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CONSTRUCTION OF SHIFT SPACES OF INFINITE TYPE AND DEVANEY'S CHAOS

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Abstract. In this paper we present two new examples of shift spaces of infinite type over two symbols. One of the spaces is shown to be densely Li-Yorke chaotic and Robinson chaotic, but not chaotic in the sense of Devaney, while the other one is found out to be Devaney chaotic.

1. Introduction

A **topological dynamical system** (or simply dynamical system) is a pair (X, f), where f is a continuous self map on a compact metric space X. We define the **forward orbit** $O^+(x) = \bigcup_{n\in\mathbb{N}} f^n(x)$, where $\mathbb{N} = \{0, 1, 2, \cdots\}$. A point $x \in X$ is **periodic** if $f^n(x) = x$ for some $n \in \mathbb{Z}^+$, where $\mathbb{Z}^+ = \{1, 2, \cdots\}$. If $f^n(x)$ is periodic for some $n > 0$, then x is called **eventually periodic**. We use $tr(f)$ to denote the set of all those points whose forward orbit is dense in X.

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A point $y \in X$ is an **ω-limit point** of x if there is a sequence of strictly increasing positive integers $\{n_k\}$ such that $f^{n_k}(x) \to y$ as $k \to \infty$. The ω-limit set of x is the set $\omega(x)$ of all ω-limit points of x and is

$$\omega(x) = \bigcap_{n \in \mathbb{Z}^+} \overline{\bigcup_{i \geq n} f^i(x)}.$$

We say f or the dynamical system (X, f) is **topologically transitive** or simply **transitive** if for any non-empty open subsets U and V of X, there exist $n \in \mathbb{Z}^+$, such that $f^n(U) \cap V \neq \emptyset$ [1]. The dynamical system (X, f) is said to be **topologically mixing** or simply **mixing** if for any non-empty open subsets U and V of X, there exist $k \in \mathbb{Z}^+$, such that $f^n(U) \cap V \neq \emptyset, \forall\ n \geq k$. The dynamical system (X, f) is said to be **totally transitive** if (X, f^n) is transitive for all $n \in \mathbb{Z}^+$. It is clear that mixing implies totally transitive. The dynamical system (X, f) is **weakly blending** if for any pair of non-empty open sets Ü·U and V there exists an ÝŠ$n \in \mathbb{Z}^+$, such that $f^n(U) \cap f^n(V) \neq \emptyset$ [4]. The dynamical system (X, f) has **sensitive dependence on initial conditions** or simply **sensitive** if there exists an $\epsilon > 0$ such that for every $x \in X$ and every neighborhood U of x, there exists $y \in U$ and $n \in \mathbb{Z}^+$ with $d(f^n(x), f^n(y)) > \epsilon$.
A dynamical system (X, f) is said to be **minimal** if $\omega(x) = X$ for any $x \in X$. A point $x \in X$ is **recurrent** if $x \in \omega(x)$. A point $x \in X$ is **almost periodic** if for any open set $U \subset X$ containing x, we can find $k \in \mathbb{Z}^+$ such that for any $n \in \mathbb{Z}^+$ there is some $i \in \{n, n+1, \cdots, n+k\}$ such that $f^i(x) \in U$. Infinite minimal systems will not contain any periodic points and they are precisely those systems which are the forward orbit closure of non-periodic, almost periodic points [3]. Every almost periodic point is a recurrent point but, the converse is not true. We give some of the well known and useful results and for reader's convenience we provide some of the proofs.

Proposition 1. [3] *If (X, f) is transitive dynamical system, then $tr(f) \neq \emptyset$*

Corollary 2. *If the system (X, f) is transitive then either X is a single periodic orbit or it contains no isolated points.*

Proof. If X is finite then $O^+(x)$ is finite for every $x \in X$, also by transitivity we have $\overline{O^+(x)} = X$ for some $x \in X$, so $O^+(x) = X$. It

is well known that orbit of a non-periodic, eventually periodic point cannot be transitive. Therefore, $O^+(x)$ is a periodic orbit.
Suppose X is infinite and if $x \in X$ is an isolated point then $\overline{O^+(x)} = X$. Also $f^n(\{x\}) \cap \{x\} \neq \emptyset$ for some $n \in \mathbb{Z}^+$, that is X is finite, which is a contradiction. ∎

Lemma 3 ([2]). *If (X, f) is a transitive dynamical system, then $f(X) = X$.*

Proof. If X is a single periodic orbit then $f(X) = X$.
Suppose that X is infinite. Since (X, f) is transitive, we can find $x \in X$ such that $\overline{O^+(x)} = X$. Let $y \in X$ be an arbitrary element. If $y \in O^+(x)$ then the result follows, else there is a sequence $\{f^{n_k}(x)\}$ such that $f^{n_k}(x) \to x$. Using the compactness of X, without loss of generality we can take $f^{n_k-1}(x) \to z$ for some $z \in X$, therefore $f^{n_k}(x) \to f(z)$ and hence $y = f(z)$. ∎

Theorem 4. *The dynamical system (X, f) is transitive if and only if there exists a point $x \in X$ such that $\omega(x) = X$.*

Proof. Suppose, $\omega(x) = X$ for some $x \in X$. Let U and V be any two nonempty open sets. Then there exist $m \in \mathbb{N}$ and $n \in \mathbb{N}$ with $n > m$ such that $f^m(x) \in U$ and $f^n(x) \in V$, therefore, $f^{n-m}(U) \cap V \neq \phi$.
Conversely, let f be transitive. Then there is a point $x \in X$ such that $O^+(x)$ is dense in X, i.e. $X = O^+(x) \cup \omega(x)$. Then by **Lemma 3**, there is a $y \in X$ such that $f(y) = x$. If $y \in O^+(x)$ then x is periodic and $\omega(x) = X$, otherwise $y \in \omega(x)$ and we know that $\sigma(\omega(x)) = \omega(x)$, so again $x \in \omega(x)$. Therefore, we have $\omega(x) = X$. ∎

2. Symbolic dynamical systems

Let $\mathcal{A}$ be a finite set called an **alphabet** and its elements as **symbols**. We assign the discrete topology to $\mathcal{A}$. Let $\mathcal{A}^{\mathbb{N}}$ denote the set of all one sided sequences in $\mathcal{A}$. Then $\mathcal{A}^{\mathbb{N}}$ is the space obtained with the product topology and is called the **full shift**. A **shift** is a continuous map $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ given by $\sigma(x)_i = x_{i+1}$. The metric

$$d(x, y) = \begin{cases} 0, & if \;\; x = y \\ 2^{-l}, & if \;\; x \neq y, \;\; where \;\; l = \min\{i : x_i \neq y_i\} \end{cases}$$

generates the product topology on $\mathcal{A}^{\mathbb{N}}$. It is well known that $\mathcal{A}^{\mathbb{N}}$ is compact, perfect and totally disconnected with the given topology. For $x = (x_i) \in \mathcal{A}^{\mathbb{N}}$ and $i, j \in \mathbb{N}$ with $i \leq j$ define $x_{[i,j]} = x_i x_{i+1} \cdots x_j$. The open ball $B(x, 2^{-l})$ in $\mathcal{A}^{\mathbb{N}}$ is the symmetric cylinder

$$C(x_{[0,l]}) = \{y \in \mathcal{A}^{\mathbb{N}} : y_{[0,l]} = x_{[0,l]}\}.$$

A **block** (or **word**) over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. For a block u over $\mathcal{A}$ we use $|u|$ to denote the number of symbols contained in u and it is called the length of u. The **empty block** λ is the block with length 0. A *k-block* is simply a block of length k. The set of all k-blocks over $\mathcal{A}$ is denoted by $\mathcal{A}^k$ and $\mathcal{A}^* = \bigcup_{k=0}^{\infty} \mathcal{A}^k$. Two blocks u and v can be concatenated by writing u first and then v, forming a new block uv. By convention, $\lambda u = u\lambda = u$ for all blocks u. If $n \geq 1$, then u^n denote the concatenation of n copies of u, and we put $u^0 = \lambda$. The point $uuu\cdots$ is denoted by u^∞. We say a block u is a subblock of v or u is contained (or occurs) in v if there are blocks x, y such that $v = xuy$. We say a block u is a subblock or contained or occuring in $x \in \mathcal{A}^{\mathbb{N}}$ if $u = x_{[i,j]}$ for some $i, j \in \mathbb{N}$ and $i \leq j$.
Let $\mathcal{F}$ be a collection of blocks over $\mathcal{A}$. For any such $\mathcal{F}$, define $X_{\mathcal{F}}$ to be the set of all sequences in $\mathcal{A}^{\mathbb{N}}$ which do not contain any block in $\mathcal{F}$. A **shift space** is the subset X of a full shift $\mathcal{A}^{\mathbb{N}}$ such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden blocks over $\mathcal{A}$. If $\mathcal{F}$ is finite then X is called a **shift space of finite type** and if we cannot find any finite set $\mathcal{F}$ such that $X = X_{\mathcal{F}}$ then X is called **shift space of infinite type**. With the restriction of the shift map $\sigma_X = \sigma_{|X}$ on X, the system (X, σ_X) is called a **symbolic dynamical system**.
Let $X \subset \mathcal{A}^{\mathbb{N}}$ be any non-empty set, then $\mathcal{B}_n(X)$ denote the set of all n-blocks that occured in points in X. The **language** of X is the collection

$$\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X) \subseteq \mathcal{A}^*.$$

A block u is said to be allowed or appearing in X if $u \in \mathcal{B}(X)$. For an element $x \in \mathcal{A}^{\mathbb{N}}$, we use $\mathcal{B}(x)$ to denote the collection of all blocks contained in x. For a block $u \in B(X)$, we define $C_X(u) = X \cap C(u)$, where $C(u) = \{x \in \mathcal{A}^{\mathbb{N}} : x_{[o,l]} = u\}$ and $l = |u| - 1$. The following proposition and results are easily derived from their respective definions and also from the results given in *Lind* and *Marcus* book [11]. For reader's convenience we give proof of some of the results.

Proposition 5 ([11])**.** (1) *Let $X \subseteq \mathcal{A}^{\mathbb{N}}$ be a shift space, and $\mathcal{B}(X)$ be its language. If $w \in \mathcal{B}(X)$, then*
(a) *every subblock of w belongs to $\mathcal{B}(X)$, and*
(b) *there is a nonempty block $v \in \mathcal{B}(X)$ so that $wv \in \mathcal{B}(X)$.*

(2) *If $\mathcal{L}$ is a collection of blocks over $\mathcal{A}$, then $\mathcal{L} = \mathcal{B}(X)$ for some shift space X if and only if $\mathcal{L}$ satisfies condition* (1)*.*
(3) *For any shift space X we have $X = X_{\mathcal{B}(X)^c}$. Thus two shift spaces are equal if and only if they have the same language.*

Corollary 6. *Let $X \subset \mathcal{A}^{\mathbb{N}}$. Then X is a shift space if and only if whenever $x \in \mathcal{A}^{\mathbb{N}}$ and each $x_{[i,j]} \in \mathcal{B}(X)$ then $x \in X$.*

Proof. Let X be a shift space then $X = X_{\mathcal{B}(X)^c}$. So, if $x \in \mathcal{A}^{\mathbb{N}}$ and each $x_{[i,j]} \in \mathcal{B}(X)$ that is no subword in x is contained in $\mathcal{B}(X)^c$, that means $x \in X_{\mathcal{B}(X)^c}$.
Conversely, whenever $x \in \mathcal{A}^{\mathbb{N}}$ and each $x_{[i,j]} \in \mathcal{B}(X)$ then $x \in X$. We claim, $X = X_{\mathcal{B}(X)^c}$.
If $x \in X$, then no subblock in x is in $\mathcal{B}(X)^c$, so $x \in X_{\mathcal{B}(X)^c}$. If $x \in X_{\mathcal{B}(X)^c}$ then each $x_{[i,j]} \in \mathcal{B}(X)$. So, $x \in X$. ∎

Lemma 7. *$X \subseteq \mathcal{A}^{\mathbb{N}}$ is a shift space if and only if X is closed and $\sigma(X) \subseteq X$.*

Proof. First part easily follows.
Conversely, let X is closed and $\sigma(X) \subseteq X$.
We claim, $X = X_{\mathcal{B}(X)^c}$. Let $x \in X_{\mathcal{B}(X)^c}$.
Suppose, $x \in \mathcal{A}^{\mathbb{N}} \backslash X$ then by closedness of X we can find $k = k(x) \in \mathbb{N}$ such that the cylinder set $C(u) \subset \mathcal{A}^{\mathbb{N}} \setminus X$, where $u = x_{[0,k]}$.
We know $u \in \mathcal{B}(X)$, and therefore exist some $y \in X$ such that $y_{[l,m]} = u$ for some $l, m \in \mathbb{N}$ and $l \leq m$. Using, $\sigma(X) \subseteq X$ we have $\sigma^l(y) \in X$ and $\sigma^l(y)_{[0,k]} = u$,i.e. $\sigma^l(y) \in C(u)$ a contradiction and the rest follows easily.
Suppose, $x \in X$, then each $x_{[i,j]} \in \mathcal{B}(X)$,i.e. $x \in \mathcal{B}(X)$. ∎

Proposition 8 ([11])**.** *The dynamical system (X, σ_X) where $X \subseteq \mathcal{A}^{\mathbb{N}}$ is transitive if and only if for any two non empty words $u, v \in B(X)$ there is a non empty word $w \in B(X)$ so that $uwv \in B(X)$.*

Proof. Let (X, σ_X) be transitive. For $u, v \in B(X)$, let $|u| = l$ and $|v| = m$ and let $U = X \cap C(x_{[0,l]} = u)$ and $V = \sigma_X^{-l}(X \cap C(x_{[0,m]} = v))$. As X is shift space we can see that U and V are non empty open sets. So, using transitivity of X we can find $n \in \mathbb{Z}^+$ such that there exist $y \in U \cap \sigma_X^{-n}(V)$. Let $w \neq \lambda$ be such that $y_{[0,k]} = uwv$. Converse part is clear. ∎

The following results follow in a straightforward manner from the respective definitions.

(1) A point $x \in \mathcal{A}^{\mathbb{N}}$ is recurrent if and only if for any $k \in \mathbb{Z}^+$, the initial block $x_{[0,k]}$ is contained in x infinitely many times.
(2) A point $x \in \mathcal{A}^{\mathbb{N}}$ is almost periodic if and only if for any $k \in \mathbb{Z}^+$, the initial block $x_{[0,k]}$ is contained in x infinitely many times with bounded gaps.
(3) If $X \subseteq \mathcal{A}^{\mathbb{N}}$ and (X, σ_X) is a shift space, then (X, σ_X) is mixing if and only if for any two non empty words $u, v \in B(X)$ there is a $N \in \mathbb{Z}^+$ such that for any $n \geq N$ there is a word $w_n \in B_n(X)$ such that $uw_nv \in B(X)$.
(4) If $x \in \mathcal{A}^{\mathbb{N}}$ and $X = \overline{O^+(x)}$, then $\mathcal{B}(X) = \mathcal{B}(x)$.
(5) If $x \in \mathcal{A}^{\mathbb{N}}$, then $y \in \omega(x)$ if and only if any initial block $y_{[0,k]}$ is occured in x infinitely many times.

3. Chaos

The term *chaos* is one of the most frequently used term in the study of dynamical systems. There are several different definitions of what it means for a function f from a compact metric space X to itself to be chaotic [5, 10, 8]. Most frequently used definition of a chaotic dynamical system is Devaney's [5]. A map $f : X \to X$ is said to be **Devaney chaotic** on X if

(1) f is transitive,
(2) the periodic points of f are dense in X,
(3) f has sensitive dependence on initial conditions.

If X is an infinite metric space then the first two conditions implies the third [1]. f is said to be **Robinson chaotic** if f is transitive and sensitive dependent to initial condition. Following the definition on [10], we define scrambled pairs as follows. Two points $a, b \in X$ form a **chaotic pair** (or a **scrambled pair**) for the map f, if a and b satisfy:

(1) $\lim_{n\to+\infty} \sup d(f^n(a), f^n(b)) > 0$
(2) $\lim_{n\to+\infty} \inf d(f^n(a), f^n(b)) = 0$.

A subset $S \subseteq X$ is called a *scrambled* set, if for any $a, b \in S$, with $a \neq b$, then (a, b) is a chaotic pair. We say f is **Li-Yorke chaotic**, if there exists an uncountable, scrambled set; further, if S is dense in X then we say (X, f) is **densely Li-Yorke chaotic**.
It is well-known that if $X_{\mathcal{F}} \subseteq \mathcal{A}^{\mathbb{N}}$ is a transitive shift space of finite type, then $X_{\mathcal{F}}$ contains countably many periodic points and they are dense [9]. Existence of minimal and existence of non-minimal shift spaces of infinite type are discussed in many papers [6, 7, 12].

Very few studies have been published regarding the chaotic nature of shift space of infinite type. So, it is quite an interesting area to be analysed.
One of the most popular shift of infinite type is the **even shift**. Even shift space X is the set of all sequences on $\{0,1\}^{\mathbb{N}}$ so that between any two 1's there is an even number of 0's. Each even shift space is Devaney chaotic and mixing. There exist shift spaces that are Li-Yorke chaotic without being Devaney chaotic and such an example is given below.

Theorem 9 ([8]). *Assume that $f : X \to X$ is transitive with X infinite and contains a periodic point. Then there is an uncountable scrambled set for f. Moreover, if f is totally transitive, then f is densely Li-Yorke chaotic. Particularly, chaos in the sense of Devaney is stronger than that in the sense of Li-Yorke.*

Example 1. *Take $B_1 = 1$ and $B_n = B_{n-1}0^{n-1}B_{n-1}$, $n \geq 2$ where 0^{n-1} is the concatenation of 0, $n-1$ times. We construct a point $x = (x_n) \in \{0,1\}^{\mathbb{N}}$ inductively as $x = B_1 0 B_1 00 B_2 \cdots B_n 0^{n+1} B_{n+1} \cdots$. First few terms of x are*

$$x = 1010010100010100101000 0\cdots$$

x is a recurrent, non almost periodic point. The shift space $X_1 = \overline{O^+(x)} = \omega(x)$ is transitive and non minimal containing uncountably many elements.

The shift space X_1 has the following properties:

1. X_1 is a shift space of infinite type. If X_1 is shift space of finite type, then there is $2 \leq N \in \mathbb{Z}^+$ such that $\mathcal{F}$ is a collection of forbidden N-blocks and $X_1 = X_{\mathcal{F}}$. Then every N-block in $B_{N-1}0^N B_{N-1}0^\infty$ is allowed forcing it to be an element of X_1, which is not true.
2. For any open sets U and V in X_1 there is a $k \in \mathbb{N}$ such that $\sigma_X^n(U) \cap \sigma_X^n(V) \neq \emptyset$ for all $n \geq k$. In other words X_1 is somewhat more than weakly blending. Let $U = C_{X_1}(u)$ and $V = C_{X_1}(v)$ be any two basic open sets in X_1, then we can find two words B_l and B_m such that $B_l = x_1 u y_1$ and $B_m = x_2 v y_2$ for some $x_1, x_2, y_1, y_2 \in \mathcal{B}(X_1)$. Then the open sets $C_{X_1}(uy_1) \subset U$ and $C_{X_1}(vy_2) \subset V$. Also we know that $uy_1 0^\infty \in C_{X_1}(uy_1)$ and $vy_2 0^\infty \in C_{X_1}(vy_2)$, thus we can find some $k \in \mathbb{N}$ such that $\sigma_{X_1}^n(C_{X_1}(uy_1)) \cap \sigma_{X_1}^n(C_{X_1}(vy_2)) \neq \emptyset$.

(3) 0^∞ is the only periodic point contained in X_1. Suppose that u^∞ is a periodic point contained in X_1 different from 0^∞. Then u will contain at least one 1. Let B_l contain u such that l is the smallest one. Take $w = uu\cdots u$, n-times for some $n \in \mathbb{Z}^+$ such that n exceeds the number of 1 contained in $B_{l+|u|}$, i.e, $n > 2^{l+|u|-1}$. Also, if $h = m + j$, then $B_h = B_m 0^m B_m 0^{m+1} B_m 0^m B_m 0^{m+2} \cdots 0^{m+j-1} B_m 0^m B_m 0^{m+1} B_m 0^m B_m \cdots B_m$ and $\mathcal{B}_{l+|u|+1} = \mathcal{B}_{l+|u|} 0^{l+|u|} \mathcal{B}_{l+|u|}$, so w is not contained in $\mathcal{B}_{l+|u|+1}$.
Therefore, we can see that any B_k such that $k > l + |u|$ will not contain w. This contradicts the assumption that u^∞ is contained in X_1.

(4) (X_1, σ_{X_1}) is mixing. Let u and v be any two blocks occuring in $\mathcal{B}(X_1)$. Since σ_{X_1} is transitive we can find $b \in \mathcal{B}(X_1)$ such that $ubv \in \mathcal{B}(X_1)$, then ubv occurs in B_k for some $k \in \mathbb{N}$, and suppose $aubvc = B_k$. Now as we can see $x = B_k 0^k B_k 0^{k+1} 0^{k+2} B^{k+2} \cdots$ and
$B_{k+l} = B_k 0^k B_k 0^{k+1} B_k 0^k B_k 0^{k+2} \cdots 0^{k+l-1} B_k 0^k B_k 0^{k+1} B_k 0^k B_k \cdots B_k$, $l = 1, 2\cdots$. Thus for any $m \geq k, B_k 0^m B_k \in \mathcal{B}(X_1)$. Taking $K = |bvc0^k aub|$, we have for any $n \geq K$ that there is a $w_n \in \mathcal{B}_n(X_1)$ such that $uw_n v \in \mathcal{B}(X_1)$.
Therefore, we conclude that X_1 is mixing.

(5) (X_1, σ_{X_1}) is densely Li-Yorke chaotic and Robinson chaotic.

Example 2. *Take $B_1 = 1$, $B_2 = 101$, $B_3 = B_2 00 B_2 B_1^2$ and $B_n = B_{n-1} 0^{n-1} B_{n-1} B_{n-2}^{n-1} \cdots B_1^{n-1}$, $n \geq 3$. Inductively we construct a point in $\{0,1\}^{\mathbb{N}}$ as $x = B_1 0 B_1 00 B_2 B_1^2 000 B_3 B_2^3 B_1^3 \cdots B_{n-1} B_{n-2}^{n-1} \cdots B_1^{n-1} 0^n B_n \cdots$. First few terms of x are*
$x = 1010010111000101001011110110110111110000\cdots$.
The shift space $X_2 = \overline{O^+(x)} = \omega(x)$ is transitive and non minimal containing uncountably many elements.

The shift space X_2 possesses the following properties.

(1) X_2 is a shift space of infinite type.
If X_2 is a shift space of finite type, then there is $2 \leq N \in \mathbb{Z}^+$ such that $\mathcal{F}$ is a collection of forbidden N-blocks and $X_2 = X_{\mathcal{F}}$. Then every N-block in $B_N 0^\infty$ is allowed, forcing it to be an element of X_2, which is not true.

(2) X_2 contains infinitely many periodic points. The set of periodic points is dense in X_2.

If $C(u) \cap X_2$ is a nonempty basic open set, then u is a subblock of B_k for some $k \in \mathbb{Z}^+$, therefore we can find $x, y \in \mathcal{B}(X_2)$ such that $B_k = xuy$. Also, we know that the periodic point $B_k^\infty \in X_2$, and thus the periodic point $\sigma^{|x|}(B_k^\infty) = (uyx)^\infty$ belongs to $C(u) \cap X_2$. Therefore, X_2 is chaotic in the sense of Devaney.

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