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SPLINES FOR THE SET FUNCTIONS

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Abstract. Following the multiple possibilities to approximate any set function using proper sequences of countable additive set - functions and our main conclusions given in some previous research works concerning the best approximation, we present the adequate splines for the countable additive set - functions. This work is a sequel by completion of the most important results contained in the corresponding references.

1. INTRODUCTION

We introduced the concept of *Spline Function (Postolică, V., 1981)*, in an arbitrary *H - locally convex space*, concept defined by *(Precupanu, T., 1968) and also studied by Krammar, E., 1981)* as any Hausdorff locally convex space $(X, P = \{p_\alpha : \alpha \in A\})$ with every semi-norms p_α satisfying the parallelogram law:

$$p_\alpha^2(x+y) + p_\alpha^2(x-y) = 2\left[p_\alpha^2(x) + p_\alpha^2(y)\right], \forall x, y \in X.$$

This is *equivalent* to say that *each semi-norm* p_α is generated by *a scalar semi-product* by $p_\alpha(x) = \sqrt{[x,x]_\alpha}, \forall x \in X$ *(Jordan-von Neumann, 1935).* We established and developed the basic properties of Approximation and Optimal interpolation for these Splines, with the appropriate extensions in *((Isac, G., Postolică, V., 1993, Postolică, V., 1981, 1998, 2001, 2002 and other connected works).*

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Our splines are natural *generalizations* in *the H - locally convex spaces* of *the usual Abstract Splines* which appear in *Hilbert's Spaces* like *the minimizing elements* for *an arbitrary semi – norm* subject to *the restrictions* given by a set of *linear and continuous functionals* and they were constructed as follows. Let $(X, P = \{p_\alpha : \alpha \in I\})$be a H - locally convex space with each semi - norm $p_\alpha \in P$ being induced by a scalar semi - product $(.,.)_\alpha (\alpha \in I)$. Let M be a (closed) linear subspace of X for which there exists a H - locally convex space

$(Y, Q = \{q_\alpha : \alpha \in I\})$with every semi - norm $q_\alpha \in Q$ generated by a scalar semi-product $< .,. >_\alpha$ $(\alpha \in I)$ and a linear (possibly continuous) operator U:X→Y such that

$M = \{x \in X\text{: } (x, y)_\alpha =< U_x, U_y >_\alpha \ \forall \alpha \in I\}$. *The space of spline functions with respect to U* was defined in *(Postolică, V., 1981)* as the *U - orthogonal of M,* that is, $M^\perp = \{x \in X :< U_x, U_\zeta >_\alpha = 0, \forall \zeta \in M, \alpha \in I\}$.

Clearly, $M^\perp$ is *the orthogonal* of M *in the H - locally convex sense* because $(x, y)_\alpha = 0, \forall x \in M, y \in M^\perp, \alpha \in I$.

Let us consider the direct sum *X'=M⊕*$M^\perp$ and for every *x∈X'* we denote its projection onto $M^\perp$ by s_x. Then, taking into account the **Theorem 4** in *(Postolică, V., 1981),* it follows that this spline is *the only P-best simultaneous U - approximation* of x with respect to $M^\perp$ since it satisfies the condition

$p_\alpha(x - s_x) \le p_\alpha(x - y) \forall y \in M^\perp, p_\alpha \in P$. Moreover, following the results given in **Chapter 3** of *(Isac, G., Postolică, V., 1993)* and the **Theorem 3** in *(Postolică, V., 1981)*, we have

Theorem 1. *(Postolică, V., 2000, 2002).*

(i) for every $x \in X'$ *the only elements of best simultaneous, vectorial approximation and optimal interpolation with respect to any family of seminorms which generates the*

H - locally convex topology on X by the linear subspace of splines with respect to the operator U are the spline functions sx: $q_\alpha(Us_x) = \min\{q_\alpha(Uy) : y - s_x \in M\} = \min\{q_\alpha(Uy) : y - x \in M\}, \forall \alpha \in I$;

Moreover, if M and $M^\perp$ *supply an orthogonal decomposition for X, that is X=M⊕*$M^\perp$*, then* $M^\perp$ *is simultaneous and vectorial proximinal, that is, for each* $x \in X$, $(p_\alpha(x - s_x)) \le (p_\alpha(x - y)), \forall y \in M^\perp$ *(the P-simultaneous proximinality) and there exists no* $t \in M^\perp$*such that* $(p_\alpha(x - t)) < (p_\alpha(x - s_x))$ *(theP-vectorial proximinality), where* $a = ((a_\alpha)) \le b = ((b_\alpha))$ *in* R^I *means* $a_\alpha \le b_\alpha, \forall \alpha \in I$ *and* $a < b \Leftrightarrow a \le b$ *with* $a \ne b$ *;*

(ii) $q_\alpha(U(s_x - x)) \leq q_\alpha(U(\eta - x)), \forall\alpha \in I, \forall\eta \in M^\perp$, *with equality if and only if* $q_\alpha(U(\eta - s_x)) = 0$ *for every* $\alpha \in I$.

(iii) $\min\limits_{y\in M^\perp} \sup\limits_{\alpha\in I} p_\alpha(x - y) = \sup\limits_{\alpha\in I} p_\alpha(x - s_x), \forall x \in X.$

Finally, let us consider a significant application to *the best approximation* and *the optimal interpolation* of *the countable additive set functions* of *bounded 2 – variations* on *a separable subspace* of *an arbitrary metric space* following the paragraph 3 in **Chapter 1** of *(Isac, G., Postolică, V., 1993)* and (*Postolică, V., 2001)* and with the main property that the subspaces M and $M^\perp$ realize a *natural orthogonal decomposition.* Let X be *a metric space*, $Y \subseteq X$ *a separable subspace*, $B(Y)$ the σ - *algebra* of all *Borel subsets* in Y and $\mu : B(Y) \to R_+$ *a measure.* If one denotes by $BV_p(Y, \mu)$ *the class of all countable additive set functions* $F : B(Y) \to R$ with$p \geq 1$ bounded variation on Y with respec to μ , then

Theorem 2. *(Postolică, V., 1981). A function F belongs to $BV_p(Y, \mu)$ whenever $p > 1$ if and only if there exists a function $f \in L^p(Y)$ such that*

$$F(A) = \int_A f d\mu, \forall A \in B(Y).$$

In all these cases, the total p – variation of the set function F is

$$V_p(F, Y, \mu) = (\int_Y |f|^p \, d\mu)^{1/p}.$$

Moreover, the linear space $BV_2^C(Y, \mu)$ of all countable set functions with bounded 2 – variation on Y with respect to μ is a genuine Hilbert space (Postolică, V.,2000, 2002) following the scalar product below indicated.

Simple examples show that, if the set function F is not countable additive or $p = 1$, then the above theorem loses its validity in the sense that not any real extended values set function defined on an arbitrary σ*-algebra* Ω and having *p-bounded variation* on a non-empty set $T \in \Omega$ allows such an integral representation as this and the next considerations concerning the set functions fail.

Let now consider a (possible closed) linear subspace Y_0 of $L^2(Y)$. Therefore, Y_0 and its orthogonal

$Y_0^\perp = \left\{ f \in L^2(Y) : \int\limits_Y f \cdot g d\mu = 0, \forall g \in Y_0 \right\}$ realize an *orthogonal decomposition* of the space $L^2(Y)$. Hence, for every function $f \in L^2(Y)$

we have $f = f_0 + f_0^{\perp}$ with $f_0 \in Y_0$ and $f_0^{\perp} \in Y_0^{\perp}$. If one considers *(Isac, G., Postolică, V., 1993,Postolică, V., 2001)* $BV_2^C(Y,\mu)$ endowed with the topology of Hilbert space generated by the scalar product $(.,.)$ defined by $(F,G) = \int\limits_Y f \cdot g d\mu$ for every $F, G \in BV_2^c(Y,\mu)$ given by $F(A) = \int\limits_A f d\mu$, $G(A) = \int\limits_A g d\mu$, $\forall A \in B(Y)$ with $f, g \in L^2(Y)$, $L^2(Y)$ endowed with the usual topology generated also by the scalar product $\langle f, g\rangle = \int\limits_Y f_0^{\perp} \cdot g_0^{\perp} d\mu + \int\limits_Y f \cdot g d\mu$, $\forall f = f_0 + f_0^{\perp}$, $g = g_0 + g_0^{\perp} \in L^2(Y)$, $\left(f_0, g_0 \in Y_0, f_0^{\perp}, g_0^{\perp} \in Y_0^{\perp}\right)$ and the *linear continuous operator* $U : BV_2^C(Y) \to L^2(Y)$ is defined by $U(F) = f$ for every $F \in BV_2^C(Y,\mu)$ with $F(A) = \int\limits_A f d\mu, \forall A \in B(Y)$, $f \in L^2(Y)$, then

Theorem 3 (*Postolică, V., 2002)*.

(i) For every set function $F \in BV_2^C(Y,\mu)$, $F(A) = \int\limits_A f d\mu, \forall A \in B(Y)$, *with* $f \in L^2(Y)$, *we have* $F = F_0 + F_0^{\perp}$ *where* $F_0(A) = \int\limits_A f_0 d\mu$, $F_0^{\perp}(A) = \int\limits_A f_0^{\perp} d\mu$, $A \in B(Y)$, $f = f_0 + f_0^{\perp}$, $f_0 \in Y_0$, $f_0^{\perp} \in Y_0^{\perp}$:

(ii) $M = \left\{F \in BV_2^C(Y) : F(A) = \int\limits_A f d\mu, f \in Y_0, A \in B(Y)\right\}$

$$M^{\perp} = \left\{G \in BV_2^C(Y) : G(A) = \int_A g d\mu, g \in Y_0^{\perp}, A \in B(Y)\right\}$$

realize an orthogonal decomposition of the Hilbert space $BV_2^C(Y,\mu)$ *that is,*
$BV_2^C(Y,\mu) = M \oplus M^{\perp}$.

(iii) $\int\limits_Y |f_0|^2 d\mu = \min\left\{\int\limits_Y |f - g|^2 d\mu : g \in Y_0^{\perp}\right\}, \forall f = f_0 + f_0^{\perp}$ *with* $f_0 \in Y_0$ *and* $f_0^{\perp} \in Y_0^{\perp}$.

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