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FIXED POINTS FOR COMPATIBLE MAPPINGS IN S - METRIC SPACES

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Abstract. The purpose of this paper is to prove a general fixed point theorem for two pairs of compatible mappings in S - metric spaces, which extend Theorem 2 [11] to S - metric spaces and generalize Theorems 2.2 and 2.7 [20] and other results for a pair of mappings and for a single mapping..

1. Introduction

Let f and g be two self mappings of a metric space (X, d). Sessa [21] defines f and g be weakly commuting if

$$d(fgx, gfx) \leq d(fx, gx), \text{ for all } x \in X.$$

In 1986, Jungck [4] defines f and g to be compatible if

$$d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \text{ for some } t \in X.$$

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Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but neither implication is reversible (Example 13 [22] and Example 2.2 [4]).

A generalization of metric space, named D - metric space, is introduced in [1], [2].

Mustafa and Sims [10], [11] proved that most of the claims concerning the fundamental topological structures on D - metric spaces are incorrect and introduced appropriate notion of generalized metric spaces, named G - metric space.

In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings in G - metric spaces.

Recently in [16], the authors introduced a "generalization" of G - metric spaces, named S - metric space. In [3], the authors proved that the notion of S - metric space is not a generalization of G - metric space or vice versa. Hence, the notions of G - metric space and S - metric space are independent.

Other results in the study of fixed points in S - metric space are obtained in [6], [9], [14], [15], [17], [19] and in other papers.

Several classical fixed point theorems and common fixed point theorems in metric spaces have been unified in [10], [11], considering a general condition by implicit function.

The study of fixed point for two pairs of compatible mappings satisfying an implicit relation is initiated in [11].

The study of fixed point for mappings satisfying an implicit relation in G - metric spaces is initiated in [12], [13].

Quite recently, new type of implicit relations in S - metric spaces is introduced in [17] and [18].

The purpose of this paper is to prove a general fixed point theorems for two pairs of compatible mappings in S - metric spaces which extend Theorem 2 [11] to S - metric spaces, generalizing Theorem 2.2 and 2.7 [20] and other results for a pair of mappings and for a single mapping.

2. Preliminaries

Definition 2.1 ([16], [17]). Let X be a nonempty set. A S - metric on X is a function $S : X^3 \to \mathbb{R}_+$ such that:
(S_1): $S(x, y, z) = 0$ if and only if $x = y = z$;
(S_2): $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

The pair (X, S) is called a S - metric space.

Example 2.2. *Let $X = \mathbb{R}$ and $S(x,y,z) = |x-z| + |y-z|$. Then, $S(x,y,z)$ is a S - metric on $\mathbb{R}$, which is named a usual S - metric on X.*

Lemma 2.3 ([16], [17])**.** *If S is a S - metric on a nonempty set X, then $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$.*

Definition 2.4. Let (X,S) be a S - metric space. For $r > 0$ and $x \in X$ we define the open ball with center x and radius r, the set

$$B_S(x,r) = \{y \in X : S(x,x,y) < r\}.$$

The topology induced by S is the topology determined by the base of all open balls in X.

Definition 2.5 ([16], [17])**.** a) A sequence $\{x_n\}$ in a S - metric space (X,S) is convergent to x, denoted $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, if $S(x_n,x_n,x) \to 0$ as $n \to \infty$.

b) A sequence $\{x_n\}$ in (X,S) is a Cauchy sequence if $S(x_n,x_n,x_m) \to 0$ as $n,m \to \infty$.

c) (X,S) is complete if every Cauchy sequence is convergent.

Example 2.6. (X,S) *with the usual S - metric is complete.*

Lemma 2.7 ([16], [17])**.** *Let (X,S) be a S - metric space. If $x_n \to x$ and $y_n \to y$, then $S(x_n,x_n,y_n) \to S(x,x,y)$.*

Lemma 2.8 ([16], [17])**.** *The limit of a convergent sequence is unique.*

Lemma 2.9 ([20])**.** *Let (X,S) be a S - metric space. If there exist $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} S(x_n,x_n,y_n) = 0$ and $\lim_{n\to\infty} x_n = t$, then $\lim_{n\to\infty} y_n = t$.*

Definition 2.10 ([5])**.** Let (X,S) and (X',S') be S - metric spaces and $f : (X,S) \to (X',S')$ be a function. Then f is continuous at a point $a \in X$ if every sequence $\{x_n\}$ in X with $S(x_n,x_n,a) \to 0$ implies $S(fx_n,fx_n,fa) \to 0$. f is continuous if has this property in all $x \in X$.

Definition 2.11 ([20])**.** Let (X,S) be a S - metric space. A pair $\{f,g\}$ of self mappings of (X,S) is said to be compatible if $\lim_{n\to\infty} S(fgx_n,fgx_n,gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_x = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Quite recently, the following two theorems are proved.

Theorem 2.12 (Theorem 2.2 [20])**.** *Suppose that f,g,R and T be self mappings of a complete S - metric space (X,S) with $f(X) \subset$*

$T(X)$, $g(X) \subset R(X)$ and the pairs $\{f,R\}$ and $\{g,T\}$ are compatible. If

$$S(fx,fy,gz) \leq q \max\left\{ \begin{array}{c} S(Rx,Ry,Tz), S(fx,fx,Rx), \\ S(gz,gz,Tz), S(fy,fy,gz) \end{array} \right\} \tag{2.1}$$

for each $x,y,z \in X$, with $q \in (0,1)$.

Then f,g,R and T have a unique common fixed point in X provided that R and T are continuous.

Theorem 2.13 (Theorem 2.7 [20]). *Let $\{f,R\}$ and $\{g,T\}$ be compatible mappings of a complete S - metric space (X,S) and for all $x,y,z \in X$ satisfying*

$$\begin{array}{c} S(fx,fy,gz) \leq a_1 S(Rx,Ry,Tz) + a_2 S(fx,fx,Tz) + \\ a_3 S(Rx,Ry,gz) + a_4 S(fy,fy,Tz) + a_5 S(gz,gz,Tz) \end{array} \tag{2.2}$$

where $a_i \geq 0$, $i = 1,2,3,4,5$, are real constants with $a_1 + 3a_2 + 3a_3 + 3a_4 + a_5 < 1$.

If $f(X) \subset T(X)$ and $g(X) \subset R(X)$ and R and T are continuous, then f,g,R and T have a unique common fixed point.

Remark 2.14. *In the proofs of Theorems 2.12 and 2.13 is used only $x = y$. Hence instead inequalities (2.1) and (2.2) we obtain*

$$S(fx,fx,gz) \leq q \max\left\{ \begin{array}{c} S(Rx,Rx,Tz), \\ S(fx,fx,Rx), \\ S(gz,gz,Tz) \end{array} \right\} \tag{2.3}$$

$$\begin{array}{c} S(fx,fx,gz) \leq a_1 S(Rx,Rx,Tz) + a_2 S(fx,fx,Tz) + \\ a_3 S(Rx,Rx,gz) + a_4 S(fx,fx,Tz) + a_5 S(gz,gz,Tz) \end{array} \tag{2.4}$$

Remark 2.15. *It is known that $q(x,y) = G(x,x,y)$ is a quasi - metric on X. By Lemma 2.3 and (S_1) we have that $s(x,y) = S(x,x,y)$ is a symmetric on X.*

3. Implicit relations

Let $\mathcal{F}_{CS}$ be the set of all real continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) : F is nonincreasing in variables t_5 and t_6;

(F_2) : There exists $h \in [0,1)$ such that for all $u,v \geq 0$,

(F_{2a}) : $F(u,v,v,u,0,2u+v) \leq 0$

or

(F_{2b}) : $F(u,v,u,v,2u+v,0) \leq 0$

implies $u \leq hv$;
(F_3) : $F(t,t,0,0,t,t) > 0, \ \forall t > 0.$

In all the following examples, condition (F_1) is obviously.

Example 3.1. *$F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$, where $k \in \left[0, \frac{1}{3}\right)$.*

(F_2) : (F_{2a}) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - k \max\{u, v, 2u + v\} \leq 0$. If $u > v$, then $u(1 - 3k) \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = 3k < 1$.
(F_{2b}) : The proof is similar to the proof of (F_{2a}).
(F_3) : $F(t,t,0,0,t,t) = t(1-k) > 0, \ \forall t > 0.$

Example 3.2. *$F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a, b, c, d, e \geq 0$ and $a + b + c + 3d + 3e < 1$.*

(F_2) : (F_{2a}) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - av - bv - cu - e(2u + v) \leq 0$. If $u > v$, then $u[1 - (a + b + c + 3e)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq h_1 v$, where $0 \leq h_1 = a + b + c + 3e < 1$.
(F_{2b}) : The proof is similar to the proof of (F_{2a}), with $u \leq h_2 v$, where $0 \leq h_2 = a + b + c + 3d < 1$.

If $h = \max\{h_1, h_2\}$, then (F_2) is proved.
(F_3) : $F(t,t,0,0,t,t) = t[1 - (a + d + e)] > 0, \ \forall t > 0.$

Example 3.3. *$F(t_1, ..., t_6) = t_1 - at_2 - b \max\{t_3, t_4, t_5, t_6\}$, where $a, b \geq 0$ and $a + 3b < 1$.*

(F_2) : (F_{2a}) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - av - b \max\{u, v, 2u + v\} \leq 0$. If $u > v$, then $u[1 - (a + 3b)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a + 3b < 1$.
(F_{2b}) : The proof is similar to the proof of (F_{2a}).
(F_3) : $F(t,t,0,0,t,t) = t[1 - (a + b)] > 0, \ \forall t > 0.$

Example 3.4. *$F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - d \max\{t_5, t_6\}$, where $a, b, c, d \geq 0$ and $a + b + c + 3d < 1$.*

(F_2) : (F_{2a}) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - av - bv - cu - d(2u + v) \leq 0$. If $u > v$, then $u[1 - (a + b + c + 3d)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a + b + c + 3d < 1$.
(F_{2b}) : The proof is similar to the proof of (F_{2a}).
(F_3) : $F(t,t,0,0,t,t) = t[1 - (a + d)] > 0, \ \forall t > 0.$

Example 3.5. $F(t_1,...,t_6) = t_1 - at_2 - d\max\{t_3,t_4\} - bt_5 - ct_6$, *where* $a,b,c,d \geq 0$ *and* $a+d+3(b+c) < 1$.

(F_2) : (F_{2a}) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u - av - d\max\{u,v\} - c(2u+v) \leq 0$. If $u > v$, then $u[1-(a+d+3c)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq h_1 v$, where $0 \leq h_1 = a+d+3c < 1$.
(F_{2b}) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u - av - d\max\{u,v\} - b(2u+v) \leq 0$. If $u > v$, then $u[1-(a+d+3b)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq h_2 v$, where $0 \leq h_2 = a+d+3b < 1$.

If $h = \max\{h_1,h_2\}$, then (F_2) is proved.
(F_3) : $F(t,t,0,0,t,t) = t[1-(a+b+c)] > 0, \ \forall t > 0$.

Example 3.6. $F(t_1,...,t_6) = t_1 - a(t_5+t_6) - bt_2 - c\max\{t_3,t_4\}$, *where* $a,b,c \geq 0$ *and* $3a+b+c < 1$.

(F_2) : (F_{2a}) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u - a(2u+v) - bv - c\max\{u,v\} \leq 0$. If $u > v$, then $u[1-(3a+b+c)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = 3a+b+c < 1$.
(F_{2b}) : The proof is similar to the proof of (F_{2a}).
(F_3) : $F(t,t,0,0,t,t) = t[1-(2a+b)] > 0, \ \forall t > 0$.

Example 3.7. $F(t_1,...,t_6) = t_1 - a(t_3+t_4) - bt_2 - c\max\{t_5,t_6\}$, *where* $a,b,c \geq 0$ *and* $2a+b+3c < 1$.

(F_2) : (F_{2a}) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u - a(u+v) - bv - c(2u+v) \leq 0$. If $u > v$, then $u[1-(2a+b+3c)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = 2a+b+3c < 1$.
(F_{2b}) : The proof is similar to the proof of (F_{2a}).
(F_3) : $F(t,t,0,0,t,t) = t[1-(b+c)] > 0, \ \forall t > 0$.

Example 3.8. $F(t_1,...,t_6) = t_1 - a\max\{t_4+t_5, t_3+t_6\} - bt_2$, *where* $a,b \geq 0$ *and* $4a+b < 1$.

(F_2) : (F_{2a}) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u - a\max\{u, 2u+2v\} - 2bv \leq 0$. If $u > v$, then $u[1-(4a+b)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = 4a+b < 1$.
(F_{2b}) : The proof is similar to the proof of (F_{2a}).
(F_3) : $F(t,t,0,0,t,t) = t[1-(a+b)] > 0, \ \forall t > 0$.

Example 3.9. $F(t_1,...,t_6) = t_1^2 - t_1(at_2+bt_3+ct_4) - dt_5t_6$, *where* $a,b,c,d \geq 0$, $a+b+c < 1$ *and* $a+d < 1$.

(F_2) : (F_{2a}) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u^2 - u(av + bv + cu) \leq 0$. If $u > v$, then $u^2[1 - (a + b + c)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = \sqrt{a + b + c} < 1$.
(F_{2b}) : The proof is similar to the proof of (F_{2a}).
(F_3) : $F(t, t, 0, 0, t, t) = t^2[1 - (a + d)] > 0, \ \forall t > 0$.

Example 3.10. *$F(t_1, ..., t_6) = t_1^2 - at_1t_2 - bt_3t_4 - ct_5t_6$, where $a, b, c \geq 0$, $a + b < 1$ and $a + c < 1$.*

(F_2) : (F_{2a}) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u^2 - auv - buv \leq 0$. If $u > v$, then $u^2[1 - (a + b)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = \sqrt{a + b} < 1$.
(F_{2b}) : The proof is similar to the proof of (F_{2a}).
(F_3) : $F(t, t, 0, 0, t, t) = t^2[1 - (a + c)] > 0, \ \forall t > 0$.

Example 3.11. *$F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{3}\right\}$, where $k \in [0, 1)$..*

(F_2) : (F_{2a}) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - k \max\left\{u, v, \dfrac{2u + v}{3}\right\} \leq 0$. If $u > v$, then $u(1 - k) \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = k < 1$.
(F_{2b}) : The proof is similar to the proof of (F_{2a}).
(F_3) : $F(t, t, 0, 0, t, t) = t(1 - k) > 0, \ \forall t > 0$.

4. Main results

Theorem 4.1. *Let f, g, R and T be self mappings of a complete S - metric space (X, S) with*
1) $f(X) \subset T(X)$ and $g(X) \subset R(X)$,
2) R and T are continuous,
and

$$F\left(\begin{array}{c} S(fx, fx, gy), S(Rx, Rx, Ty), S(fx, fx, Rx), \\ S(gy, gy, Ty), S(fx, fx, Ty), S(gy, gy, Rx) \end{array}\right) \leq 0 \tag{4.1}$$

for all $x, y \in X$ and some $F \in \mathcal{F}_{CS}$.

If (f, R) and (g, T) are compatible, then f, g, R and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point of X. Since $f(X) \subset T(X)$, there exists $x_1 \in X$ such that $fx_0 = Tx_1$ and also, as $gx_1 \in R(X)$, we choose $x_2 \in X$ such that $gx_1 = Rx_2$. Continuing this process, if

$x_{2n+1} \in X$ is chosen such that $fx_{2n} = Tx_{2n+1}$ and $x_{2n+2} \in X$ such that $gx_{2n+1} = Rx_{2n+2}$, we obtain a sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = gx_{2n+1} = Rx_{2n+2}.$$

By (4.1) for $x = x_{2n}$ and $y = x_{2n+1}$ we have

$$F\left(\begin{array}{c} S\left(fx_{2n}, fx_{2n}, gx_{2n+1}\right), S\left(Rx_{2n}, Rx_{2n}, Tx_{2n+1}\right), \\ S\left(fx_{2n}, fx_{2n}, Rx_{2n}\right), S\left(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}\right), \\ S\left(fx_{2n}, fx_{2n}, Tx_{2n+1}\right), S\left(gx_{2n+1}, gx_{2n+1}, Rx_{2n}\right) \end{array}\right) \leq 0$$

$$F\left(\begin{array}{c} S\left(y_{2n}, y_{2n}, y_{2n+1}\right), S\left(y_{2n-1}, y_{2n-1}, y_{2n}\right), \\ S\left(y_{2n}, y_{2n}, y_{2n-1}\right), S\left(y_{2n+1}, y_{2n+1}, y_{2n}\right), \\ 0, S\left(y_{2n+1}, y_{2n+1}, y_{2n-1}\right) \end{array}\right) \leq 0 \tag{4.2}$$

By Lemma 2.3,

$$\begin{array}{c} S\left(y_{2n}, y_{2n}, y_{2n-1}\right) = S\left(y_{2n-1}, y_{2n-1}, y_{2n}\right), \\ S\left(y_{2n+1}, y_{2n+1}, y_{2n}\right) = S\left(y_{2n}, y_{2n}, y_{2n+1}\right). \end{array}$$

By (S_2) and Lemma 2.3 we have

$$\begin{aligned} S\left(y_{2n+1}, y_{2n+1}, y_{2n-1}\right) &\leq 2S\left(y_{2n+1}, y_{2n+1}, y_{2n}\right) + S\left(y_{2n-1}, y_{2n-1}, y_{2n}\right) \\ &= 2S\left(y_{2n}, y_{2n}, y_{2n+1}\right) + S\left(y_{2n-1}, y_{2n-1}, y_{2n}\right). \end{aligned}$$

Then, by (4.2) we obtain

$$F\left(\begin{array}{c} S\left(y_{2n}, y_{2n}, y_{2n+1}\right), S\left(y_{2n-1}, y_{2n-1}, y_{2n}\right), \\ S\left(y_{2n-1}, y_{2n-1}, y_{2n}\right), S\left(y_{2n}, y_{2n}, y_{2n+1}\right), \\ 0, 2S\left(y_{2n}, y_{2n}, y_{2n+1}\right) + S\left(y_{2n-1}, y_{2n-1}, y_{2n}\right) \end{array}\right) \leq 0$$

By (F_{2a}) we obtain

$$S\left(y_{2n}, y_{2n}, y_{2n+1}\right) \leq hS\left(y_{2n-1}, y_{2n-1}, y_{2n}\right).$$

If $x = x_{2n}$ and $y = x_{2n-1}$, by (4.1) we obtain

$$F\left(\begin{array}{c} S\left(fx_{2n}, fx_{2n}, gx_{2n-1}\right), S\left(Rx_{2n}, Rx_{2n}, Tx_{2n-1}\right), \\ S\left(fx_{2n}, fx_{2n}, Rx_{2n}\right), S\left(gx_{2n-1}, gx_{2n-1}, Tx_{2n-1}\right), \\ S\left(fx_{2n}, fx_{2n}, Tx_{2n-1}\right), 0 \end{array}\right) \leq 0$$

$$F\left(\begin{array}{c} S\left(y_{2n}, y_{2n}, y_{2n-1}\right), S\left(y_{2n-1}, y_{2n-1}, y_{2n-2}\right), \\ S\left(y_{2n}, y_{2n}, y_{2n-1}\right), S\left(y_{2n-1}, y_{2n-1}, y_{2n-2}\right), \\ S\left(y_{2n}, y_{2n}, y_{2n-2}\right), 0 \end{array}\right) \leq 0 \tag{4.3}$$

By (S_2) we have

$$S\left(y_{2n}, y_{2n}, y_{2n-2}\right) \leq 2SS\left(y_{2n-1}, y_{2n-1}, y_{2n}\right) + S\left(y_{2n-2}, y_{2n-2}, y_{2n}\right).$$

By (4.3) and Lemma 2.3 we obtain

$$F\left(\begin{array}{c} S\left(y_{2n-1},y_{2n-1},y_{2n}\right),S\left(y_{2n-2},y_{2n-2},y_{2n-1}\right), \\ S\left(y_{2n-1},y_{2n-1},y_{2n}\right),S\left(y_{2n-1},y_{2n-1},y_{2n}\right), \\ 2S\left(y_{2n-1},y_{2n-1},y_{2n}\right)+S\left(y_{2n-2},y_{2n-2},y_{2n-1}\right),0 \end{array}\right)\leq 0\ .$$

By (F_{2b}) we obtain

$$S\left(y_{2n-1},y_{2n-1},y_{2n}\right)\leq hS\left(y_{2n-2},y_{2n-2},y_{2n-1}\right).$$

Hence for all $n\in\mathbb{N}$ we have

$$S\left(y_{n},y_{n},y_{n-1}\right)\leq hS\left(y_{n-1},y_{n-1},y_{n}\right)$$

for all $n=1,2,...$, which implies

$$S\left(y_{n},y_{n},y_{n-1}\right)\leq h^{n}S\left(y_{1},y_{1},y_{0}\right).$$

By a routine calculation, see for example [17], we obtain that $\{y_n\}$ is a Cauchy sequence. Since (X,S) is complete, there exists $z\in X$ such that

$$\lim_{n\to\infty} fx_n=\lim_{n\to\infty} Tx_{2n+1}=\lim_{n\to\infty} gx_{2n+1}=\lim_{n\to\infty} Rx_{2n+2}=z.$$

Suppose that R is continuous. Then

$$\lim_{n\to\infty} Rfx_{2n}=R\lim_{n\to\infty} fx_{2n}=Rz$$

and

$$\lim_{n\to\infty} R^2x_{2n+2}=Rz.$$

Since (f,R) are compatible, them by (S_3)

$$S\left(fRx_{2n},fRx_{2n},Rz\right)\leq 2S\left(fRx_{2n},fRx_{2n},Rfx_{2n}\right)+S\left(Rz,Rz,Rfx_{2n}\right).$$

Letting n tend to infinity we obtain

$$\lim_{n\to\infty} S\left(fRx_n,fRx_n,Rz\right)=0.$$

By Lemma 2.9,

$$\lim_{n\to\infty} fRx_n=Rz.$$

By (4.1) for $x=Rx_{2n}$, $y=x_{2n+1}$, we have

$$F\left(\begin{array}{c} S\left(fRx_{2n},fRx_{2n},gx_{2n+1}\right),S\left(R^2x_{2n},R^2x_{2n},Tx_{2n+1}\right), \\ S\left(fRx_{2n},fRx_{2n},R^2x_{2n}\right),S\left(gx_{2n+1},gx_{2n+1},Tx_{2n+1}\right), \\ S\left(fRx_{2n},fRx_{2n},Tx_{2n+1}\right),S\left(gx_{2n+1},gx_{2n+1},R^2x_{2n}\right) \end{array}\right)\leq 0\ .$$

Letting n tend to infinity by Lemma 2.3, we obtain

$$F\left(S\left(Rz,Rz,z\right),S\left(Rz,Rz,z\right),0,0,S\left(Rz,Rz,z\right),S\left(z,z,Rz\right)\right)\leq 0\ .$$

By Lemma 2.3,

$$S(z,z,Rz)=S(Rz,Rz,z).$$

Hence,

$$F(S(Rz,Rz,z),S(Rz,Rz,z),0,0,S(Rz,Rz,z),S(Rz,Rz,z))\leq 0,$$

a contradiction of (F_4) if $S(Rz,Rz,z)>0$. Hence, $S(Rz,Rz,z)=0$ and by (S_1), $z=Rz$.

In a similar way, since T is continuous, we obtain

$$\lim_{n\to\infty} T^2x_{2n+1}=Tz \text{ and } \lim_{n\to\infty} Tgx_{2n+1}=Tz$$

and by compatibility of g and T we obtain

$$\lim_{n\to\infty} gTx_{2n+1}=Tz.$$

By (4.1) for $x=x_{2n}$ and $y=Tx_{2n+1}$ we obtain that

$$F\left(\begin{array}{c} S(fx_{2n},fx_{2n},gTx_{2n+1}),S(Rx_{2n},Rx_{2n},T^2x_{2n+1}), \\ S(fx_{2n},fx_{2n},Rx_{2n}),S(gTx_{2n+1},gTx_{2n+1},T^2x_{2n+1}), \\ S(fx_{2n},fx_{2n},T^2x_{2n+1}),S(gTx_{2n+1},gTx_{2n+1},Rx_{2n}) \end{array}\right)\leq 0\ .$$

Letting n tend to infinity we obtain

$$F(S(z,z,Tz),S(z,z,Tz),0,0,S(z,z,Tz),S(z,z,Tz))\leq 0,$$

a contradiction of (F_3) if $S(z,z,Tz)>0$. Hence, $S(z,z,Tz)=0$, which implies, by (S_1) that $z=Tz$.

Also, by (4.1) for $x=z$ and $y=x_{2n+1}$ we obtain

$$F\left(\begin{array}{c} S(fz,fz,gx_{2n+1}),S(Rz,Rz,Tx_{2n+1}), \\ S(fz,fz,Rz),S(gx_{2n+1},gx_{2n+1},Tx_{2n+1}), \\ S(fz,fz,Tx_{2n+1}),S(gx_{2n+1},gx_{2n+1},Rz) \end{array}\right)\leq 0,$$

$$F(S(fz,fz,z),0,S(fz,fz,z),0,S(fz,fz,z),0)\leq 0\ .$$

By (F_1) we have

$$F(S(fz,fz,z),0,S(fz,fz,z),0,2S(fz,fz,z),0)\leq 0\ .$$

By (F_{2b}),

$$S(fz,fz,z)=0$$

which implies

$$z=fz.$$

By (4.1) for $x=z$ and $y=z$ we obtain that

$$F\left(\begin{array}{c} S(fz,fz,gz),S(Rz,Rz,Tz),S(fz,fz,Rz), \\ S(gz,gz,Tz),S(fz,fz,Tz),S(gz,gz,Rz) \end{array}\right)\leq 0,$$

$$F(S(z,z,gz),0,0,S(z,z,gz),0,S(z,z,gz))\leq 0,$$

By (F_1) we obtain

$$F\left(S\left(z,z,gz\right),0,0,S\left(z,z,gz\right),0,2S\left(z,z,gz\right)\right)\leq 0.$$

By (F_{2a}), $S\left(z,z,gz\right)=0$ which implies by (S_1), $z=gz$. Hence, z is a common fixed point of f,g,R and T.

Suppose that u is other common fixed point of f,g,R and T. By (4.1) we obtain

$$F\left(\begin{array}{c} S\left(fu,fu,gv\right),S\left(Ru,Ru,Tv\right),S\left(fu,fu,Ru\right), \\ S\left(gv,gv,Tv\right),S\left(fu,fu,Tv\right),S\left(gv,gv,Ru\right) \end{array}\right)\leq 0\,,$$

$$F\left(S\left(u,u,v\right),S\left(u,u,v\right),0,0,S\left(u,u,v\right),S\left(v,v,u\right)\right)\leq 0\,.$$

By Lemma 2.3,

$$S\left(u,u,v\right)=S\left(v,v,u\right).$$

Hence,

$$F\left(S\left(u,u,v\right),S\left(u,u,v\right),0,0,S\left(u,u,v\right),S\left(u,u,v\right)\right)\leq 0\,,$$

a contradiction of (F_4). Hence, $S\left(u,u,v\right)=0$. By (S_1), $u=v$. □

By Theorem 4.1 and Example 3.11 we obtain

Theorem 4.2. *Let f,g,R and T be self mappings of a complete S - metric space (X,S) with*
1) $f\left(X\right)\subset T\left(X\right)$ and $g\left(X\right)\subset R\left(X\right)$,
2) R and T are continuous,
and

$$S\left(fx,fx,gy\right)\leq q\max\left\{\begin{array}{c} S\left(Rx,Rx,Ty\right), \\ S\left(fx,fx,Rx\right), \\ S\left(gy,gy,Ty\right), \\ \dfrac{S\left(fx,fx,Ty\right)+S\left(gy,gy,Rx\right)}{3} \end{array}\right\}$$

for all $x,y\in X$, where $q\in(0,1)$.

If (f,R) and (g,T) are compatible, then f,g,R and T have a unique common fixed point.

Corollary 4.3. *Let f,g,R and T be self mappings of a complete S - metric space (X,S) with*
1) $f\left(X\right)\subset T\left(X\right)$ and $g\left(X\right)\subset R\left(X\right)$,
and

$$S\left(fx,fx,gy\right)\leq q\max\left\{S\left(Rx,Rx,Ty\right),S\left(fx,fx,Rx\right),S\left(gy,gy,Ty\right)\right\}$$

for all $x,y\in X$, where $q\in(0,1)$.

If (f,R) and (g,T) are compatible, then f,g,R and T have a unique common fixed point.

Proof. The proof follows by Theorem 4.2 because

$$\max\left\{S\left(Rx,Rx,Ty\right),S\left(fx,fx,Rx\right),S\left(gy,gy,Ty\right)\right\}\leq$$
$$\max\left\{\begin{array}{c} S\left(Rx,Rx,Ty\right),S\left(fx,fx,Rx\right),S\left(gy,gy,Ty\right), \\ \dfrac{S\left(fx,fx,Ty\right)+S\left(gy,gy,Rx\right)}{3}\end{array}\right\}.$$

□

Remark 4.4. *This corollary is a new form of Theorem 2.12.*

Theorem 4.5. *Let f,g,R and T be self mappings of a complete S - metric space (X,S) with*
1) $f(X)\subset T(X)$ and $g(X)\subset R(X)$,
2) R and T are continuous,
and

$$S\left(fx,fx,gy\right)\leq a_1S\left(Rx,Rx,Ty\right)+a_2S\left(fx,fx,Rx\right)+$$
$$a_3S\left(gy,gy,Ty\right)+a_4S\left(fx,fx,Ty\right)+a_5S\left(gy,gy,Rx\right),$$

where $a_1,...,a_5\geq 0$ and $a_1+a_2+a_3+3a_4+3a_5<1$, for all $x,y\in X$.
If (f,R) and (g,T) are compatible, then f,g,R and T have a unique common fixed point.

Proof. The proof follows by Theorem 4.1 and Example 3.2. □

Corollary 4.6. *Let f,g,R and T be self mappings of a complete S - metric space (X,S) with*
1) $f(X)\subset T(X)$ and $g(X)\subset R(X)$,
2) R and T are continuous,
and

$$S\left(fx,fx,gy\right)\leq a_1S\left(Rx,Rx,Ty\right)+a_2S\left(gy,gy,Ty\right)+$$
$$+a_3S\left(fx,fx,Ty\right)+a_4S\left(gy,gy,Rx\right),$$

for all $x,y\in X$, where $a_1,...,a_4\geq 0$ and $a_1+a_2+3a_3+a_4<1$.
If (f,R) and (g,T) are compatible, then f,g,R and T have a unique common fixed point.

Proof. The proof is similar to the proof of Corollary 4.3. □

If R and T are identity mappings on X by Theorem 4.1 we obtain

Theorem 4.7. *Let f and g be self mappings of a complete S - metric space (X,S) such that*

$$F\left(\begin{array}{c} S\left(fx,fx,gy\right),S\left(x,x,y\right),S\left(fx,fx,x\right), \\ S\left(gy,gy,y\right),S\left(fx,fx,y\right),S\left(gy,gy,x\right)\end{array}\right)\leq 0$$

for all $x,y\in X$ and some $F\in\mathcal{F}_{CS}$.
Then f and g have a unique common fixed point.

By Theorem 4.1 and Example 3.11 we obtain

Theorem 4.8. *Let (X,S) be a complete S - metric space and f,g be self mappings of X such that*

$$S(fx,fx,gy) \leq q \max \left\{ \begin{array}{c} S(x,x,y), S(fx,fx,x), S(gy,gy,y), \\ \dfrac{S(fx,fx,y)+S(gy,gy,x)}{3} \end{array} \right\}$$

for all $x,y \in X$ and some $F \in \mathcal{F}_{CS}$.

Then f and g have a unique common fixed point.

If $f=g$, then by Theorem 4.7 we obtain

Theorem 4.9. *Let (X,S) be a complete S - metric space and $f : X \to X$ be a mapping nonincreasing in t_6 such that*

$$F\left(\begin{array}{c} S(fx,fx,fy), S(x,x,y), S(fx,fx,x), \\ S(fy,fy,y), S(fx,fx,y), S(fy,fy,x) \end{array} \right) \leq 0$$

for all $x,y \in X$ and some F satisfying properties $(F_1), (F_{2a})$ and (F_3).

Then f has a unique fixed point.

By Example 3.11 we obtain a new Ćirić type theorem in S - metric spaces.

Corollary 4.10. *Let (X,S) be a complete S - metric space and $f : X \to X$ be a mapping such that*

$$S(fx,fx,fy) \leq q \max \left\{ \begin{array}{c} S(x,x,y), S(fx,fx,x), S(fy,fy,y), \\ \dfrac{S(fx,fx,y)+S(fy,fy,x)}{3} \end{array} \right\}$$

for all $x,y \in X$ and $q \in [0,1)$.

Then f has a unique fixed point.

Remark 4.11. *1) By Examples 3.1 - 3.8 we obtain all the results from* [9].
2) By Example 3.2 we obtain Corollary 2.19 [17], *Theorems 2.3, 2.4* [13] *and Theorems 3.2, 3.3, 3.4* [14].

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