"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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ON THE GROWTH OF SOLUTIONS OF HOMOGENEOUS AND NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS

MANSOURIA SAIDANI AND BENHARRAT BELAÏDI

Abstract. In this paper, we investigate the growth of solutions of higher order linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0$$

and

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z),$$

where $A_0(z) \not\equiv 0$, $A_1(z), \cdots, A_{k-1}(z)$ and $F(z) \not\equiv 0$ are meromorphic functions of finite iterated p-order. We improve and extend some results of papers [1] and [5] by using the concept of the iterated order and considering the growth of some arbitrary dominant coefficient A_s $(s = 0, 1, \cdots, k-1)$ instead of A_0.

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1. Introduction and main results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We shall use the standard notations in Nevanlinna value distribution theory of meromorphic functions [12, 20, 21], such as $T(r,f)$, $N(r,f)$, $m(r,f)$. For the definition of iterated order of meromorphic function, we use the same definition as in [15], [17]. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$.

Definition 1.1 [15, 17] Let f be a meromorphic function. Then the iterated p-order $\rho_p(f)$ of f is defined by

$$\rho_p(f) := \limsup_{r\to\infty} \frac{\log_p^+ T(r,f)}{\log r}, \quad (p \geq 1 \text{ is an integer}).$$

For $p = 1$, this notation is called order and for $p = 2$ hyper-order.

Definition 1.2 [15] The finiteness degree of the order of a meromorphic function f is defined by

$$i(f) := \begin{cases} 0, & \text{for } f \text{ rational}, \\ \min\{j \in \mathbb{N} : \rho_j(f) < \infty\}, & \text{for } f \text{ transcendental} \\ \quad \text{for which some } j \in \mathbb{N} \text{ with } \rho_j(f) < \infty \text{ exists}, & \\ +\infty, & \text{for } f \text{ with } \rho_j(f) = +\infty,\ \forall j \in \mathbb{N}. \end{cases}$$

Definition 1.3 [15] Let $n(r,a,f)$ be the unintegrated counting function for the sequence of a-points of a meromorphic function f. Then the iterated convergence exponent of a-points of f is defined by

$$\lambda_p(f,a) := \limsup_{r\to\infty} \frac{\log_p^+ n(r,a,f)}{\log r}, \quad (p \geq 1 \text{ is an integer}).$$

In the definition of the iterated convergence exponent, we may replace $n(r,a,f)$ with the integrated counting function $N(r,a,f)$, and we have

$$\lambda_p(f,a) := \limsup_{r\to\infty} \frac{\log_p^+ N(r,a,f)}{\log r}, \quad (p \geq 1 \text{ is an integer}),$$

where $N(r,a,f) = N\left(r, \frac{1}{f-a}\right)$. If $a = 0$, the iterated convergence exponent of the zero-sequence of f is defined by

$$\lambda_p(f) := \limsup_{r\to\infty} \frac{\log_p^+ N\left(r, \frac{1}{f}\right)}{\log r}, \quad (p \geq 1 \text{ is an integer}),$$

where $N\left(r, \frac{1}{f}\right)$ is the integrated counting of zeros of f in $\{z : |z| \leq r\}$. If $a = \infty$, the iterated convergence exponent of the pole-sequence of f is defined by

$$\lambda_p\left(\frac{1}{f}\right) := \limsup_{r\to\infty} \frac{\log_p^+ N(r, f)}{\log r}, \quad (p \geq 1 \text{ is an integer}).$$

Similarly, the iterated convergence exponent of distinct zero-sequence of f is defined by

$$\overline{\lambda}_p(f) := \limsup_{r\to+\infty} \frac{\log_p^+ \overline{N}\left(r, \frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the integrated counting of distinct zeros of f in $\{z : |z| \leq r\}$, and the iterated convergence exponent of distinct pole-sequence of f is defined by

$$\overline{\lambda}_p\left(\frac{1}{f}\right) := \limsup_{r\to+\infty} \frac{\log_p^+ \overline{N}(r, f)}{\log r}.$$

Definition 1.4 [18] The iterated lower p-order $\mu_p(f)$ of a meromorphic function f is defined by

$$\mu_p(f) := \liminf_{r\to\infty} \frac{\log_p^+ T(r, f)}{\log r}, \quad (p \geq 1 \text{ is an integer}).$$

Definition 1.5 [13] The Lebesgue linear measure of a set $E \subset [0, +\infty)$ is $m(E) = \int\limits_E dt$, and the logarithmic measure of a set $F \subset [1, +\infty)$ is $m_l(F) = \int\limits_F \frac{dt}{t}$. The upper density of $E \subset [0, +\infty)$ is given by

$$\overline{dens}(E) = \limsup_{r\to+\infty} \frac{m(E \cap [0, r])}{r}.$$

The upper logarithmic density of the set $F \subset (1, +\infty)$ is defined by

$$\overline{\log dens}(F) = \limsup_{r\longrightarrow+\infty} \frac{m_l(F \cap [1, r])}{\log r}.$$

Proposition 1.1 [1, 5] *For all $H \subset [1, +\infty)$ the following statements hold :*

(i) *If $lm(H) = \infty$, then $m(H) = \infty$;*

(ii) *If $\overline{dens} H > 0$, then $m(H) = \infty$;*

(iii) *If $\overline{\log dens} H > 0$, then $m_l(H) = \infty$.*

In this paper, we consider for $k \geq 2$ the homogeneous and non-homogeneous linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.1}$$

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z), \tag{1.2}$$

where $A_0(z) \not\equiv 0$, $A_1(z), \cdots, A_{k-1}(z)$ and $F(z) \not\equiv 0$ are meromorphic functions of finite iterated $p-$order. Several authors [3, 4, 7, 8, 16, 19] have investigated the growth of solutions of second order and higher order homogeneous and non-homogeneous linear differential equations with entire or meromorphic coefficients. In [1], Andasmas and Belaïdi, investigated the zeros and growth of meromorphic solutions of equations (1.1), (1.2) and obtained the following results.

Theorem A [1] *Let $H \subset [0, +\infty)$ be a set with infinite linear measure, and let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be meromorphic functions with finite order. If there exist positive constants $\sigma > 0$, $\alpha > 0$ such that $\rho = \max\{\rho(A_j) : j = 1, 2, \cdots, k-1\} < \sigma$ and $|A_0(z)| \geq \exp\{\alpha |z|^{\sigma}\}$ as $|z| = r \in H$, $r \to +\infty$, then every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies $\mu(f) = \rho(f) = +\infty$ and $\rho_2(f) \geq \sigma$. Furthermore, if $\lambda(1/f) < \infty$, then $\sigma \leq \rho_2(f) \leq \rho(A_0)$.*

Theorem B [1] *Let $H \subset [0, +\infty)$ be a set with a positive upper density, and let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$, and $F(z) \not\equiv 0$ be meromorphic functions with finite order. If there exist positive constants $\sigma > 0, \alpha > 0$ such that $\rho = \max\{\rho(A_j) : j = 1, 2, \cdots, k-1, \rho(F)\} < \sigma$ and $|A_0(z)| \geq \exp\{\alpha |z|^{\sigma}\}$ as $|z| = r \in H, r \to +\infty$, then meromorphic solution f with $\lambda(1/f) < \sigma$ of equation (1.2) is of infinite order and*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty, \ \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f).$$

Furthermore, if $\lambda(1/f) < \min\{\mu(f), \sigma\}$, then $\sigma \leq \rho_2(f) \leq \rho(A_0)$.

Recently, Belaïdi in [5] considered the growth of meromorphic solutions of equations (1.1) and (1.2) with meromorphic coefficients of finite iterated order and obtained some results which improve and generalize some previous results.

Theorem C [5] *Let $H \subset [0, +\infty)$ be a set with a positive upper density, and let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be meromorphic functions with finite iterated $p-$order. If there exist positive constants $\sigma > 0, \alpha > 0$ such that $\rho = \max\{\rho_p(A_j) : j = 1, \cdots, k-1)\} < \sigma$ and*

$|A_0(z)| \geq \exp_p(\alpha r^\sigma)$ as $|z| = r \in H$, $r \to +\infty$, then every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies

$$\mu_p(f) = \rho_p(f) = +\infty, \ \rho_{p+1}(f) \geq \sigma.$$

Furthermore, if $\lambda_p\left(\frac{1}{f}\right) < \infty$, then $i(f) = p+1$ and

$$\sigma \leq \rho_{p+1}(f) \leq \rho_p(A_0).$$

Theorem D [5] *Let $H \subset [0, +\infty)$ be a set with a positive upper density, and let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ and $F(z) \not\equiv 0$ be meromorphic functions with finite iterated $p-$order. If there exist positive constants $\sigma > 0, \alpha > 0$ such that $|A_0(z)| \geq \exp_p(\alpha r^\sigma)$ as $|z| = r \in H$, $r \to +\infty$, and $\rho = \max\{\rho_p(A_j)\ (j = 1, \cdots, k-1), \rho_p(F)\} < \sigma$, then every meromorphic solution f of equation (1.2) with $\lambda_p\left(\frac{1}{f}\right) < \sigma$ satisfies*

$$\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = +\infty, \ \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f).$$

Furthermore, if $\lambda_p\left(\frac{1}{f}\right) < \min\{\mu_p(f), \sigma\}$, then $i(f) = p+1$ and

$$\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) \leq \rho_p(A_0).$$

There exists a natural question: How about the growth of meromorphic solutions of equations (1.1) and (1.2) with meromorphic coefficients of finite p-order if we replace the dominant fixed coefficient A_0 by the arbitrary coefficient A_s? The main purpose of this paper is to consider the above question. Now, we show our main results. For the homogeneous linear differential equation (1.1), we obtain the following result.

Theorem 1.1 *Let $H \subset (1, +\infty)$ be a set with a positive upper logarithmic density (or $m_l(H) = \infty$), and let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be meromorphic functions with finite iterated $p-$order. If there exist positive constants $\sigma > 0, \alpha > 0$ and an integer s, $0 \leq s \leq k-1$, such that $|A_s(z)| \geq \exp_p(\alpha r^\sigma)$ as $|z| = r \in H$, $r \to +\infty$, and $\rho = \max\{\rho_p(A_j)\ (j \neq s)\} < \sigma$, then every non-transcendental solution $f(z) \not\equiv 0$ of (1.1) is a polynomial with $\deg f \leq s-1$ and every transcendental meromorphic solution f of (1.1) with $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$ satisfies $i(f) = p+1$*

$$\mu_p(f) = \rho_p(f) = +\infty$$

and

$$\sigma \leq \rho_{p+1}(f) \leq \rho_p(A_s).$$

Corollary 1.1 *Under the hypotheses of Theorem 1.1, suppose further that* φ *be a transcendental meromorphic function satisfying* $i(\varphi) < p+1$ *or* $\rho_{p+1}(\varphi) < \sigma$*. Then every transcendental meromorphic solution* f *of equation* (1.1) *with* $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$ *satisfies*

$$\sigma \leq \overline{\lambda}_{p+1}(f-\varphi) = \lambda_{p+1}(f-\varphi) = \rho_{p+1}(f-\varphi) = \rho_{p+1}(f) \leq \rho_p(A_s).$$

Considering the non-homogeneous linear differential equation (1.2), we obtain the following result.

Theorem 1.2 *Let* $H \subset (1,+\infty)$ *be a set with a positive upper logarithmic density (or* $m_l(H) = \infty$*), and let* $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ *and* $F(z) \not\equiv 0$ *be meromorphic functions with finite iterated* $p-$*order. If there exist positive constants* $\sigma > 0, \alpha > 0$ *and an integer* s, $0 \leq s \leq k-1$, *such that* $|A_s(z)| \geq \exp_p(\alpha r^\sigma)$ *as* $|z| = r \in H$, $r \to +\infty$, *and* $\max\{\rho_p(A_j)\ (j \neq s),\ \rho_p(F)\} < \sigma$, *then every non-transcendental solution* f *of* (1.2) *is a polynomial with* $\deg f \leq s-1$ *and every transcendental meromorphic solution* f *of* (1.2) *with* $\lambda_p\left(\frac{1}{f}\right) < \min\{\sigma, \mu_p(f)\}$ *satisfies* $i(f) = p+1$

$$\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = \mu_p(f) = +\infty$$

and

$$\sigma \leq \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) \leq \rho_p(A_s).$$

Corollary 1.2 *Let* $A_j(z)$ $(j = 0, 1, \cdots, k-1)$, $F(z)$, H *satisfy all of the hypothesis of Theorem 1.2, and let* φ *be a transcendental meromorphic function satisfying* $i(\varphi) < p+1$ *or* $\rho_{p+1}(\varphi) < \sigma$*. Then every transcendental meromorphic solution* f *with* $\lambda_p\left(\frac{1}{f}\right) < \min\{\sigma, \mu_p(f)\}$ *of equation* (1.2) *satisfies*

$$\sigma \leq \overline{\lambda}_{p+1}(f-\varphi) = \lambda_{p+1}(f-\varphi) = \rho_{p+1}(f-\varphi) \leq \rho_p(A_s).$$

Remark 1.1. Obviously, Theorem 1.1 and Theorem 1.2 are generalization of Theorems A, B, C and D.

2. Auxiliary lemmas

In order to prove our theorems, we need the following lemmas.

Lemma 2.1 [9] *Let f be a transcendental meromorphic function in the plane, and let $\eta > 1$ be a given constant. Then there exist a set $E_1 \subset (1, +\infty)$ that has a finite logarithmic measure, and a constant $B > 0$ depending only on η and (m, n) $(m, n \in \{0, 1, \cdots, k\})$ $m < n$ such that for all z with $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leq B\left(\frac{T(\eta r, f)}{r}(\log^{\eta} r)\log T(\eta r, f)\right)^{n-m}.$$

Lemma 2.2 [11] *Let $p \geq 1$ be an integer and let $f(z) = \frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$, $d(z)$ are entire functions satisfying $\mu_p(g) = \mu_p(f) = \mu \leq \rho_p(f) = \rho_p(g) \leq +\infty$, $i(d) < p$ or $i(d) = p$ and $\rho_p(d) = \rho < \mu$. Then there exists a set E_2 of finite logarithmic measure such that $|z| = r \notin E_2$, $|g(z)| = M(r, g)$, we have*

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq r^{2s} \ \ (s \geq 1 \text{ is an integer}).$$

Lemma 2.3 [18] *Let $p \geq 1$ be an integer. Suppose that $f(z)$ is a meromorphic function such that $i(f) = p$, $\rho_p(f) = \rho < +\infty$. Then, there exist entire functions $\pi_1(z)$, $\pi_2(z)$ and $D(z)$ such that*

$$f(z) = \frac{\pi_1(z)\, e^{D(z)}}{\pi_2(z)} \text{ and } \rho_p(f) = \max\left\{\rho_p(\pi_1), \rho_p(\pi_2),\ \rho_p(e^{D(z)})\right\}.$$

Moreover, for any given $\varepsilon > 0$, we have

$$\exp\left\{-\exp_{p-1}\left(r^{\rho+\varepsilon}\right)\right\} \leq |f(z)| \leq \exp_p\left(r^{\rho+\varepsilon}\right),\ r \notin E_3,$$

where $E_3 \subset (1, +\infty)$ is a set of r of finite linear measure.

To avoid some problems caused by the exceptional set, we recall the following lemmas.

Lemma 2.4 [2] *Let $g : [0, +\infty) \to \mathbb{R}$ and $h : [0, +\infty) \to \mathbb{R}$ be monotone nondecreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E_4 of finite linear measure. Then for any $\lambda > 1$ there exists $r_0 > 0$ such that $g(r) \leq h(\lambda r)$ for all $r > r_0$.*

Lemma 2.5 [10] *Let $\varphi : [0, +\infty) \to \mathbb{R}$ and $\psi : [0, +\infty) \to \mathbb{R}$ be monotone nondecreasing functions such that $\varphi(r) \leq \psi(r)$ for all $r \notin (E_5 \cup [0, 1])$, where E_5 is a set of finite logarithmic measure. Let $\gamma > 1$ be a given constant. Then there exists an $r_1 = r_1(\gamma) > 0$ such that $\varphi(r) \leq \psi(\gamma r)$ for all $r > r_1$.*

Lemma 2.6 [5] *Assume that $k \geq 2$ and A_0, $A_1, \cdots, A_{k-1}$, F are meromorphic functions. Let*

$$\rho = \max\left\{\rho_p\left(A_j\right), (j = 0, 1, \cdots, k-1), \rho_p\left(F\right)\right\} < \infty$$

and let f be a meromorphic solution of infinite iterated p-order of equation (1.2) *with $\lambda_p\left(\frac{1}{f}\right) < \mu_p\left(f\right)$. Then $\rho_{p+1}(f) \leq \rho$.*

Lemma 2.7 *Under the assumptions of Theorem 1.1 or Theorem 1.2, we have $\rho_p\left(A_s\right) = \beta \geq \sigma$.*

Proof. Suppose that $\rho_p\left(A_s\right) = \beta < \sigma$. Then, by using Lemma 2.3, there exists a set $E_3 \subset (1, +\infty)$ that has finite linear measure (and so of finite logarithmic measure) such that when $|z| = r \notin E_3$, we have for any given ε with $0 < \varepsilon < \sigma - \beta$

$$|A_s\left(z\right)| \leq \exp_p\left(r^{\beta+\varepsilon}\right). \tag{2.1}$$

On the other hand, by the hypotheses of Theorems 1.1 or 1.2, there exist positive constants $\sigma > 0, \alpha > 0$ such that

$$|A_s\left(z\right)| \geq \exp_p\left(\alpha r^{\sigma}\right) \tag{2.2}$$

as $|z| = r \in H$, $r \to +\infty$, where $H \subset (1, +\infty)$ is a set with a positive upper logarithmic density (by Proposition 1.1, we have $m_l\left(H\right) = \infty$). From (2.1) and (2.2), we obtain for $|z| = r \in H \backslash E_3$, $r \to +\infty$

$$\exp_p\left(\alpha r^{\sigma}\right) \leq |A_s\left(z\right)| \leq \exp_p\left(r^{\beta+\varepsilon}\right)$$

and by ε is arbitrary with $0 < \varepsilon < \sigma - \beta$, this is a contradiction as $r \to +\infty$. Hence $\rho_p\left(A_s\right) = \beta \geq \sigma$.

Lemma 2.8 *Let $H \subset (1, +\infty)$ be a set with a positive upper logarithmic density (or infinite logarithmic measure), and let $A_j\left(z\right)$ $(j = 0, 1, \cdots, k-1)$, $F\left(z\right)$ be meromorphic functions with finite iterated $p-$order. If there exist positive constants $\sigma > 0$, $\alpha > 0$ and an integer s, $0 \leq s \leq k-1$, such that $|A_s\left(z\right)| \geq \exp_p\left(\alpha r^{\sigma}\right)$ as $|z| = r \in H$, $r \to +\infty$, and $\rho = \max\left\{\rho_p\left(A_j\right) \ (j \neq s), \ \rho_p\left(F\right)\right\} < \sigma$, then every transcendental meromorphic solution f of equation* (1.2) *satisfies $\rho_p(f) \geq \sigma$.*

Proof. Assume that f is a transcendental meromorphic solution of equation (1.2). From (1.2), we have

$$A_s = \frac{F}{f^{(s)}} - \frac{f^{(k)}}{f^{(s)}} - \sum_{\substack{j=0 \\ j\neq s}}^{k-1} A_j \frac{f^{(j)}}{f^{(s)}} \tag{2.3}$$

Combining the formula (2.3) and the first main theory in Nevanlinna theory, we get

$$T(r, A_s) \leq \sum_{\substack{j=0 \\ j\neq s}}^{k} T\left(r, f^{(j)}\right) + \sum_{\substack{j=0 \\ j\neq s}}^{k-1} T(r, A_j)$$

$$+ (k+1)\, T\left(r, f^{(s)}\right) + T(r, F) + O(1). \tag{2.4}$$

For every integer $j \in [1, k]$, we have the estimate, see ([12], p. 56),

$$T\left(r, f^{(j)}\right) \leq (j+1)\, T(r, f\ ) + S(r, f), \tag{2.5}$$

where $S(r, f) = O(\log T(r, f) + \log r)$, possibly outside a set $E \subset [0, +\infty)$ of a finite linear measure. Combining the two inequalities (2.4) and (2.5), we obtain

$$T(r, A_s) \leq \sum_{\substack{j=0 \\ j\neq s}}^{k-1} T(r, A_j) + cT(r, f) + T(r, F) + S(r, f) + O(1), \tag{2.6}$$

with $c > 0$. It follows from $\max\{\rho_p(A_j)\ \ (j \neq s),\ \rho_p(F)\} < \sigma \leq \rho_p(A_s)$, Lemma 2.4 and Lemma 2.7 that (2.6) gives

$$\rho_p(f) \geq \rho_p(A_s) \geq \sigma.$$

Lemma 2.9 [6] *Let $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ be finite iterated p-order meromorphic functions. If f is a meromorphic solution with $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho < +\infty$ of equation (1.2), then $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = +\infty$ and $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho$.*

Lemma 2.10 [14] *Let $A_j(z)\ \ (j = 0, 1, \cdots, k-1)$, $F(z)\,(\not\equiv 0)$ be meromorphic functions and let $f(z)$ be a meromorphic solution of (1.2) satisfying one of the following conditions:*
(i) $\max\{i(F) = p,\ i(A_j)\ (j = 0, 1, \cdots, k-1)\} < i(f) = p + 1 (0 < p < +\infty)$,
(ii) $b = \max\{\rho_{p+1}(F),\ \rho_{p+1}(A_j)\ \ (j = 0, 1, \cdots, k-1)\} < \rho_{p+1}(f)$.
Then $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f)$.

3. Proof of Theorem 1.1

Assume that $f(z) \not\equiv 0$ is a rational solution of (1.1). If either $f(z)$ is a rational function, which has a pole at z_0 of degree $m \geq 1$, or $f(z)$ is a polynomial with $\deg f \geq s$, then $f^{(s)}(z) \not\equiv 0$. By (1.1), we have

$$A_s(z) f^{(s)}(z) = -\left(f^{(k)}(z) + \sum_{\substack{j=0 \\ j\neq s}}^{k-1} A_j(z) f^{(j)}(z) \right).$$

Then, by using Lemma 2.7

$$\begin{aligned} \sigma \leq \rho_p(A_s) = \rho_p(A_s f^{(s)}) &= \rho_p\left(-\left(f^{(k)} + \sum_{j=0,\ j\neq s}^{k-1} A_j f^{(j)} \right)\right) \\ &\leq \max_{j=0,1,\cdots,k-1,\ j\neq s} \{\rho_p(A_j)\}, \end{aligned}$$

which is a contradiction. Therefore, $f(z)$ must be a polynomial with $\deg f \leq s-1$.

Now, we assume that $f(z)$ is a transcendental meromorphic solution of (1.1) such that $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$. By (1.1), we have

$$|A_s| \leq \left|\frac{f}{f^{(s)}}\right| \left(\left|\frac{f^{(k)}}{f}\right| + |A_0| + \sum_{\substack{j=1 \\ j\neq s}}^{k-1} |A_j| \left|\frac{f^{(j)}}{f}\right| \right). \tag{3.1}$$

By the hypotheses of Theorem 1.1, there exists a set $H \subset (1,+\infty)$ with $m_l(H) = \infty$, such that for all z satisfying $|z| = r \in H$, $r \to +\infty$, we have

$$|A_s(z)| \geq \exp_p(\alpha r^\sigma). \tag{3.2}$$

By using Lemma 2.1, for $\eta = 2$, there exists a set $E_1 \subset (1,+\infty)$ with $m_l(E_1) < \infty$ and constant $B > 0$, such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left[T(2r,f)\right]^{k+1},\ j = 1,2,\cdots,k,\ j \neq s. \tag{3.3}$$

By Lemma 2.2, there exists a set $E_2 \subset (1,+\infty)$ of finite logarithmic measure such that $|z| = r \notin E_2$, $|g(z)| = M(r,g)$ and for r sufficiently large, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \le r^{2s} \quad (s \ge 1 \text{ is an integer}). \tag{3.4}$$

And by Lemma 2.3, for any given ε with $0 < \varepsilon < \sigma - \rho$ there exists a set $E_3 \subset (1, +\infty)$ with a finite linear measure (and so of finite logarithmic measure) such that

$$|A_j(z)| \le \exp_p\left(r^{\rho+\varepsilon}\right), \ j = 0, 1, \cdots, k-1, \ j \ne s \tag{3.5}$$

holds for all z satisfying $|z| = r \notin E_3$. Hence it follows from (3.1), (3.2), (3.3), (3.4) and (3.5) that for all z satisfying $|z| = r \in H \backslash ([0,1] \cup E_1 \cup E_2 \cup E_3)$, $r \to +\infty$, we have

$$\exp_p(\alpha r^\sigma) \le Bkr^{2s} \exp_p\left(r^{\rho+\varepsilon}\right) [T(2r, f)]^{k+1}. \tag{3.6}$$

By $0 < \varepsilon < \sigma - \rho$, it follows from Lemma 2.5 and (3.6) that

$$\mu_p(f) = \rho_p(f) = +\infty \text{ and } \rho_{p+1}(f) \ge \sigma. \tag{3.7}$$

By using Lemma 2.7, we have

$$\max\{\rho_p(A_j) : j = 0, 1, \cdots, k-1\} = \rho_p(A_s) = \beta < +\infty.$$

Since f is a transcendental meromorphic solution of equation (1.1) with $\rho_p(f) = +\infty$ and $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$, then by Lemma 2.6, we get

$$\rho_{p+1}(f) \le \rho_p(A_s). \tag{3.8}$$

By (3.7) and (3.8), we conclude that $i(f) = p+1$, $\mu_p(f) = \rho_p(f) = +\infty$ and $\sigma \le \rho_{p+1}(f) \le \rho_p(A_s)$.

4. Proof of Corollary 1.1

Setting $h = f - \varphi$ where φ is such that $i(\varphi) < p+1$ or $\rho_{p+1}(\varphi) < \sigma$. Using the properties of iterated order, we get $\rho_{p+1}(h) = \rho_{p+1}(f)$, so $\sigma \le \rho_{p+1}(h) \le \rho_p(A_s)$. By substituting $f = h + \varphi$ into (1.1), we obtain

$$h^{(k)} + A_{k-1}(z) h^{(k-1)} + \cdots + A_1(z) h' + A_0(z) h$$

$$= -\left(\varphi^{(k)} + A_{k-1}(z)\varphi^{(k-1)} + \cdots + A_1(z)\varphi' + A_0(z)\varphi\right). \tag{4.1}$$

Set $K(z) = \varphi^{(k)} + A_{k-1}(z)\varphi^{(k-1)} + \cdots + A_1(z)\varphi' + A_0(z)\varphi$. If $i(\varphi) < p+1$ or $\rho_{p+1}(\varphi) < \sigma$, then by Theorem 1.1, we deduce that φ is not a solution of equation (1.1), implying that $K(z) \not\equiv 0$, and in this case we have $\rho_{p+1}(K) \le \max\{\rho_{p+1}(\varphi), \rho_{p+1}(A_j) \ \ (j = 0, 1, \cdots, k-1)\} < \sigma$, so $\max\{\rho_{p+1}(K), \ \rho_{p+1}(A_j) \ \ (j = 0, 1, \cdots, k-1)\} < \sigma \le \rho_{p+1}(f)$

and by Lemma 2.10, we obtain $\sigma \leq \overline{\lambda}_{p+1}(f-\varphi) = \lambda_{p+1}(f-\varphi) = \rho_{p+1}(f-\varphi) = \rho_{p+1}(f) \leq \rho_p(A_s)$.

5. Proof of Theorem 1.2

Assume that $f(z)$ is a rational solution of (1.2). If either $f(z)$ is a rational function, which has a pole at z_0 of degree $m \geq 1$, or $f(z)$ is a polynomial with $\deg f \geq s$, then $f^{(s)}(z) \not\equiv 0$. By (1.2), we have

$$A_s(z) f^{(s)}(z) = F(z) - \left(f^{(k)}(z) + \sum_{\substack{j=0 \\ j\neq s}}^{k-1} A_j(z) f^{(j)}(z) \right).$$

Then, by using Lemma 2.7

$$\begin{aligned} \sigma \leq \rho_p(A_s) = \rho_p(A_s f^{(s)}) &= \rho_p\left(F - \left(f^{(k)} + \sum_{j=0,\ j\neq s}^{k-1} A_j f^{(j)} \right)\right) \\ &\leq \max_{j=0,1,\cdots,k-1,\ j\neq s} \{\rho_p(A_j),\ \rho_p(F)\}, \end{aligned}$$

which is a contradiction. Therefore, $f(z)$ must be a polynomial with $\deg f \leq s-1$.

Now, we assume that $f(z)$ is a transcendental meromorphic solution of (1.2) such that $\lambda_p\left(\frac{1}{f}\right) < \min\{\mu_p(f), \sigma\}$. By (1.2), we have

$$|A_s| \leq \left|\frac{f}{f^{(s)}}\right| \left(|A_0| + \left|\frac{f^{(k)}}{f}\right| + \sum_{\substack{j=1 \\ j\neq s}}^{k-1} |A_j| \left|\frac{f^{(j)}}{f}\right| + \left|\frac{F}{f}\right| \right). \tag{5.1}$$

By Lemma 2.8, we know that f satisfies $\rho_p(f) \geq \sigma$. By the hypothesis $\lambda_p\left(\frac{1}{f}\right) < \min\{\mu_p(f), \sigma\}$ and Hadamard factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions satisfying

$$\mu_p(g) = \mu_p(f) = \mu \leq \rho_p(g) = \rho_p(f),$$
$$\rho_p(d) = \lambda_p\left(\frac{1}{f}\right) = \beta < \min\{\mu_p(f), \sigma\}.$$

By the definition of the iterated lower $p-$order, for sufficiently large r

$$M(r,g) \geq \exp_p\{r^{\mu_p(g)-\varepsilon}\}. \tag{5.2}$$

Set

$$\rho_1 = \max\left\{\rho_p\left(A_j\right),\ j \neq s, \rho_p\left(F\right)\right\} < \sigma.$$

Then by Lemma 2.3, we have by using (5.2), for any given ε with $0 < \varepsilon < \min\{\sigma - \rho_1, \frac{\mu_p(g)-\rho_p(d)}{2}\}$, there exists a set $E_3 \subset (1, +\infty)$ with a finite logarithmic measure such that for all z satisfying $|z| = r \notin E_3$ at wich $|g\left(z\right)| = M(r, g)$,

(5.3)

$$\left|\frac{F\left(z\right)}{f\left(z\right)}\right| = \left|\frac{F\left(z\right)d\left(z\right)}{g\left(z\right)}\right| \leq \frac{\exp_p\left(r^{\rho_1+\varepsilon}\right)\exp_p\left(r^{\rho_p(d)+\varepsilon}\right)}{\exp_p\left(r^{\mu_p(g)-\varepsilon}\right)} \leq \exp_p\left(r^{\rho_1+\varepsilon}\right).$$

By using the same arguments as in the proof of Theorem 1.1, for any given ε with $0 < \varepsilon < \min\{\sigma - \rho_1, \frac{\mu_p(g)-\rho_p(d)}{2}\}$ and for all z satisfying $|z| = r \in H\setminus\left([0,1] \cup E_1 \cup E_2 \cup E_3\right)$, $r \to +\infty$ at wich $|g\left(z\right)| = M(r, g)$, we have

$$\left|\frac{f^{(j)}\left(z\right)}{f\left(z\right)}\right| \leq B\left[T\left(2r, f\right)\right]^{k+1},\ j = 1, 2, ..., k,\ j \neq s, \tag{5.4}$$

$$\left|\frac{f\left(z\right)}{f^{(s)}\left(z\right)}\right| \leq r^{2s}\ \left(s \geq 1 \text{ is an integer}\right), \tag{5.5}$$

$$|A_j\left(z\right)| \leq \exp_p\left(r^{\rho_1+\varepsilon}\right),\ j = 0, 1, ..., k-1,\ j \neq s, \tag{5.6}$$

$$|A_s\left(z\right)| \geq \exp_p\left(\alpha r^{\sigma}\right). \tag{5.7}$$

Hence, it follows from (5.1), (5.3), (5.4), (5.5), (5.6) and (5.7) that for all z satisfying $|z| = r \in H\setminus\left([0,1] \cup E_1 \cup E_2 \cup E_3\right)$, $r \to +\infty$, at which $|g\left(z\right)| = M\left(r, g\right)$, that for any given ε with $0 < \varepsilon < \min\{\sigma - \rho_1, \frac{\mu_p(g)-\rho_p(d)}{2}\}$

$$\exp_p\left(\alpha r^{\sigma}\right) \leq r^{2s}\Bigg(\exp_p\left(r^{\rho_1+\varepsilon}\right) + B\left[T\left(2r, f\right)\right]^{k+1}$$

$$+\sum_{\substack{j=1\\ j\neq s}}^{k-1}\exp_p\left(r^{\rho_1+\varepsilon}\right)B\left[T\left(2r, f\right)\right]^{k+1} + \exp_p\left(r^{\rho_1+\varepsilon}\right)\Bigg)$$

$$\leq r^{2s}B\left(k+1\right)\left[T\left(2r, f\right)\right]^{k+1}\exp_p\left(r^{\rho_1+\varepsilon}\right). \tag{5.8}$$

By $0 < \varepsilon < \sigma - \rho_1$, it follows from Lemma 2.5 and (5.8) that

$$\mu_p\left(f\right) = \rho_p(f) = +\infty \text{ and } \rho_{p+1}(f) \geq \sigma. \tag{5.9}$$

Since $F \not\equiv 0$, from Lemma 2.9, we have

$$\begin{aligned}\overline{\lambda}_p(f) = \lambda_p(f) = \mu_p(f) = \rho_p(f) = +\infty,\\ \sigma \leq \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f).\end{aligned} \tag{5.10}$$

By Lemma 2.7, we have

$$\max\{\rho_p(A_j), (j = 0, 1..., k-1), \rho_p(F)\} = \rho_p(A_s) = \beta < +\infty.$$

Since f is a meromorphic solution of (1.2) with $\rho_p(f) = +\infty$, $\lambda_p\left(\frac{1}{f}\right) < \sigma < \mu_p(f) = +\infty$, then by Lemma 2.6, we get

$$\rho_{p+1}(f) \leq \rho_p(A_s). \tag{5.11}$$

By (5.10) and (5.11), we obtain $i(f) = p + 1$

$$\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = +\infty,$$

$$\sigma \leq \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) \leq \rho_p(A_s).$$

6. **Proof of Corollary 1.2**

Setting $h = f - \varphi$ such that $i(\varphi) < p+1$ or $\rho_{p+1}(\varphi) < \sigma$. By Theorem 1.2, we have $\sigma \leq \rho_{p+1}(f) \leq \rho_p(A_s)$. Using the properties of iterated order, we get $\sigma \leq \rho_{p+1}(h) = \rho_{p+1}(f) \leq \rho_p(A_s)$. By substituting $f = h + \varphi$ into (1.2), we get

$$h^{(k)} + A_{k-1}(z) h^{(k-1)} + \cdots + A_1(z) h' + A_0(z) h$$

$$= F(z) - \left(\varphi^{(k)} + A_{k-1}(z)\varphi^{(k-1)} + \cdots + A_1(z)\varphi' + A_0(z)\varphi\right). \tag{6.1}$$

Set $G(z) = F(z) - \left(\varphi^{(k)} + A_{k-1}(z)\varphi^{(k-1)} + \cdots + A_1(z)\varphi' + A_0(z)\varphi\right)$. If $i(\varphi) < p + 1$ or $\rho_{p+1}(\varphi) < \sigma$, then by Theorem 1.2, we deduce that φ is not a solution of equation (1.2), implying that $G(z) \not\equiv 0$, and in this case we have

$$\rho_{p+1}(G) \leq \max\{\rho_{p+1}(\varphi),\ \rho_{p+1}(F),\ \rho_{p+1}(A_j)\ \ (j = 0, 1, \cdots, k-1)\} < \sigma,$$

so

$$\max\{\rho_{p+1}(G),\ \rho_{p+1}(A_j)\ \ (j = 0, 1, \cdots, k-1)\} < \sigma \leq \rho_{p+1}(f)$$

and by Lemma 2.10, we obtain

$$\sigma \leq \overline{\lambda}_{p+1}(f - \varphi) = \lambda_{p+1}(f - \varphi) = \rho_{p+1}(f - \varphi) \leq \rho_p(A_s).$$

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Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (UMAB), B. P. 227 Mostaganem-Algeria.
e-mails: saidaniman@yahoo.fr; benharrat.belaidi@univ-mosta.dz