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SUPRA GENERALIZED PRE-REGULAR SEPARATION AXIOMS

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Abstract: In this paper, new types of separation axioms in supratopological spaces, namely gpr^{μ}-separation axioms are introduced and discussed. Several characterizations and consequences of the properties given by these axioms are studied.

1. Introduction

In 1970, Levine [2] defined generalized closed sets in topological spaces and also introduced a class of topological spaces called $T_{1/2}$ spaces. The extended study in the field of generalized closed sets introduced several new separation axioms. In 1975, Maheswari and Prasad [3] defined the three new separation axioms called semi-T_2, semi-T_1 and semi-T_0 using semi-open sets. After that many researchers introduced different types of separation axioms. In 1983, Mashhour et.al [4] derived supra topological spaces, investigated S-continuous maps and S^*- continuous maps. Extensive research in supra topological space made many topologists to introduce different types of open and closed mappings. These innovations further developed and in 2012, Mustafa and Qoqazeh [1] introduced the notions of supra-T_i for spaces $i = 0, 1, 2$. In 2012, Menon [6] introduced the notion of gpr^{μ}-open sets.

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In this paper, we define gpr^{μ}-separation axioms namely gpr^{μ} -T_2, gpr^{μ} -T_1 and gpr^{μ} -T_0 and study the properties of these spaces. The separation axioms supra-R_0 and supra-R_1 were introduced and studied by Mashhour et. al [5]. In this paper, we use the concept of gpr^{μ}-open sets to introduce gpr^{μ} -R_0 and gpr^{μ} -R_1 spaces. Throughout this paper, (X, τ), (Y, σ) and (Z, η) represent nonempty topological spaces on which no separation axioms are assumed unless explicitly stated.

2. Preliminaries

Definition 2.1. [1] Let X be a non empty set. A subcollection $\mu \subset \mathcal{P}(X)$ is called a *supra topology* on X if
(i) $\emptyset$, $X \in \mu$ and
(ii) μ is closed under arbitrary union.
Then (X, μ) is called a *supra topological space*. The elements of μ are said to be *supra open sets* in (X, μ) and the complement of a supra open set is called *supra closed set*. The *supra closure* of a set A, denoted by $cl^{\mu}(A)$, is the intersection of all supra closed sets including A. The *supra interior* of a set A, denoted by $int^{\mu}(A)$, is the union of all supra open sets included in A. We call μ a supra topology associated with a topology τ if $\tau \subset \mu$.
A set A is called *supra pre-closed* [6] if $cl^{\mu}(int^{\mu}(A)) \subseteq A$. The complement of supra pre-closed set is called *supra pre-open set* [6]. The *supra pre- closure* of A, denoted by $pcl^{\mu}(A)$ is the intersection of the supra pre-closed sets including A. The *supra pre-interior* of A, denoted by $pint^{\mu}(A)$ is the union of the supra pre-open sets included in A.

Definition 2.2. [7] A subset A of a space (X, μ) is called
(i) *supra generalized pre-regular closed* (briefly, *gpr^{μ}-closed*) if $pcl^{\mu}(A) \subseteq U$ whenever $A \subseteq U$ and U is supra regular open in (X, μ).
(ii) *supra generalized pre-regular open* (briefly, *gpr^{μ}-open*) if $U \subseteq pint^{\mu}(A)$ whenever $U \subseteq A$ and U is supra regular closed in (X, μ).
The collections of all supra generalized pre-regular closed and supra generalized pre-regular open subsets of X are denoted by $GPRC^{\mu}(X)$ and $GPRO^{\mu}(X)$ respectively.

Definition 2.3. Let (X, τ) and (Y, σ) be two topological spaces with supra topologies λ and μ associated with τ and σ respectively. A function $f : (X, \tau) \to (Y, \sigma)$ is called
(i) *gpr^{λ}-continuous* [6] if $f^{-1}(V)$ is gpr^{λ}-closed (resp. gpr^{λ}-open) in X for every closed set (resp. open set) V of Y.
(ii) *supra gpr-irresolute* [8] if $f^{-1}(V)$ is gpr^{λ}-closed (resp. gpr^{λ}-open) in X for every gpr^{μ} -closed set (resp. gpr^{μ}-open set) V of Y.

Definition 2.4. [7] A space (X, μ) is called *supra preregular* $T_{1/2}$ space if every gpr^{μ}-closed set is supra pre-closed.

Definition 2.5. [6] Let A be a subset of (X, μ). Then
(i) $gpr^{\mu}\text{-}cl(A) = \bigcap[F : A \subset F, F$ is a gpr^{μ}-closed set in (X, μ).
(ii)$gpr^{\mu}\text{-}int(A) = \bigcup[M : M \subset A, M$ is a gpr^{μ}-open set in (X, μ).

Definition 2.6. [6] Let (X, μ) be a supra topological space. If Y is a subset of X, the collection $\mu_Y = \{Y \cap U : U \in \mu\}$ is a supra topology on Y called the supra subspace topology. With this supra topology, Y is called a *supra subspace* of X.

Now we recall some separation axioms in a supra topological space [1].

Definition 2.7. [1] Let (X, μ) be a supra topological space. Then
(i) X is S-T_0 if for every two distinct points x and y in X, there exists a supra open set U containing one of them but not the other.
(ii) X is S-T_1 if for every two distinct points x and y in X , there exists a pair of supra open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.
(iii) X is S-T_2 if for every two distinct points x and y in X , there exists a pair of disjoint supra-open sets U and V such that $x \in U$ and $y \in V$.

3. GPRμ-SEPARATION AXIOMS

Definition 3.1. Let (X, μ) be a supra topological space with supra topology μ. Then
(i)X is gpr^{μ}-T_0 if for every two distinct points x and y in X, there exists a gpr^{μ}- open set U containing one of them but not the other,
(ii) X is gpr^{μ}-T_1 if for every two distinct points x and y in X, there exists a pair of gpr^{μ}- open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$,
(iii) X is gpr^{μ}-T_2 if for every two distinct points x and y in X, there exists a pair of disjoint gpr^{μ}-open sets U and V such that $x \in U$ and $y \in V$.

Definition 3.2. [8] Let (X, τ) and (Y, σ) be two topological spaces with supra topologies λ and μ associated with τ and σ respectively. A function $f : (X, \tau) \to (Y, \sigma)$ is called *gpr^{μ}-open* (resp. *gpr^{μ}-closed*) if the image of every open set (resp. closed set) in X is gpr^{μ}-open (resp. gpr^{μ}-closed) in Y.

Theorem 3.3. *Every supra topological space (X, μ) is gpr^{μ} - T_0.*
Proof. Let x and y be any two points in X such that $x \neq y$. If

$int^\mu\{x\} \neq \emptyset$, then $\{x\}$ is supra open, hence $\{x\}$ is gpr^μ-open. Thus (X, μ) is gpr^μ - T_0. Now, if $int^\mu\{x\} = \emptyset$, then $\{x\}$ is supra pre-closed . Thus $X - \{x\}$ is supra pre-open. But every supra pre-open set is gpr^μ-open, so $X - \{x\}$ is gpr^μ-open. Therefore (X, μ) is gpr^μ-T_0.

Theorem 3.4. *If (X, μ) is any supra topological space, then any gpr^μ-closures of distinct points in X are distinct.*

Proof. Let $x, y \in X$ and $x \neq y$. We prove that gpr^μ-$cl(\{x\}) \neq gpr^\mu$-$cl(\{y\})$. Assume $A = X - \{x\}$. Then $cl^\mu(A) = A$ or $cl^\mu(A) = X$. If $cl^\mu(A) = A$, then A is supra closed and so gpr^μ-closed. Therefore $\{x\} = X - A$ is a gpr^μ-open set which contains x but not y. Hence $y \notin gpr^\mu$-$cl(\{x\})$. But $x \in gpr^\mu$-$cl(\{x\})$, therefore gpr^μ-$cl(\{x\}) \neq gpr^\mu$-$cl(\{y\})$.
If $cl^\mu(A) = X$, then A is supra pre-open and so $\{x\} = X - A$ is supra pre-closed. Therefore $\{x\}$ is gpr^μ- closed and thus gpr^μ-$cl(\{x\}) = \{x\}$. Since $y \notin gpr^\mu$-$cl(\{x\})$ and $y \in gpr^\mu$-$cl(\{y\})$, gpr^μ-$cl(\{x\}) \neq gpr^\mu$-$cl(\{y\})$.

Theorem 3.5. *For a supra topological space (X, μ) every S-T_i space is gpr^μ - T_i for $i = 0, 1, 2$.*

Proof. Let (X, μ) be a S-T_0 space. Let x and y be two disjoint points in X. The assumption that X is S-T_0 implies that there exists a supra open set U in X such that $x \in U$, $y \notin U$ or $x \notin U$, $y \in U$. Let $x \in U$ and $y \notin U$. Since every supra open set is gpr^μ-open, U is a gpr^μ-open set in X. Thus, for any two distinct points x, y in X there exists a gpr^μ-open set U in X such that $x \in U, y \notin U$ or $x \notin U, y \in U$. Hence (X, μ) is a gpr^μ - T_0 space.
Similarly we can prove that each S-T_i space is gpr^μ-T_i for $i = 1, 2$.

Theorem 3.6. *Every gpr^μ- T_i space is gpr^μ - T_{i-1} for each $i = 1, 2$.*

Proof. Let X be a gpr^μ- T_2 space and let x and y be any two points of X such that $x \neq y$. By definition, there exists a pair of disjoint gpr^μ-open sets U and V such that $x \in U$ and $y \in V$. This implies that U and V are gpr^μ-open sets such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$. Hence X is a gpr^μ- T_1 space.
Similarly every gpr^μ- T_1 space is a gpr^μ- T_0 space.

Remark 3.7. The converse of the Theorem 3.5 need not be true as seen in the following example.

Example 3.8 Let (X, μ) be a supra topological space.
(i) Let $X = \{a, e, f\}$, $\mu = \{\emptyset, X, \{a\}\}$.
Then $GPRO^\mu(X) = \{\emptyset, X, \{a\}, \{e\}, \{f\}, \{a, e\}, \{e, f\}, \{a, f\}\}$, hence (X, μ) is a gpr^μ - T_0 space, but not a S-T_0 space.
(ii) Let $X = \{a, e, f\}$, $\mu = \{\emptyset, X, \{a, e\}, \{e, f\}\}$.

Then $GPRO^{\mu}(X) = \{\emptyset, X, \{a\}, \{e\}, \{f\}, \{a,e\}, \{e,f\}, \{a,f\}\}$, hence (X,μ) is a gpr^{μ}-T_1 and gpr^{μ}-T_2 space. But (X,μ) is not a S-T_1 and S-T_2 space.

Remark 3.9. A supra topological space (X,μ) which is a gpr^{μ}-T_0 space need not be a supra pre-regular $T_{1/2}$ space, as the following example shows.

Example 3.10. Let $X = \{0,1,2\}, \mu = \{\emptyset, X, \{0,1\}, \{1,2\}\}$. Then (X,μ) is a gpr^{μ} - T_0 space but not a supra pre-regular $T_{1/2}$ space.

Theorem 3.11. *Let (X,μ) be a supra topological space such that each one point set is gpr^{μ}-closed. Then X is gpr^{μ} - T_1.*

Proof. Let $x, y \in X$ such that $x \neq y$. By our assumption, $\{x\}, \{y\}$ are gpr^{μ}-closed sets in X. Then $X - \{x\}$ and $X - \{y\}$ are gpr^{μ}-open sets such that $X - \{x\}$ contains y but not x and $X - \{y\}$ contains x but not y. Thus (X,μ) is a gpr^{μ} - T_1 space.

Definition 3.12. A supra topological space (X,μ) is called s*upra symmetric space* if for x and y in X, $x \in cl^{\mu}(\{y\})$ implies $y \in cl^{\mu}(\{x\})$.

Definition 3.13. A supra topological space (X,μ) is called *gpr^{μ}-symmetric space* if for x and y in X , $x \in gpr^{\mu}\text{-}cl(\{y\})$ implies $y \in gpr^{\mu}\text{-}cl(\{x\})$.

Theorem 3.14. *Let (X,μ) be a gpr^{μ}-symmetric space. Then the following are equivalent:*
(i) (X,μ) is gpr^{μ}-T_0.
(ii) (X,μ) is gpr^{μ}-T_1.

Proof. (i)$\Longrightarrow$ (ii): Let $x \neq y$. Since (X,μ) is gpr^{μ}-T_0, we have $x \in U \subset X - \{y\}$ for some gpr^{μ}-open set U. Thus $X - U$ is a gpr^{μ}-closed set containing y but not x. Now, $gpr^{\mu}\text{-}cl(\{y\}) \subset X - U$ and therefore $x \notin gpr^{\mu}\text{-}cl(\{y\})$. This implies $y \notin gpr^{\mu}\text{-}cl(\{x\})$. Thus there exist a gpr^{μ}-open set V such that $y \in V \subset X - \{x\}$ and this shows that(X,μ) is a gpr^{μ}-T_1 space.
(ii)$\Longrightarrow$(i): Obvious, by definitions.

Theorem 3.15. *If (X,μ) is a supra topological space, then the following are equivalent:*
(i) (X,μ) is a gpr^{μ}-symmetric space.
(ii) $\{x\}$ is gpr^{μ}-closed, for each $x \in X$.

Proof. (i)$\Longrightarrow$(ii): Let $\{x\} \subseteq U$, where U is a supra regular open set such that $pcl^{\mu}(\{x\})$ not contained in U. Then $pcl^{\mu}(\{x\}) \cap X - U \neq \emptyset$. Now, let $y \in pcl^{\mu}(\{x\}) \cap X - U$. Then $pcl^{\mu}(\{y\}) \subseteq X - U$.

Since every supra pre-closed set is gpr^{μ}- closed, gpr^{μ}-$cl(\{y\}) \subseteq pcl^{\mu}(\{y\}) \subseteq X - U$. By hypothesis, $x \in gpr^{\mu}$-$cl(\{y\}) \subseteq X - U$. Thus $x \notin U$, which is a contradiction. Therefore, $\{x\}$ is gpr^{μ}-closed, for each $x \in X$.
(ii)$\Longrightarrow$ (i): Assume that $x \in gpr^{\mu}$-$cl(\{y\})$ and $y \notin gpr^{\mu}$-$cl(\{x\})$.
Then $\{y\} \subseteq X - gpr^{\mu}$-$cl(\{x\})$ and hence gpr^{μ}-$cl(\{y\}) \subseteq X - gpr^{\mu}$-$cl(\{x\})$. Therefore $x \in X - gpr^{\mu}$-$cl(\{x\})$, which is a contradiction. Thus $y \in gpr^{\mu}$-$cl(\{x\})$.

Theorem 3.16. *If $GPRO^{\mu}(X)$ is closed under arbitrary union for a supra topological space (X, μ), then the following statements are equivalent:*
(i) (X, μ) is gpr^{μ}-T_1.
(ii) (X, μ) is gpr^{μ}-T_0.
(iii) Each singleton is gpr^{μ}-closed in X.
(iv) Each subset A of X is the intersection of all gpr^{μ}-open sets including A.
(v) The intersection of all gpr^{μ}-open sets containing the point $x \in X$ is the set $\{x\}$.

Proof. (i)$\Longrightarrow$ (ii): Obvious.
(ii)$\Longrightarrow$ (iii): Let (X, μ) be a gpr^{μ}-T_0 space and $x \in X$.
Then for any $y \in X$, $y \neq x$ there exist a gpr^{μ}-open set U containing y not x or U containing x but not y. Let $y \in U$, then $y \in U \subset X - \{x\}$. Therefore $X - \{x\} = \bigcup\{U : y \subset X - \{x\}\}$. Thus $X - \{x\}$ is a gpr^{μ}-open set. Hence $\{x\}$ is gpr^{μ}-closed.
(iii)$\Longrightarrow$ (iv): Let each one point set is gpr^{μ}-closed in X. If $A \subset X$, then for each point $y \notin A$, there exist a set $X - \{y\}$ such that $A \subset X - \{y\}$ and each of these sets is gpr^{μ}-open. Therefore $A = \bigcap\{X - \{y\} : y \in X - A\}$. This implies intersection of all gpr^{μ}-open sets containing A is the set A itself.
(iv)$\Longrightarrow$ (v): Obvious.
(v)$\Longrightarrow$(i): Let $x, y \in X$ such that $x \neq y$. Then by (v), the intersection of all gpr^{μ}- open sets containing the point x is the set $\{x\}$ and the intersection of all gpr^{μ}-open sets containing the point y is the set $\{y\}$. By hypothesis, for each $x \in X$ there exist a gpr^{μ}-open set U such that $x \in U$ and $y \notin U$ and for each $y \in X$ there exist a gpr^{μ}-open set V such that $y \in V$ and $x \notin V$. Therefore (X, μ) is a gpr^{μ}-T_1 space.

Theorem 3.17. *If $GPRC^{\mu}(X)$ is closed under arbitrary intersection for a supra topological space (X, μ) , then following statements are equivalent:*
(i) (X, μ) is gpr^{μ} - T_2.

(ii) If $x \in X$, then for each $y \neq x$, there exist a gpr^{μ}-open set U such that $x \in U$ and $y \notin gpr^{\mu}$-$cl(U)$.
(iii) For each $x \in X$, $\bigcap\{gpr^{\mu}\text{-}cl(U) : U$ is a gpr^{μ}-open set with $x \in U\} = \{x\}$.
Proof. (i)$\Longrightarrow$(ii): Let $x \in X$ and $y \neq x$. Then there exist two disjoint gpr^{μ}-open sets U and V such that $x \in U$ and $y \in V$. Then $X - V$ is gpr^{μ}-closed with gpr^{μ}-$cl(U) \subseteq X - V$ and $y \notin X - V$. Therefore $y \notin gpr^{\mu}$-$cl(U)$.
(ii)$\Longrightarrow$ (iii): If $y \neq x$, then there exist a gpr^{μ}-open set U such that $x \in U$ and $y \notin gpr^{\mu}$-$cl(U)$. Therefore $y \notin \bigcap\{gpr^{\mu}\text{-}cl(U) : U$ is a gpr^{μ}-open set with $x \in U\}$. This implies that $\bigcap\{gpr^{\mu}\text{-}cl(U); U$ is a gpr^{μ}-open set with $x \in U\} = \{x\}$.
(iii)$\Longrightarrow$(i): Let $y \neq x$. Then $y \notin \{x\} = \bigcap\{gpr^{\mu}\text{-}cl(U) : U$ is a gpr^{μ}-open set with $x \in U\}$. This implies that there exist a gpr^{μ}-open set U such that $x \in U$ and $y \notin gpr^{\mu}$-$cl(U)$. Let $V = X - gpr^{\mu}$-$cl(U)$. Then V is a gpr^{μ}-open set with $y \in V$ and $U \cap V = \emptyset$. Therefore (X, μ) is gpr^{μ}-T_2.

Theorem 3.18. *For every supra topological space (X, μ) which is an S-T_i* space, *each supra subspace is a S-T_i space, for $i = 0, 1, 2$.*

Proof. Let (X, μ) be a S-T_0 space and (Y, μ_Y) be a supra subspace of (X, μ).

Let $x, y \in Y$ be arbitrary such that $x \neq y$. Since $Y \subset X$, $x, y \in X$ with $x \neq y$.

a) Now, if X is S-T_0, then there exists a supra open set G such that $x \in G$ and $y \notin G$. Since μ_Y is a supra subspace topology, $G \cap Y \in \mu_Y$. So, $x \in G \cap Y$ and $y \notin G \cap Y$. Therefore (Y, μ_Y) is a S-T_0 space.
b) Let (X, μ) be a S-T_1 space and (Y, μ_Y) be a supra subspace of (X, μ). The assumption that X is S-T_1 implies that there exist supra open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Since μ_Y is a supra subspace topology, $U \cap Y \in \mu_Y$ and $V \cap Y \in \mu_Y$.This implies $x \in U \cap Y$, $y \notin U \cap Y$ and $y \in V \cap Y$, $x \notin V \cap Y$. Therefore (Y, μ_Y) is a S-T_1 space.
c) Let (X, μ) be a S-T_2 space and (Y, μ_Y) be a supra subspace of (X, μ). The assumption that X is S-T_2 implies that there exist supra open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Since μ_Y is a supra subspace topology, $U \cap Y \in \mu_Y$ and $V \cap Y \in \mu_Y$. Thus implies $x \in U \cap Y$ and $y \in V \cap Y$. Now $U \cap V = \emptyset$ implies $(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \emptyset$. Therefore, (Y, μ_Y) is a S-T_2 space.

Theorem 3.19. *For a supra topological space (X,μ), each supra subspace of a S-T_i space is gpr^{μ}-T_i space for $i = 0,1,2$.*

Proof. Follows from Theorem 3.5 and Theorem 3.18.

Theorem 3.20. *Let (X,τ) and (Y,σ) be two topological spaces and let μ be a supra topology associated with σ. If $f : (X,\tau) \to (Y,\sigma)$ is a bijective, gpr^{μ}-open map and X is T_1, then Y is a gpr^{μ}-T_1 space.*

Proof. Let y_1, $y_2 \in Y$ with $y_1 \neq y_2$. Since f is bijective, there exist x_1, $x_2 \in X$ with $x_1 \neq x_2$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. By hypothesis X is T_1, then there exist a pair of open sets U and V in X such that $x_1 \in U$, $x_2 \notin U$ and $x_2 \in V$, $x_1 \notin V$. Now, f is gpr^{μ}-open implies $f(U)$ and $f(V)$ are gpr^{μ}-open sets in Y. Thus $y_1 \in f(U)$, $y_2 \notin f(U)$ and $y_2 \in f(V)$, $y_1 \notin f(V)$. Hence Y is a gpr^{μ}-T_1.

Theorem 3.21. *Let (X,τ) and (Y,σ) be two topological spaces with supra topologies λ and μ associated with τ and σ respectively. If $f : (X,\tau) \to (Y,\sigma)$ is a bijective, supra gpr-irresolute map and Y is a gpr^{μ}-T_1 space, then X is also a gpr^{λ}-T_1 space.*

Proof. Let x_1, $x_2 \in X$ with $x_1 \neq x_2$. Since f is bijective, there exist y_1, $y_2 \in Y$ with $y_1 \neq y_2$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Thus $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Now, Y is gpr^{μ}-T_1 space implies there exist a pair of gpr^{μ}-open sets U and V in Y such that $y_1 \in U$, $y_2 \notin U$ and $y_2 \in V$, $y_1 \notin V$. Since f is supra *gpr*-irresolute, $f^{-1}(U)$ and $f^{-1}(V)$ are gpr^{λ}-open sets in X . Then $x_1 \in f^{-1}(U)$, $x_2 \notin f^{-1}(U)$ and $x_2 \in f^{-1}(V)$, $x_1 \notin f^{-1}(V)$. Hence X is a gpr^{λ}-T_1 space.

Theorem 3.22. *Let (X,τ) and (Y,σ) be two topological spaces with supra topology μ associated with τ. If $f : (X,\tau) \to (Y,\sigma)$ is a bijective, gpr^{μ}-continuous map and Y is T_1, then X is a gpr^{μ}-T_1 space.*

Proof. Let x_1 and x_2 be two distinct points in X. Let $y_1 = f(x_1)$ and $y_2 = fx_2)$. Since f is bijective, $y_1 \neq y_2$ and $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$. Since Y is T_1, there exist a pair of open sets U and V in Y such that $y_1 \in U$, $y_2 \notin U$ and $y_2 \in V$, $y_1 \notin V$. Now, f is also gpr^{μ}-continuous implies $f^{-1}(U)$ and $f^{-1}(V)$ are gpr^{μ}-open sets in X. Then $x_1 \in f^{-1}(U)$, $x_2 \notin f^{-1}(U)$ and $x_2 \in f^{-1}(V)$, $x_1 \notin f^{-1}(V)$. Therefore, X is a gpr^{μ}-T_1 space.

4. Supra generalized pre-regular neighbourhoods

Definition 4.1. [1] Let (X,μ) be a supra topological space and let $x \in X$. A subset N of X is said to be a *supra neighbourhood* of a point x if there exists a supra open set G such that $x \in G \subseteq N$.

Definition 4.2. Let (X, μ) be a supra topological space and let $x \in X$. A subset N of X is said to be a *supra generalized pre-regular neighbourhood* (briefly, a supra *gpr-neighbourhood*) of x if there exists a supra generalized pre-regular open set G such that $x \in G \subseteq N$.

Definition 4.3. A subset N of a supra topological space (X, μ) is called a *supra generalized pre-regular neighbourhood of a set* $A \subseteq X$ if there exist a supra generalized pre-regular open set G such that $A \subseteq G \subseteq N$.

Remark 4.4. For a supra topological space (X, μ), the supra generalized pre-regular neighborhood of $x \in X$ need not be a supra generalized pre-regular open set in X.

Example 4.5. Consider the supra topological space (X, μ) ,where $X = \{0, 1, 2, 3\}$ with supra topology $\mu = \{\emptyset, X, \{0\}, \{1\}, \{0, 1\}\}$. Now $\{0, 2, 3\}$ is a supra *gpr*-neighbourhood of $\{0\}$,$\{2\}$ and $\{3\}$ but $\{0, 2, 3\}$ is not a gpr^{μ}-open set in (X, μ).

Theorem 4.6. *For a supra topological space (X, μ), every supra neighbourhood N of $x \in X$ is a supra gpr-neighbourhood of x.*

Proof. Let N be a supra neighbourhood of a point $x \in X$. Then there exists a supra open set G such that $x \in G \subseteq N$. Since every supra open set is gpr^{μ}-open, G is a gpr^{μ}-open set such that $x \in G \subseteq N$. Hence N is a supra *gpr*-neighbourhood of x.

Remark 4.7. In general, a supra *gpr*-neighbourhood of $x \in X$ in (X, μ) need not be a supra neighbourhood of x in (X, μ).

Example 4.8. Consider the supra topological space (X, μ), where $X = \{a, b, c, d\}$ with supra topology $\mu = \{\emptyset, X, \{a, b\}, \{a\}, \{b\}\}$. Then $GPRO^{\mu}(X) = \{\emptyset, X, \{a, b, c\}, \{a, b, d\}, \{a, d\}, \{c, d\}, \{a, b\}, \{b, c\}, \{a, c\}, \{b, d\}, \{d\}, \{c\}, \{a\}, \{b\}\}$.
The set $\{a, c, d\}$ is the supra *gpr*-neighbourhood of the point c but not a supra neighbourhood of point c.

Theorem 4.9. *If a subset N of a supra topological space (X, μ) is gpr^{μ}-open, then N is a supra gpr-neighbourhood of each of its points.*

Proof. Let (X, μ) be a supra topological space and let N be gpr^{μ}-open. Fix $x \in N$. Since N is a gpr^{μ}-open set and $x \in N \subseteq N$, we see that N is a supra *gpr*-neighbourhood of x. Since x is an arbitrary point of N, it follows that N is a supra *gpr*-neighborhood of each of its points.

Remark 4.10. The converse of the Theorem 4.9 is not true.

Example 4.11. Consider the supra topological space (X, μ) ,where $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, X, \{a, b\}, \{a\}, \{b\}\}$.

Then $GPRO^{\mu}(X) = \{\emptyset, X, \{a,b,c\}, \{a,b,d\}, \{a,d\}, \{c,d\}, \{a,b\}, \{b,c\}, \{a,c\}, \{d\}, \{c\}, \{a\}, \{b\}\}$.

The set $\{a,c,d\}$ is a supra *gpr*-neighbourhood of each of its points but the set $\{a,c,d\}$ is not a gpr^{μ}-open set in X.

Theorem 4.12. *If $GPRC^{\mu}(X)$ is closed under arbitrary intersection for a supra topological space (X,μ), then followings statements are equivalent.*
(i)(X,μ) is gpr^{μ}-T_2.
(ii) If $x \in X$, then for each $y \neq x$, there exist a supra gpr-neighbourhood $N(x)$ of x such that $y \notin gpr^{\mu}$-$cl(N(x))$.

Proof. (i) $\Longrightarrow$ (ii): Let X be a gpr^{μ} - T_2 space and let $x \in X$. Then for each $y \neq x$ there exist two disjoint gpr^{μ}-open sets U and V such that $x \in U$ and $y \in V$. Then $x \in U \subset X \setminus V$. Thus, $X - V$ is a supra *gpr*-neighbourhood of x. Let $N(x) = X \setminus V$. Then $N(x)$ is gpr^{μ}-closed set in X and $y \notin X \setminus V$ implies $y \notin N(x)$. Thus $y \notin gpr^{\mu}$-$cl(N(x))$.
(ii) $\Longrightarrow$ (i): Let $x, y \in X$, $x \neq y$. Then by hypothesis there exists a supra *gpr*-neighbourhood $N(x)$ of x such that $y \notin gpr^{\mu}$-$cl(N(x))$. Thus $y \in X - gpr^{\mu}$-$cl(N(x))$ and $x \notin X - gpr^{\mu}$-$cl(N(x))$. But $X - gpr^{\mu}$-$cl(N(x))$ is a gpr^{μ}-open set. Also there exists a gpr^{μ}-open set A such that $x \in A \subset N(x)$ and $A \cap (X - gpr^{\mu}$-$cl(N(x)) = \emptyset$. It follows that (X,μ) is gpr^{μ}-T_2.

5. GPRμ-R_0 AND GPRμ-R_1 SPACES

Definition 5.1. A space (X,μ) is said to be a gpr^{μ}-R_0 space if for each gpr^{μ}-open set G and $x \in G$, gpr^{μ}-$cl(\{x\}) \subset G$.

Definition 5.2. A space (X,μ) is said to be a gpr^{μ}-R_1space if for $x, y \in X$ with gpr^{μ}-$cl(\{x\}) \neq gpr^{\mu}$-$cl(\{y\})$ there exist disjoint gpr^{μ}-open sets U and V such that gpr^{μ}-$cl(\{x\}) \subset U$ and gpr^{μ}-$cl(\{y\}) \subset V$.

Theorem 5.3. *If $GPRC^{\mu}(X)$ is closed under arbitrary intersection for a supra topological space (X,μ), then each of the following properties are equivalent.*
(i) (X,μ) is gpr^{μ}-R_0.
(ii) For any gpr^{μ}-closed set $F \subset X$ and any $x \notin F$, there exists a gpr^{μ}-open set U such that $F \subset U$ and $x \notin U$.
(iii) For any gpr^{μ}-closed set F and $x \notin F$ implies $F \cap gpr^{\mu}$-$cl(\{x\}) = \emptyset$.

Proof. (i)$\Longrightarrow$(ii): Let (X,μ) be a gpr^{μ}-R_0 space and $F \subset X$ be a gpr^{μ}-closed set such that $x \notin F$. Then $X - F$ is a gpr^{μ}-open set and $x \in X - F$. Since X is a gpr^{μ}-R_0 space, gpr^{μ}-$cl(\{x\}) \subset X - F$. This implies $F \subset X - gpr^{\mu}$-$cl(x)$. Let $U = X - gpr^{\mu}$-$cl(\{x\})$. Then U is

gpr^{μ}-open such that $F \subset U$ and $x \notin U$.
(ii) $\Longrightarrow$ (i): Let $x \in U$, where U is a gpr^{μ}-open set in X. Then $X \setminus U$ is gpr^{μ}-closed set and $x \notin X \setminus U$. By hypothesis there exist a gpr^{μ}-open set W such that $X \setminus U \subset W$ and $x \notin W$. Now $X - W \subset U$ and $x \in X - W$. But $X - W$ is gpr^{μ}-closed and hence gpr^{μ}-$cl(\{x\}) \subset X \setminus W \subset U$. Therefore, X is a gpr^{μ}-R_0 space.
(ii) $\Longrightarrow$ (iii): Let F be a gpr^{μ}-closed and $x \notin F$. There exist a gpr^{μ}-open set U such that $F \subset U$ and $x \notin U$. Also $U = X \setminus gpr^{\mu}$-$cl(\{x\})$ by hypothesis, $U \cap X - gpr^{\mu}$-$cl(\{x\}) = \emptyset$. Therefore $F \cap gpr^{\mu}$-$cl(\{x\}) = \emptyset$.
(iii) $\Longrightarrow$ (i): By hypothesis, for any gpr^{μ}-closed set F with $x \notin F$ it follows that $F \cap gpr^{\mu}$-$cl(\{x\}) = \emptyset$. Thus $X \setminus F$ is a gpr^{μ}-open set and $x \in X \setminus F$. Now, $F \cap gpr^{\mu}$-$cl(\{x\}) = \emptyset$ implies $F \subset X \setminus gpr^{\mu}$-$cl(\{x\})$, that is, gpr^{μ}-$cl(\{x\}) \subset X \setminus F$. Therefore, (X, μ) is gpr^{μ}-R_0.

Theorem 5.4. *For a supra topological space (X, μ), the following are equivalent.*
(i) (X, μ) is gpr^{μ}-R_0.
(ii) $x \in gpr^{\mu}$-$cl(\{y\})$ if and only if $y \in gpr^{\mu}$-$cl(\{x\})$, for any points x and y in X.

Proof. (i)$\Longrightarrow$ (ii): Let $x \in gpr^{\mu}$-$cl(\{y\})$ and let U be any gpr^{μ}-open set such that $y \in U$. Now X is gpr^{μ}-R_0 implies $x \in U$. Thus, every gpr^{μ}-open set which is containing y also contains x. Hence $y \in gpr^{\mu}$-$cl(\{x\})$.
(ii)$\Longrightarrow$(i): Let U be any gpr^{μ}- open set such that $x \in U$. If $y \notin U$, then $x \notin gpr^{\mu}$-$cl(\{y\})$ and hence $y \notin gpr^{\mu}$-$cl(\{x\})$. This implies gpr^{μ}-$cl(\{x\}) \subset U$. Hence (X, μ) is gpr^{μ}-R_0.

Theorem 5.5. *Every gpr^{μ}-R_1 space is a gpr^{μ}-R_0 space.*

Proof. Let U be a gpr^{μ}-open set such that $x \in U$. If $y \notin U$, then $x \notin gpr^{\mu}$-$cl(\{y\})$ and thus gpr^{μ}-$cl(\{x\}) \neq gpr^{\mu}$-$cl(\{y\})$. Since X is gpr^{μ}-R_1, there exists a gpr^{μ}-open set V containing y such that gpr^{μ}-$cl(\{y\}) \subset V$ and $x \notin V$ which implies $y \notin gpr^{\mu}$-$cl(\{x\})$. Therefore, gpr^{μ}-$cl(\{x\}) \subset U$. Thus, (X, μ) is gpr^{μ}-R_0.

Theorem 5.6. *If a supra topological space (X, μ) is both gpr^{μ}-R_1 and gpr^{μ}-T_0, then it is a gpr^{μ}-T_2 space.*

Proof. Let $x \neq y$. Since X is gpr^{μ}-T_0 , there exist a gpr^{μ}- open set U such that $x \in U$ and $y \notin U$. This implies $x \notin gpr^{\mu}$-$cl(\{y\})$. Therefore gpr^{μ}-$cl(\{x\}) \neq gpr^{\mu}$-$cl(\{y\})$. Now X is gpr^{μ}-R_1 implies that there exist disjoint gpr^{μ}-open sets U and V such that gpr^{μ}-$cl(\{x\}) \subset U$ and gpr^{μ}-$cl(\{y\}) \subset V$. Thus $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. It follows that (X, μ) is a gpr^{μ}-T_2 space.

Theorem 5.7. *Assume that $GPRC^{\mu}(X)$ is closed under arbitrary intersection for a supra topological space (X, μ). If (X, μ) is both gpr^{μ}-R_0 and gpr^{μ}-T_0, then it is a gpr^{μ}-T_1 space.*

Proof. Let $x, y \in X$ be any pair of disjoint points. Since X is gpr^{μ}-T_0 , there exists a gpr^{μ}- open set U such that $x \in U$ and $y \notin U$ or there exists a gpr^{μ}- open set V such that $y \in V$ and $x \notin V$.

Suppose $x \in U$ and $y \notin U$. Then gpr^{μ}-$cl(\{x\}) \subset U$ and $y \notin gpr^{\mu}$-$cl(\{x\})$. Hence, $y \in V = X - gpr^{\mu}$-$cl(\{x\})$. Also $x \notin X - gpr^{\mu}$-$cl(\{x\})$. Therefore there exists gpr^{μ}-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

The case where there exists a gpr^{μ}- open set V such that $y \in V$ and $x \notin V$ is similar to the above.

We proved that X is a gpr^{μ}-T_1 space.

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