(P_4) : p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is called a partial metric space.

If $p(x,y) = 0$ then by (P_1) and (P_2), $x = y$, but the converse does not always hold.

Each partial metric p on X generates a T_0 - topology τ_p which has as base the family of open p - balls $\{B_p(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ is a metric on X.

A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$ with respect to τ_p if and only if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.

Lemma 2.2 ([2], [14]). *Let (X,p) be a partial metric space and $\{x_n\}$ a sequence in X such that $x_n \to z$ as $n \to \infty$, where $p(z,z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z,y)$ for all $y \in X$.*

Definition 2.3 ([16], [21]). a) A sequence $\{x_n\}$ in a partial metric space (X,p) is called Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
b) A partial metric space (X,p) is said to be complete if every Cauchy sequence in X converges with respect to τ_p to a point $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = p(x,x)$.
c) A sequence $\{x_n\}$ in a partial metric space (X,p) is called 0 - Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.
d) A partial metric space (X,p) is said to be 0 - complete if every 0 - Cauchy sequence in X converges with respect to τ_p to a point $x \in X$ such that $p(x,x) = 0$.

Lemma 2.4 ([16], [21], [22]). *Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X.*
a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if $\{x_n\}$ is a Cauchy sequence in metric space (X,d_p).
b) (X,p) is complete if and only if (X,d_p) is complete.
Further more, $\lim_{n\to\infty} d_p(x_n, x) = 0$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n.m\to\infty} p(x_n, x_m)$.
c) Every 0 - Cauchy sequence in (X,p) is Cauchy in (X,d_p).
d) If (X,p) is complete, then is 0 - complete.

The converses of assertions c) and d) are not true.

Definition 2.5. A function $\varphi : [0,\infty) \to [0,\infty)$ is a ϕ - function, $\varphi \in \phi$ if φ is continuous, nondecreasing such that $\sum\limits_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$ and $\varphi(0) = 0$.

Remark 2.6. *Since $\sum\limits_{n=1}^{\infty} \varphi^n(t) < \infty$, $\lim_{n\to\infty} \varphi^n(t) = 0$, then as in [16], $\varphi(t) < t$, $\forall t > 0$.*

3. ϕ - IMPLICIT RELATIONS

Definition 3.1. Let $\mathcal{F}_\phi$ be the set of all continuous functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ such that:
(F_1) : F is nonincreasing in variables $t_2, t_3, ..., t_6$,
(F_2) : There exists $\varphi \in \phi$ such that for all $u, v \geq 0$
(F_{2a}) : $F(u, v, v, u, u+v, v) \leq 0$
and

$(F_{2b}) : F(u,v,u,v,v,u+v) \leq 0$
implies $u \leq \varphi(v)$.

In the following examples, the proofs of (F_1) are obviously.

Example 3.2. *$F(t_1,...,t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a,b,c,d,e \geq 0$ and $a+b+c+2d+2e < 1$.*
(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,u+v,v) = u - av - bv - cu - d(u+v) - ev \leq 0$. If $u > v$, then $u[1-(a+b+c+2d+e)] \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq (a+b+c+2d+e)v \leq (a+b+c+2d+2e)v$ and (F_2) is satisfied for $\varphi(t) = (a+b+c+2d+2e)t$.

Example 3.3. *$F(t_1,...,t_6) = t_1 - k\max\{t_2,t_3,...,t_6\}$, where $k \in [0,\frac{1}{2})$.*
(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,u+v,v) = u - k(u+v) \leq 0$ which implies $u \leq \frac{k}{1-k}v$. Similarly, $F(u,v,u,v,v,u+v) \leq 0$ implies $u \leq \frac{k}{1-k}v$ and (F_2) is satisfied for $\varphi(t) = \frac{k}{1-k}t$.

Example 3.4. *$F(t_1,...,t_6) = t_1 - k\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{3}\right\}$, where $k \in (0,1)$.*
(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,u+v,v) = u - k\max\left\{u,v,\frac{2v+u}{3}\right\} \leq 0$. If $u > v$, then $u(1-k) \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq kv$. Similarly, $F(u,v,u,v,v,u+v) \leq 0$ implies $u \leq kv$ and (F_2) is satisfied for $\varphi(t) = kt$.

Example 3.5. *$F(t_1,...,t_6) = t_1 - \max\{at_2, b(t_3+t_4), b(t_5+t_6)\}$, where $a \in (0,1)$ and $b \in \left[0,\frac{1}{3}\right)$.*
(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,u+v,v) = u - \max\{av, b(v+2u), b(u+2v)\} \leq 0$. If $u > v$, then $u[1-\max\{a,3b\}] \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq \max\{a,3b\}v$. Similarly, $F(u,v,u,v,v,u+v) \leq 0$ implies $u \leq \max\{a,3b\}v$ and (F_2) is satisfied for $\varphi(t) = \max\{a,3b\}t$.

Example 3.6. *$F(t_1,...,t_6) = t_1^2 - a\max\{t_2^2,t_3^2,t_4^2\} - bt_5t_6$, where $a,b \geq 0$ and $a+2b < 1$.*
(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,u+v,v) = u^2 - a\max\{u^2,v^2\} - bv(u+v) \leq 0$. If $u > v$, then $u^2[1-(a+2b)] \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq \sqrt{a+2b}v$. Similarly, $F(u,v,u,v,v,u+v) \leq 0$ implies $u \leq \sqrt{a+2b}v$ and (F_2) is satisfied for $\varphi(t) = \sqrt{a+2b}t$.

Example 3.7. *$F(t_1,...,t_6) = t_1 - \varphi(at_2+bt_3+ct_4+dt_5+et_6)$, where $a,b,c,d,e \geq 0$, $a+b+c+2d+e < 1$and $\varphi \in \phi$.*

(F_2) : *Let* $u, v \geq 0$ *and* $F(u, v, v, u, u+v, v) = u - \varphi(av + bv + cu + d(u+v) + ev) \leq 0$. *If* $u > v$, *then* $u[1 - \varphi((a+b+c+2d+e)u)] \leq 0$, *a contradiction. Hence,* $u \leq v$ *which implies* $u \leq \varphi((a+b+c+2d+e)v) = \varphi(v)$. *Similarly,* $F(u, v, u, v, v, u+v) \leq 0$ *implies* $u \leq \varphi(v)$.

Example 3.8. $F(t_1, ..., t_6) = t_1 - \varphi\left(\max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{3}\right\}\right)$, *where* $\varphi \in \phi$.

The proof is similar to the proof from Example 3.7.

4. Main results

Lemma 4.1 ([1]). *Let* f, g *be weakly compatible mappings of a nonempty set* X. *If* f *and* g *have a unique point of coincidence* $w = fx = gx$ *for some* $x \in X$, *then* w *is the unique common fixed point of* f *and* g.

Theorem 4.2. *Let* (X, p) *be a partial metric space and* A, B, S *and* T *self mappings of* X *satisfying the condition*

$$F\left(\begin{array}{c} p(Ax, By), p(Sx, Ty), p(Sx, Ax), \\ p(Ty, By), p(Sx, By), p(Ty, Ax) \end{array}\right) \leq 0 \tag{4.1}$$

for all $x, y \in X$ *and* $F \in \mathcal{F}_\phi$.

If there exists $u, v \in X$ *such that* $Su = Au$ *and* $Tv = Bv$, *then there exists* $t \in X$ *such that* t *is the unique point of coincidence of* A *and* S, *as well is the unique point of coincidence of* B *and* T.

Proof. First we prove that $Su = Tv$. We suppose that $Su \neq Tv$. Then by (4.1) we get

$$F\left(\begin{array}{c} p(Au, Bv), p(Su, Tv), p(Su, Au), \\ p(Tv, Bv), p(Su, Bv), p(Tv, Au) \end{array}\right) \leq 0, \tag{4.2}$$

$$F\left(\begin{array}{c} p(Su, Tv), p(Su, Tv), p(Su, Su), \\ p(Tv, Tv), p(Su, Tv), p(Tv, Su) \end{array}\right) \leq 0. \tag{4.3}$$

By (P_2),

$$\begin{aligned} p(Su, Su) &\leq p(Su, Tv), \\ p(Tv, Tv) &\leq p(Su, Tv). \end{aligned}$$

By (F_1) we have

$$F\left(\begin{array}{c} p(Su, Tv), p(Su, Tv), p(Su, Tv), \\ p(Su, Tv), 2p(Su, Tv), p(Su, Tv) \end{array}\right) \leq 0 \tag{4.4}$$

which implies

$$p(Su,Tv) \leq \varphi(p(Su,Tv)) < p(Su,Tv), \tag{4.5}$$

a contradiction. Hence, $p(Su,Tv) = 0$, i.e. $Su = Tv$. Therefore,

$$Au = Su = Tv = Bv = t \tag{4.6}$$

for some $t \in X$.

Assume that there exists $w \neq u$ such that $Aw = Sw \neq Au$. Then by (4.1) we obtain

$$F\left(\begin{array}{c} p(Aw,Bv), p(Sw,Tv), p(Sw,Aw), \\ p(Tv,Bv), p(Sw,Bv), p(Tv,Aw) \end{array}\right) \leq 0, \tag{4.7}$$

$$F\left(\begin{array}{c} p(Sw,Tv), p(Sw,Tv), p(Sw,Sw), \\ p(Tv,Tv), p(Sw,Tv), p(Tv,Sw) \end{array}\right) \leq 0. \tag{4.8}$$

By (P_2),

$$\begin{array}{rcl} p(Sw,Sw) & \leq & p(Sw,Tv), \\ p(Tv,Tv) & \leq & p(Sw,Tv). \end{array}$$

By (F_1) we obtain

$$F\left(\begin{array}{c} p(Sw,Tv), p(Sw,Tv), p(Sw,Tv), \\ p(Sw,Tv), 2p(Sw,Tv), p(Sw,Tv) \end{array}\right) \leq 0. \tag{4.9}$$

By (F_{2a}) we obtain

$$p(Sw,Tv) \leq \varphi(p(Sw,Tv)) < p(Sw,Tv), \tag{4.10}$$

a contradiction if $\varphi(p(Sw,Tv)) > 0$. Hence

$$p(Sw,Tv) = 0, \tag{4.11}$$

which implies

$$Sw = Aw = Su = Au = Tv = Bv = t. \tag{4.12}$$

Hence t is the unique point of coincidence of A and S. Similarly, t is the unique point of coincidence of B and T. □

Theorem 4.3. *Let (X,p) be a 0 - complete partial metric space and A, B, S and T be self mappings on X such that $A(X) \subset T(X)$, $B(X) \subset S(X)$ and the inequality (4.1) holds for all $x, y \in X$, where $F \in \mathcal{F}_\phi$. If one of $A(X), B(X), S(X), T(X)$ is closed then*
1) A and S have a point of coincidence,
2) B and T have a point of coincidence.

Moreover, if the pairs $\{A, B\}$ and $\{S, T\}$ are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary, but fixed point in X. Since $A(X) \subset T(X)$, there exists $x_1 \in X$ such that $Tx_1 = Ax_0$. Since $B(X) \subset S(X)$, there exists $x_2 \in X$ such that $Sx_2 = Bx_1$. Continuing this process we construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_{2n} = Tx_{2n+1} = Ax_{2n},\ y_{2n+1} = Sx_{2n+2} = Bx_{2n+1},\ n \in \mathbb{N}. \tag{4.13}$$

We prove that $\{y_n\}$ is a 0 - Cauchy sequence in (X, p).

By (4.1) for $x = x_{2n}$ and $y = x_{2n+1}$ we have

$$F\begin{pmatrix} p(Ax_{2n}, Bx_{2n+1}), p(Sx_{2n}, Tx_{2n+1}), \\ p(Sx_{2n}, Ax_{2n}), p(Tx_{2n+1}, Bx_{2n+1}), \\ p(Sx_{2n}, Bx_{2n+1}), p(Tx_{2n+1}, Ax_{2n}) \end{pmatrix} \leq 0, \tag{4.14}$$

$$F\begin{pmatrix} p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n}), p(y_{2n-1}, y_{2n}), \\ p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n+1}), p(y_{2n}, y_{2n}) \end{pmatrix} \leq 0. \tag{4.15}$$

Since by (P_4),

$$p(y_{2n-1}, y_{2n+1}) \leq p(y_{2n-1}, y_{2n}) + p(y_{2n}, y_{2n+1}) \tag{4.16}$$

and by (P_2),

$$p(y_{2n}, y_{2n}) \leq p(y_{2n-1}, y_{2n}), \tag{4.17}$$

then by (F_1) and (4.15) we obtain

$$F\begin{pmatrix} p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n}), \\ p(y_{2n-1}, y_{2n}), p(y_{2n}, y_{2n+1}), \\ p(y_{2n-1}, y_{2n}) + p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n}) \end{pmatrix} \leq 0, \tag{4.18}$$

which implies by (F_{2a}) that

$$p(y_{2n}, y_{2n+1}) \leq \varphi(p(y_{2n-1}, y_{2n})). \tag{4.19}$$

Similarly, by (4.1) for $x = x_{2n+2}$ and $y = x_{2n+1}$ we obtain

$$F\begin{pmatrix} p(Ax_{2n+2}, Bx_{2n+1}), p(Sx_{2n+2}, Tx_{2n+1}), \\ p(Sx_{2n+2}, Ax_{2n+2}), p(Tx_{2n+1}, Bx_{2n+1}), \\ p(Sx_{2n+2}, Bx_{2n+1}), p(Tx_{2n+1}, Ax_{2n+2}) \end{pmatrix} \leq 0, \tag{4.20}$$

$$F\begin{pmatrix} p(y_{2n+2}, y_{2n+1}), p(y_{2n+1}, y_{2n}), \\ p(y_{2n+1}, y_{2n+2}), p(y_{2n}, y_{2n+1}), \\ p(y_{2n+1}, y_{2n+1}), p(y_{2n}, y_{2n+2}) \end{pmatrix} \leq 0. \tag{4.21}$$

Since by (P_4),

$$p(y_{2n}, y_{2n+2}) \leq p(y_{2n}, y_{2n+1}) + p(y_{2n+1}, y_{2n+2}) \tag{4.22}$$

and by (P_2),

$$p\left(y_{2n+1},y_{2n+1}\right)\leq p\left(y_{2n},y_{2n+1}\right), \tag{4.23}$$

by (F_1) and (4.21) we obtain

$$F\left(\begin{array}{c} p\left(y_{2n+2},y_{2n+1}\right),p\left(y_{2n},y_{2n+1}\right), \\ p\left(y_{2n+2},y_{2n+1}\right),p\left(y_{2n},y_{2n+1}\right), \\ p\left(y_{2n},y_{2n+1}\right),p\left(y_{2n+2},y_{2n+1}\right)+p\left(y_{2n},y_{2n+1}\right) \end{array}\right)\leq 0. \tag{4.24}$$

By (F_{2b}) we obtain

$$\varphi\left(p\left(y_{2n+2},y_{2n+1}\right)\right)\leq\varphi\left(p\left(y_{2n},y_{2n+1}\right)\right) \tag{4.25}$$

which implies

$$p\left(y_n,y_{n+1}\right)\leq\varphi\left(p\left(y_{n-1},y_n\right)\right)\leq\ldots\leq\varphi^n\left(p\left(y_0,y_1\right)\right). \tag{4.26}$$

For $n,m\in\mathbb{N}$ with $m>n$ we obtain by (P_4) that

$$\begin{aligned} p\left(y_n,y_m\right) &\leq p\left(y_n,y_{n+1}\right)+p\left(y_{n+1},y_{n+2}\right)+\ldots+p\left(y_{m-1},y_m\right) \\ &= \sum_{k=n}^{m-1}\varphi^k\left(p\left(y_0,y_1\right)\right). \end{aligned}$$

Since

$$\sum_{k=1}^{\infty}\varphi^k\left(p\left(y_0,y_1\right)\right)<\infty, \tag{4.27}$$

then

$$\lim_{n,m\to\infty}\sum_{k=n}^{m-1}\varphi^k\left(p\left(y_0,y_1\right)\right)=0 \tag{4.28}$$

and

$$\lim_{n,m\to\infty}p\left(y_n,y_m\right)=0 \tag{4.29}$$

and so $\{y_n\}$ is a 0 - Cauchy sequence in (X,p). Since (X,p) is 0 - complete, then there exists $y\in X$ such that

$$p\left(y,y\right)=\lim_{n,m\to\infty}p\left(y_n,y_m\right)=\lim_{n\to\infty}p\left(y_n,y\right)=0. \tag{4.30}$$

Hence

$$\lim_{n\to\infty}y_{2n}=\lim_{n\to\infty}y_{2n+1}=y. \tag{4.31}$$

Now we can suppose without loss of generality that $S\left(X\right)$ is a closed subset of (X,p). Then there exists $u\in X$ such that $y=Su$.

By (4.1) with $x = u$ and $y = x_{2n+1}$ we have

$$F\begin{pmatrix} p\left(Au, Bx_{2n+1}\right), p\left(Su, Tx_{2n+1}\right), \\ p\left(Su, Au\right), p\left(Tx_{2n+1}, Bx_{2n+1}\right), \\ p\left(Su, Bx_{2n+1}\right), p\left(Tx_{2n+1}, Au\right) \end{pmatrix} \leq 0, \tag{4.32}$$

$$F\begin{pmatrix} p\left(Au, y_{2n+1}\right), p\left(Su, y_{2n}\right), p\left(Su, Au\right), \\ p\left(y_{2n}, y_{2n+1}\right), p\left(Su, y_{2n+1}\right), p\left(y_{2n}, Au\right) \end{pmatrix} \leq 0. \tag{4.33}$$

Letting n tends to infinity by Lemma 2.2 we obtain

$$F\left(p\left(Au, y\right), 0, p\left(Au, y\right), 0, 0, p\left(y, Au\right)\right) \leq 0. \tag{4.34}$$

By (F_{2b})

$$p\left(Au, y\right) \leq \varphi\left(0\right) = 0 \tag{4.35}$$

which implies

$$y = Au = Su, \tag{4.36}$$

hence A and S have a coincidence point.

Since $A\left(X\right) \subset T\left(X\right)$, $y \in T\left(X\right)$, then there exists $v \in X$ such that $y = Tv$.

By (4.1) with $x = u$ and $y = x_{2n+1}$ we obtain

$$F\begin{pmatrix} p\left(Au, Bv\right), p\left(Su, Tv\right), p\left(Su, Au\right), \\ p\left(Tv, Bv\right), p\left(Su, Bv\right), p\left(Tv, Au\right) \end{pmatrix} \leq 0, \tag{4.37}$$

$$F\left(p\left(y, Bv\right), 0, 0, p\left(y, Bv\right), p\left(y, Bv\right), 0\right) \leq 0. \tag{4.38}$$

By (F_{2a}) we have $p\left(y, Bv\right) = 0$, i.e. $y = Bv = Tv$ and B and T have a coincidence point. By Theorem 4.2, y is the unique point of coincidence of (A, S) and (B, T). If (A, S) and (B, T) are weakly compatible, by Lemma 4.1, y is the unique common fixed point of A, B, S and T. □

Remark 4.4. *By Example 3.2 and Theorem 4.3 we obtain a theorem of Hardy - Rogers type.*

For a function $f : X \to X$ we denote

$$Fix\,(f) = \{x \in X : x = fx\}.$$

Theorem 4.5. *Let A, B, S and T be self mappings of a partial metric space. If the inequality (4.2) holds for all $x, y \in X$, $F \in \mathcal{F}_\phi$, then*

$$[Fix\,(S) \cap Fix\,(T)] \cap Fix\,(A) = [Fix\,(S) \cap Fix\,(T)] \cap Fix\,(B)\,. \tag{4.39}$$

Proof. Let $x \in [Fix\,(S) \cap Fix\,(T)] \cap Fix\,(A)$. Then, by (4.1) we have

$$F\left(\begin{array}{c} p\,(Ax, Bx)\,, p\,(Sx, Tx)\,, p\,(Sx, Ax)\,, \\ p\,(Tx, Bx)\,, p\,(Sx, Bx)\,, p\,(Tx, Ax) \end{array}\right) \leq 0, \tag{4.40}$$

$$F\left(\begin{array}{c} p\,(x, Bx)\,, p\,(x, x)\,, p\,(x, x)\,, \\ p\,(x, Bx)\,, p\,(x, Bx)\,, p\,(x, x) \end{array}\right) \leq 0. \tag{4.41}$$

By (P_2),

$$p\,(x, x) \leq p\,(x, Bx)\,. \tag{4.42}$$

Then by (F_1) we have

$$F\left(\begin{array}{c} p\,(x, Bx)\,, p\,(x, Bx)\,, p\,(x, Bx)\,, \\ p\,(x, Bx)\,, 2p\,(x, Bx)\,, p\,(x, Bx) \end{array}\right) \leq 0 \tag{4.43}$$

which implies by (F_{2a}) that

$$p\,(x, Bx) \leq \varphi\,(p\,(x, Bx)) < p\,(x, Bx)\,, \tag{4.44}$$

if $p\,(x, Bx) > 0$, a contradiction.

Hence $p\,(x, Bx) = 0$ and $x = Bx$. So,

$$[Fix\,(S) \cap Fix\,(T)] \cap Fix\,(A) \subset [Fix\,(S) \cap Fix\,(T)] \cap Fix\,(B)\,. \tag{4.45}$$

Similarly,

$$[Fix\,(S) \cap Fix\,(T)] \cap Fix\,(B) \subset [Fix\,(S) \cap Fix\,(T)] \cap Fix\,(A)\,. \tag{4.46}$$

□

Theorems 4.3 and 4.5 implies the following one.

Theorem 4.6. *Let S, T and $\{A_i\}_{i \in \mathbb{N}^*}$ be self mappings of a 0 - complete partial metric space such that*

a) $A_2\,(X) \subset S\,(X)$ *and* $A_1\,(X) \subset T\,(X)$*,*
b) *one of* $S\,(X)$ *and* $T\,(X)$ *are closed,*
c) *the pairs* $\{A_1, S\}$ *and* $\{A_2, T\}$ *are weakly compatible,*
d) *the inequality*

$$F\left(\begin{array}{c} p\,(A_i x, A_{i+1} y)\,, p\,(Sx, Ty)\,, p\,(Sx, A_i x)\,, \\ p\,(Ty, A_{i+1} y)\,, p\,(Sx, A_{i+1} y)\,, p\,(Ty, A_i x) \end{array}\right) \leq 0 \tag{4.47}$$

holds for all $x, y \in X, i \in \mathbb{N}^*$ *and* $F \in \mathcal{F}_\phi$.

Then S, T *and* $\{A_i\}_{i\in\mathbb{N}^*}$ *have a unique common fixed point.*

Remark 4.7. *By Theorem 4.6 and Examples 3.2 - 3.8 we obtain new particular results.*

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