"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 28(2018), No. 2, 19-28

HERMITE-HADAMARD TYPE INEQUALITIES FOR TRIGONOMETRICALLY CONVEX FUNCTIONS

HURIYE KADAKAL

Abstract. In this paper we introduce and study the concept of trigonometrically convex function, which is a special case of h-convex functions. The class of trigonometrically convex function is large enough to include the class of non-negative convex functions. We prove two Hermite-Hadamard type inequalities for the newly introduced class of functions. We also obtain two refinements of the Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is trigonometrically convex.

1. Introduction

Throughout the paper I is a non-empty interval in $\mathbb{R}$.

Definition 1. *A function $f : I \to \mathbb{R}$ is said to be convex if the inequality*

$$f\left(tx + (1-t)y\right) \leq tf\left(x\right) + \left(1-t\right)f\left(y\right)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on the interval I.

Keywords and phrases: Convex function, trigonometrically convex function, Hermite-Hadamard inequality.
(2010)Mathematics Subject Classification: 26A51, 26D10, 26D15

Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. See articles [4, 6, 8, 9, 10] and the references therein.
Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is known as the Hermite-Hadamard inequality (for more information, see [7]). Since then, some refinements of the Hermite-Hadamard inequality for convex functions have been obtained [4, 5, 12].

Definition 2. [6] *A non-negative function $f : I \to \mathbb{R}$ is said to be a P-function if the inequality*

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. The set of P-functions on the interval I is denoted by $P(I)$.

Definition 3. [11] *Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0, 1)$ we have*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$.

2. Main Results

In this section we introduce a new concept, which is called trigonometrically convex function, as follows:

Definition 4. *A non-negative function $f : I \to \mathbb{R}$ is called trigonometrically convex if for every $x, y \in I$ and $t \in [0, 1]$,*

$$f(tx + (1-t)y) \leq \left(\sin\frac{\pi t}{2}\right) f(x) + \left(\cos\frac{\pi t}{2}\right) f(y). \tag{1}$$

We will denote by $TC(I)$ the class of all trigonometrically convex functions on interval I.

We discuss some connections between the class of trigonometrically convex functions and other classes of generalized convex functions.

Remark 1. *For $h(t) = \sin\frac{\pi t}{2}$, every trigonometrically convex function is a h-convex function.*

Remark 2. *Clearly, if $f(x)$ is a nonnegative function, then every trigonometric convex function is a P-function. Indeed, for every $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \leq \left(\sin\frac{\pi t}{2}\right) f(x) + \left(\cos\frac{\pi t}{2}\right) f(y) \leq f(x) + f(y).$$

Example 1. *Non-negative constant functions are trigonometrically convex, since $\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2} \geq 1$ for all $t \in [0,1]$.*

Lemma 1. *i) Every non-negative convex function is trigonometrically convex.*

ii) Every trigonometrically convex function is h-convex with $h(t) = \frac{\pi t}{2}$.

Proof. i.) Since the cardinal sine function $\frac{\sin x}{x}$ is decreasing on $\left[0, \frac{\pi}{2}\right]$, it follows that $t \leq \sin\frac{\pi t}{2} \leq \frac{\pi t}{2}$ for all $t \in [0,1]$. Then $1 - t \leq \cos\frac{\pi t}{2} \leq \frac{\pi}{2}(1-t)$ for all $t \in [0,1]$.

ii.) Let f be a trigonometrically convex function. If we take $h_2(t) = \sin\frac{\pi t}{2}$, then the function f is h_2-convex. In [11], it is proved that if $h_2 \leq h_1$ and a function f is h_2-convex, then f is h_1 convex. If we take as $h_1(t) = h(t) = \frac{\pi t}{2}, t \in [0,1]$, then it follows, since $h_2(t) \leq h(t)$ for all $t \in [0,1]$, that the function f is also h-convex. ∎

We can see that the space of trigonometrically convex functions is a convex cone in the vector space of functions $f : [a,b] \to \mathbb{R}$.

Theorem 1. *Let $f, g : [a,b] \to \mathbb{R}$. If f and g are trigonometrically convex functions, then*

(i) $f + g$ is trigonometrically convex function,

(ii) For $c \in \mathbb{R}$ $(c \geq 0)$ cf is trigonometrically convex function.

Proof. Theorem 1 follows from the known fact that the space of h-convex function is a convex cone, for each h (see [11], Proposition 9). ∎

Theorem 2. *If $f : I \to J$ is convex and $g : I \to J$ is trigonometrically convex and increasing, then $g \circ f : I \to \mathbb{R}$ is a trigonometrically convex function.*

Proof. For $x, y \in I$ and $t \in [0,1]$, we get

$$\begin{aligned}
(g\circ f)(tx+(1-t)y) &= g(f(tx+(1-t)y)) \\
&\leq g(tf(x)+(1-t)f(y)) \\
&\leq \left(\sin\frac{\pi t}{2}\right) g(f(x)) + \left(\cos\frac{\pi t}{2}\right) g(f(y)) \\
&= \left(\sin\frac{\pi t}{2}\right) (g\circ f)(x) + \left(\cos\frac{\pi t}{2}\right) (g\circ f)(y).
\end{aligned}$$

This completes the proof of theorem. ∎

Also, the above Theorem can be derived from Theorem 15 in [11].

Theorem 3. *Let $b > 0$ and $f_\alpha : [a,b] \to \mathbb{R}$ be an arbitrary family of trigonometrically convex functions and let $f(x) = \sup_\alpha f_\alpha(x)$. If $J = \{u \in [a,b] : f(u) < \infty\}$ is nonempty, then J is an interval and f is a trigonometrically convex function on J.*

Proof. Let $t \in [0,1]$ and $x, y \in J$ be arbitrary. Then

$$\begin{aligned}
f(tx+(1-t)y) &= \sup_\alpha f_\alpha(tx+(1-t)y) \\
&\leq \sup_\alpha \left[\left(\sin\frac{\pi t}{2}\right) f_\alpha(x) + \left(\cos\frac{\pi t}{2}\right) f_\alpha(y)\right] \\
&\leq \left(\sin\frac{\pi t}{2}\right) \sup_\alpha f_\alpha(x) + \left(\cos\frac{\pi t}{2}\right) \sup_\alpha f_\alpha(y) \\
&= \left(\sin\frac{\pi t}{2}\right) f(x) + \left(\cos\frac{\pi t}{2}\right) f(y) < \infty.
\end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is a trigonometrically convex function on J.

This completes the proof of theorem. ∎

3. HERMITE-HADAMARD INEQUALITY FOR TRIGONOMETRICALLY CONVEX FUNCTIONS

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for trigonometrically convex functions.

We will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on $[a,b]$.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a trigonometrically convex function. If $a<b<$ and $f\in L[a,b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{2}{\pi}\left[f(a)+f(b)\right].$$

Proof. By using trigonometrically convexity of the function f, if the variable is changed as $u=ta+(1-t)b$, then

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(u)du &= \int_0^1 f\left(ta+(1-t)b\right)dt\\ &\le \int_0^1\left[\left(\sin\frac{\pi t}{2}\right)f(a)+\left(\cos\frac{\pi t}{2}\right)f(b)\right]dt\\ &= f(a)\int_0^1 \sin\frac{\pi t}{2}dt+f(b)\int_0^1\cos\frac{\pi t}{2}dt\\ &= \frac{2}{\pi}\left[f(a)+f(b)\right].\end{aligned}$$

This completes the proof of theorem. ∎

The following Theorem is a special case of Theorem 5 from [2], but we will give a direct simpler proof.

Theorem 5. *Let the function $f:[a,b]\to\mathbb{R}$,be a trigonometrically convex function. If $a<b$ and $f\in L[a,b]$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\le\frac{\sqrt{2}}{b-a}\int_a^b f(x)dx.$$

Proof. By the trigonometrically convexity of the function f, we have

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &= f\left(\frac{[ta+(1-t)b]+[(1-t)a+tb]}{2}\right)\\ &= f\left(\frac{1}{2}\left[ta+(1-t)b\right]+\frac{1}{2}\left[(1-t)a+tb\right]\right)\\ &\le \sin\frac{\pi}{4}f\left(ta+(1-t)b\right)+\cos\frac{\pi}{4}f\left((1-t)a+tb\right)\\ &= \frac{\sqrt{2}}{2}\left[f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right].\end{aligned}$$

Now, if we take integral in the last inequality with respect to $t \in [0,1]$, we deduce that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\sqrt{2}}{2}\left[\int_0^1 f\left(ta+(1-t)b\right)dt + \int_0^1 f\left((1-t)a+tb\right)dt\right] \\ &= \frac{\sqrt{2}}{2}\left[\frac{1}{a-b}\int_b^a f(x)dx + \frac{1}{b-a}\int_a^b f(y)dy\right] \\ &= \frac{\sqrt{2}}{b-a}\int_a^b f(x)dx. \end{aligned}$$

This completes the proof of theorem. ∎

4. SOME NEW INEQUALITIES FOR TRIGONOMETRICALLY CONVEXITY

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is trigonometrically convex. Dragomir and Agrawal [3] used the following lemma

Lemma 2. *The following equality holds true:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'\left(ta+(1-t)b\right)dt$$

Note that we will use the following integrals in this section:

$$\begin{aligned} \int_0^1 \sin\frac{\pi t}{2}dt &= \int_0^1 \cos\frac{\pi t}{2}dt = \frac{2}{\pi}, \\ \int_0^1 |1-2t|\sin\frac{\pi t}{2}dt &= \int_0^1 |1-2t|\cos\frac{\pi t}{2}dt = \frac{2}{\pi^2}\left(\pi - 4\left(\sqrt{2}-1\right)\right), \\ \int_0^1 |1-2t|^p\, dt &= \frac{1}{p+1}. \end{aligned}$$

We denote by $A(u,v)$ the arithmetic mean of u and v.

Theorem 6. *Let $f : I \to \mathbb{R}$ be a continuously differentiable function, let $a < b$ in I and assume that $f' \in L[a,b]$. If $|f'|$ is trigonometrically* *convex function on interval* $[a,b]$*, then the following inequality*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & \frac{2}{\pi}(b-a)\left[1-\frac{4}{\pi}\left(\sqrt{2}-1\right)\right]A\left(\left|f'(a)\right|,\left|f'(b)\right|\right) \end{aligned}$$

holds for $t\in[0,1]$.

Proof. Using Lemma 2 and the inequality

$$\left|f'\left(ta+(1-t)b\right)\right| \leq \left(\sin\frac{\pi t}{2}\right)\left|f'(a)\right| + \left(\cos\frac{\pi t}{2}\right)\left|f'(b)\right|,$$

we get

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & \left|\frac{b-a}{2}\int_0^1 (1-2t)f'\left(ta+(1-t)b\right)dt\right| \\ \leq\ & \frac{b-a}{2}\int_0^1 \left|1-2t\right|\left|f'\left(ta+(1-t)b\right)\right|dt \\ \leq\ & \frac{b-a}{2}\int_0^1 \left|1-2t\right|\left[\left(\sin\frac{\pi t}{2}\right)\left|f'(a)\right|+\left(\cos\frac{\pi t}{2}\right)\left|f'(b)\right|\right]dt \\ =\ & 2(b-a)\left[\frac{1}{\pi}+\frac{4}{\pi^2}\left(1-\sqrt{2}\right)\right]A\left(\left|f'(a)\right|,\left|f'(b)\right|\right). \end{aligned}$$

This completes the proof of theorem. ∎

Theorem 7. *Let* $f:I\to\mathbb{R}$ *be a continuously differentiable function, let* $a<b$ *in* I *and assume that* $q>1$*. If* $|f'|^q$ *is a trigonometrically convex function on interval* $[a,b]$*, then the following inequality*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}2^{\frac{2}{q}}\pi^{-\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right). \end{aligned}$$

holds for $t\in[0,1]$*, where* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2, Hölder's integral inequality and inequality

$$\left|f'\left(ta+(1-t)b\right)\right|^q \leq \left(\sin\frac{\pi t}{2}\right)\left|f'(a)\right|^q + \left(\cos\frac{\pi t}{2}\right)\left|f'(b)\right|^q$$

which is the trigonometrically convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\int_0^1 |1-2t|\left|f'\left(ta+(1-t)b\right)\right|dt \\
\leq\ & \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\sin\frac{\pi t}{2}\right)|f'(a)|^q+\left(\cos\frac{\pi t}{2}\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[|f'(a)|^q\int_0^1\sin\frac{\pi t}{2}dt+|f'(b)|^q\int_0^1\cos\frac{\pi t}{2}dt\right]^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{2\left(|f'(a)|^q+|f'(b)|\right)^q}{\pi}\right]^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}2^{\frac{2}{q}}\pi^{-\frac{1}{q}}A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right).
\end{aligned}$$

This completes the proof of theorem. ∎

Theorem 8. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a continuously differentiable function, let $a<b$ in I and assume that $q\geq 1$. If $|f'|^q$ is a trigonometrically convex function on the interval $[a,b]$, then the following inequality*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{3}{q}}\left[\frac{1}{\pi}-\frac{4\left(\sqrt{2}-1\right)}{\pi^2}\right]^{\frac{1}{q}}A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right)
\end{aligned}$$

holds for $t\in[0,1]$.

Proof. Assume first that $q > 1$. From Lemma 2, Hölder integral inequality and trigonometrically convexity of $|f'|^q$, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\int_0^1 |1-2t|\left|f'\left(ta+(1-t)b\right)\right|dt \\
\leq\ & \frac{b-a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 |1-2t|\left[\left(\sin\frac{\pi t}{2}\right)|f'(a)|^q+\left(\cos\frac{\pi t}{2}\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}} \\
& \times\left(|f'(a)|^q\int_0^1 |1-2t|\sin\frac{\pi t}{2}dt+|f'(b)|^q\int_0^1 |1-2t|\cos\frac{\pi t}{2}dt\right)^{\frac{1}{q}}
\end{aligned}
$$

Since $\int_0^1 |1-2t|\,dt = \frac{1}{2}$ and $\int_0^1 |1-2t|\sin\frac{\pi t}{2}dt = \int_0^1 |1-2t|\cos\frac{\pi t}{2}dt = \frac{2}{\pi^2}\left(\pi-4\left(\sqrt{2}-1\right)\right)$, it follows that

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \\
&\tfrac{b-a}{2}\left(\tfrac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left[\frac{2}{\pi^2}\left(\pi-4\left(\sqrt{2}-1\right)\right)\right]\left(|f'(a)|^q+|f'(b)|^q\right)\right\}^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} 2^{\frac{2}{q}}\left\{\left[\frac{1}{\pi^2}\left(\pi-4\left(\sqrt{2}-1\right)\right)\right]A\left(|f'(a)|^q,|f'(b)|^q\right)\right\}^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{3}{q}}\left[\frac{1}{\pi}-\frac{4\left(\sqrt{2}-1\right)}{\pi^2}\right]^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right).
\end{aligned}
$$

For $q = 1$ we use the estimates from the proof of Theorem 6, which also follow step by step the above estimates.

This completes the proof of theorem. ∎

Corollary 1. *Under the assumption of Theorem 8 with $q = 1$, we get the conclusion of Theorem 6.*

Acknowledgement 1. *The author wishes to express her thanks to the anonymous referee and Dr. İmdat İŞCAN for their careful corrections to and valuable comments on the original version of this paper.*

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Bulancak Bahcelievler Anatolian High School
Giresun-TURKEY
e-mail: huriyekadakal@hotmail.com