"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 28(2018), No. 1, 171-176

APPLICATIONS OF WEAK-BISPLIT GRAPHS

MIHAI TALMACIU

Abstract. In this paper, using weak-decomposition, we give necessary and sufficient conditions for a graph to be weak-bisplit cograph. We also give we give some applications in optimization problems.

1. Introduction

Throughout this paper G=(V,E) is a simple (i.e. finite, undirected, without loops and multiple edges) graph [1]. Let *co*-G = $\overline{G}$ denote the complement graph of G. For $U \subseteq V$ let $G(U)$ denote the subgraph of G induced by U. By G-X we mean the graph $G(V$-$X)$, whenever $X \subseteq V$, but we often denote it simply by G-v ($\forall v \in V$) when there is no ambiguity.

If $v \in V$ is a vertex in G, the neighborhood $N_G(v)$ denotes the vertices of G-v that are adjacent to v. We write $N(v)$ when the graph G appears clearly from the context. The neighborhood of the vertex v in the complement of the graph G is denoted by $\overline{N}(v)$. For any subset S of vertices in the graph G the neighborhood of S is $N(S) = \cup_{v \in S} N(v) - S$ and $N[S]$=$S \cup N(S)$. A *stable* (or *independent*) set is a subset of V with the property that all the vertices are pairwise non-adjacent.

By P_n, C_n, K_n we mean a chordless path on $n \geq 3$ vertices, the chordless cycle on $n \geq 3$ vertices, and the complete graph on $n \geq 1$ vertices. If e=$xy \in E$, we also denote $x \sim y$; we also denote $x \nsim y$ whenever x, y

Keywords and phrases: Weak-bisplit graphs, weak-bisplit cographs.
(2010) Mathematics Subject Classification:05C85, 68R10

are not adjacent in G. A set A is totally adjacent (non adjacent) with a set B of vertices ($A \cap B = \phi$) if ab is (is not) edge, for any a vertex in A and any b vertex in B; we note denote $A \sim B$ ($A \nsim B$). A graph G is F-free if none of its induced subgraphs is in F.

The paper is organized as follows. In Section 2 we give preliminary results. In Section 3 we give a characterization of weak-bisplit graphs. In Section 4 we give some applications in optimization problems.

2. Preliminary results

At first, we recall the notion of weak component.

Definition 1. ([2],[3],[4]) *A set $A \subset V(G)$ is called a weak set of the graph G if $N_G(A) \neq V(G) - A$ and $G(A)$ are connected. If A is a weak set, maximal with respect to set inclusion, then $G(A)$ is called a weak component. For simplicity, the weak component $G(A)$ will be denoted by A.*

Definition 2. ([2],[3],[4]) *Let G=(V,E) be a connected and non-complete graph. If A is a weak set, then the partition $\{A, N(A), V\text{-}A \cup N(A)\}$ is called a weak decomposition of G with respect to A.*

The name of "*weak component*" is justified by the following result.

Theorem 1. ([2],[3],[4]) *Every connected and non-complete graph G=(V,E) admits a weak component A such that*

$$G(V - A) = G(N(A) + G(\overline{N}(A)).$$

Theorem 2. ([2],[3],[4]) *Let G=(V,E) be a connected and non-complete graph and $A \subset V$. Then A is a weak component of G if and only if $G(A)$ is connected and $N(A) \sim \overline{N}(A)$.*

The next result, based on Theorem 1, ensures the existence of a weak decomposition in a connected and non-complete graph.

Theorem 3. *If G=(V,E) is a connected and non-complete graph, then V admits a weak decomposition (A,B,C), such that $G(A)$ is a weak component and $G(V\text{-}A)=G(B)+G(C)$.*

Theorem 2 provides an $O(n+m)$ algorithm for building a weak decomposition for a non-complete and connected graph.

3. The results concerning weak-bisplit graphs

Definition 3. *A graph G is a weak bisplit graph if and only if it has an independent set S such that every connected component of G-S is a biclique (i. e. a complete bipartite subgraph).*

A graph $Star_{123}$ and the next graph *($\{a,b,c,d,e,f,g\}$, $\{ab,bc,cd,ce,ef,fg\}$)* are isomorphic.

A graph is weak bisplit ([7]) if and only if it is $\{P_7, C_{2k+1}(k \geq 2), Star_{123}\}$-free.

A graph is bi-cograph ([6]) if and only if it is $\{P_7, Star_{123}, Sun_4\}$-free.

It is known that ([7]) a graph is weak-bisplit if and only if it is $\{P_7, C_{2k+1}(k \geq 2), Star_{123}\}$-free. Because a P_4-free graph is P_7-free, any graph $\{P_4, C_{2k+1}(k \geq 2), Star_{123}\}$-free (still called *weak bisplit cograph*) is weak-bisplit.

In [5] the following result is shown, but for a complete presentation we describe it again.

Theorem 4. *Let G=(V,E) be a connected and incomplete graph and (A,N,R) be a weak decomposition with G(A)*

the weak component. G is weak-bisplit cograph if and only if:

i) $A \sim N \sim R$

ii) $G(A), G(N), G(R)$ *are weak-bisplit cographs.*

Proof. Let G be a weak-bisplit cograph and (A,N,R) be a weak decomposition with G(A) the weak component.

Then $N \sim R$, because G(A) is the weak component. Because G is P_4-free, $A \sim N \sim R$.

Vice versa, we assume that G(A), G(N), G(R) are weak-bisplit cographs and $A \sim N \sim R$. From ([11]), because

G(A), G(N), G(R) are weak-bisplit cographs and $A \sim N \sim R$, G is P_4-free. $C_{2k+1}(k \geq 2) \not\subset G(A \cup R)$, because

G(A), G(R) are weak-bisplit cographs and $G(A \cup R)$ is not connected. $C_{2k+1}(k \geq 2) \not\subset G(A \cup N)$, because G(A),

G(N) are weak-bisplit cographs and $A \sim N$. $C_{2k+1}(k \geq 2) \not\subset G(N \cup R)$, because G(N), G(R) are weak-bisplit

cographs and $N \sim R$. $C_{2k+1}(k \geq 2) \not\subset G(A \cup N \cup R)$ with $V(C_{2k+1}) \cap A \neq \Phi$, $V(C_{2k+1}) \cap N \neq \Phi$,

$V(C_{2k+1}) \cap R \neq \Phi$, because G(A), G(N), G(R) are weak-bisplit cographs and $A \sim N \sim R$. $Star_{123} \not\subset G(A \cup R)$,

because G(A), G(R) are weak bisplit cographs and $G(A \cup R)$ is not connected. $Star_{123} \not\subset G(A \cup N)$, because

G(A), G(N) are weak bisplit cographs and $A \sim N$. $Star_{123} \not\subset G(N \cup R)$, because G(N), G(R) are weak-bisplit

cographs and $N \sim R$. $Star_{123} \not\subset G(A \cup N \cup R)$ with $V(Star_{123}) \cap A \neq \Phi$, $V(Star_{123}) \cap N \neq \Phi$, $V(Star_{123}) \cap R \neq \Phi$,

because G(A), G(N), G(R) are weak-bisplit cographs and $A \sim N \sim R$. So G is weak-bisplit cograph.

4. Some applications in optimization problems

The aim of the first problem family is to determine a location that minimizes the maximum distance to any other location in the network.

The *eccentricity* of a vertex $u \in V$ is $e_G(u) = \max\{d(u,v) : v \in V\}$.

The *radius* is $r(G) = \min\{e_G(u) : u \in V\}$. The *diameter* is $diam(G) = \max_{x,y \in V(G)}\{d_G(x,y)\}$.

The *center* $C(G)$ of a graph G is $C(G) = \{u \in V : r(G) = e_G(u)\}$.

For each node i, the local clustering coefficient, $C_l(i)$, is simply defined as the fraction of pairs of neighbors of i that are themselves neighbors. The number of possible links between the neighbors of node i is simply $d_i(d_{i-1})\ /\ 2$. The number of favorable connections in the graph $G(X)$ between neighbors of node i is denoted by $n_i(G(X))$, which means, the vertex degree i in the induced subgraph by X $(X \subseteq V)$.

Thus we get

$$C_l(i) = \frac{|\{e_{jk} \in E(G) : e_{ij} \in E(G) \wedge e_{ik} \in E(G)\}|}{d_i(d_i-1)/2} = \frac{n_i(G)}{d_i(d_i-1)/2}.$$

The global clustering coefficient C$_l$ is then given by $C_l = \frac{1}{n}\sum_{i=1}^{n} C_l(i)$. A high clustering coefficient C$_l$ means (in the language of social networks), that two of your friends are likely to be also friends of each other. It also indicates a high redundancy of the network.

I still give the result:

Theorem 5. *Let G=(V,E) be a connected with at least two non-adjacent vertices. If G is weak-bisplit cograph then the radius of G is one, the diameter is two and if N is a clique the center is a clique.*

Proof. Let *(A,N,R)* a weak decomposition with A as weak component. For $\forall a \in A : d(a,n) = 1$, $d(a,r) = 2$, $\forall n \in N, \forall r \in R$. If A is a clique then $d(a,a') = 1\ \forall a' \in A - \{a\}$. So, $e_G(a) = 2$. For $\forall n \in N : d(a,n) = d(r,n) = 1$, $\forall a \in A, \forall r \in R$. If N is not a clique then $\exists n' \in N - \{n\}$ such that $d(n,n') = 2$. So $e_G(n) = 2$. For $\forall r \in R : d(r,n) = 1$, $d(a,r) = 2$,$\forall a \in A, \forall n \in N$. If R is a clique then $d(r,r') = 2, \forall r' \in R - \{r\}$. So, $e_G(r) = 2$. So, r(G)=1 and diam(G)=2. For $\forall i \in A$, $c_i = \frac{E(G(N)) + n_i(G(A))}{d_i(d_i-1)/2}$. For $\forall i \in N$, $c_i = \frac{E(G(A \cup R)) + n_i(G(N))}{d_i(d_i-1)/2}$. For $\forall i \in R$, $c_i = \frac{E(G(N)) + n_i(G(R))}{d_i(d_i-1)/2}$. If N is a clique the $C(G)$ is a clique.

5. Conclusions

In this paper, using weak decomposition, we characterize the weak-bisplit graphs, as well as some of their subclasses. Also, we give some applications in optimization problems.

References

[1] Berge, C., *Graphs*, North-Holland, Amsterdam. 1985.

[2] Talmaciu, M., *Decomposition Problems in the Graph Theory with Applications in Combinatorial Optimization* - Ph. D. Thesis, University "Al. I. Cuza" Iasi, Romania, 2002.

[3] Talmaciu, M., E.Nechita, *Recognition Algorithm for diamond-free graphs*, Informatica, 18, 2007, 457-462.

[4] Talmaciu, M; E. Nechita, G.C. Crisan, *A Recognition Algorithm for a Class of Partitionable Graphs that Satisfies the Normal Graph Conjecture*, Studies in Informatics and Control, 18, 2009, 349-354.

[5] Mihai Talmaciu, Elena Nechita, *Some Combinatorial Optimization Problems for Weak-Bisplit Graphs*, Studies in Informatics and Control, Vol. 19, No. 4, 2010 http://www.sic.ici.ro, 427-434

[6] http://wwwteo.informatik.uni-rostock.de/isgci/classes/gc_444.html

[7] http://wwwteo.informatik.uni-rostock.de/isgci/classes/gc_457.html