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A GENERAL CONNECTION ON SASAKIAN MANIFOLDS AND THE CASE OF ALMOST PSEUDO SYMMETRIC SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study necessary conditions for the existence of almost pseudo symmetric and almost pseudo Ricci symmetric Sasakian manifolds admitting a new connection, named general connection. It is worthy to mention that such connection reduces to quarter-symmetric connection, generalized Tanaka-Webster connection, Zamkovoy connection and Schouten-van Kampen connection as particular cases.

1. Introduction

Let the symbols ∇, ∇^q, ∇^T, ∇^z, ∇^s and ∇^G stands for Levi-Civita connection, quarter-symmetric metric connection, Generalized Tanaka-Webster connection, Zamkovoy connection, Schouten-Van Kampen connection and general connections respectively. Also we denote $(APS)_n$ and $(APRS)_n$ for almost pseudo symmetric Sasakian manifold and almost pseudo Ricci symmetric Sasakian manifold. In 1924, A. Friedmann and J.A. Schouten [4] introduced the idea of semi-symmetric connection in the setting of Riemmanian geometry.

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Thereafter, it is H. A. Hayden [7], who studied semi-symmetric metric-connection in 1932. The systematic study of such connection was first started by K. Yano [19] in 1970. Generalising the notion of semi-symmetric connection, Golab [6] initiated the study of quarter-symmetric connection. Thereafter, many geometers like as Rastogi ([12], [13]), Mishra et al. [10], Yano et al.[18] and others studied quarter symmetric connection. A linear connection ∇^q on an n-dimensional Riemannian manifold (M^n, g) is called a quarter-symmetric connection [6] if its torsion tensor T of the connection ∇^q satisfies

$$\begin{aligned} T(X,Y) &= \nabla^q_X Y - \nabla^q_Y X - [X,Y] \\ &= \eta(Y)\phi X - \eta(X)\phi Y, \end{aligned}$$

where η is a 1-form and ϕ is a $(1,1)$ tensor field. In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [4]. Thus the notion of quarter-symmetric connection generalizes that of the semi-symmetric connection. Furthermore, if a quarter-symmetric connection ∇^q admits the condition

$$(\nabla^q_X g)(Y,Z) = 0,$$

then ∇^q is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection. Then

$$\nabla^q_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{1.1}$$

In 1970 Tanno [17] defined and studied the concept of generalized Tanaka-Webster connection by generalizing the connection defined by Tanaka [11] and Webster [16], which coincides with the Tanaka-Webster connection if the associated Contact Riemann structure is integrable. Also J. T. Cho ([8],[9]) has studied the Generalized Tanaka-Webster connection in the context of Kahler manifold. Let M be an n-dimensional almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g. Then Generalized Tanaka-Webster connection ∇^T and the Levi-Civita connection ∇ are related by

$$\nabla^T_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y. \tag{1.2}$$

Also Zamkovoy connection ∇^z defined by Z. Zamkovoy[20], is related with Levi-Civita connection as

$$\nabla^z_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y. \tag{1.3}$$

Schouten and Van Kampen [15] introduced the Schouten-Van Kampen connection in the third decade of last century. The relation between ∇^s and ∇ is

$$\nabla^s_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi. \tag{1.4}$$

In the setting of Sasakian geometry, we would like to define a new connection called general connection and denoted by ∇^G, which is related to ∇ as

$$\nabla^G_X Y = \nabla_X Y + \lambda\left[(\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi\right] + \mu\eta(X)\phi Y, \tag{1.5}$$

for all $X, Y \in \chi(M)$ and (λ, μ) being a pair of real constants. Connection defined in (1.5) has an important characteristic because it has a flavour of quarter symmetric metric connection for $(\lambda, \mu) \equiv (0, -1)$; Tanaka Webster connection for $(\lambda, \mu) \equiv (1, -1)$; Zamkovoy connection for $(\lambda, \mu) \equiv (1, 1)$ and Schouten-Van Kampen connection for $(\lambda, \mu) \equiv (1, 0)$.

This paper is structured as follows. In section 2 we recall basic facts on Sasakian manifolds. In section 3, some properties of Sasakian manifolds associated to a general connection are studied. In section 4, we investigate the necessary conditions for the existence of almost pseudo symmetric Sasakian manifolds admitting general connection and we observe that there does not exist an almost pseudo symetric Sasakian manifold with respect to general connection unless $3A + B = 0$, provided $\overline{C} \neq 0$. We also found that there do not exist $[(APS)_n, \nabla^T]$, $[(APS)_n, \nabla^s]$ and $[(APS)_n, \nabla^z]$. In section 5 we deal with almost pseudo Ricci symmetric Sasakian manifold associate to general connection and we observe that there exists no almost pseudo Ricci symetric Sasakian manifold with respect to general connection unless $3A + B = 0$,provided $\overline{C} \neq 0$, and there do not exist $[(APRS)_n, \nabla^T]$, $[(APRS)_n, \nabla^s]$, $[(APRS)_n, \nabla^z]$. Finally, in section 6, we have studied an almost pseudo Ricci symmetric Sasakian manifold with cyclic parallel Ricci tensor admitting general connection.

2. Preliminaries

Let M be an n-dimensional almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a

Riemannian metric g. Then

$$\phi^2 Y = -Y + \eta(Y)\xi, \eta(\xi) = 1, \eta(\phi X) = 0, \ \phi\xi = 0, \tag{2.1}$$
$$g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{2.2}$$
$$g(X,\phi Y) = -g(\phi X, Y), \eta(Y) = g(Y,\xi), \text{for all } X, Y \in \chi(M), \tag{2.3}$$

where $\chi(M)$ is set of all vector fields of the manifold M. An almost contact metric manifold M is said to be (a) a contact metric manifold if

$$g(X,\phi Y) = d\eta(X,Y), \text{for all } X, Y \in \chi(M); \tag{2.4}$$

(b) a K-contact manifold if the vector field ξ is Killing, equivalently

$$\nabla_Y \xi = -\phi Y, \tag{2.5}$$

where ∇ is Riemannian connection and
(c) a Sasakian manifold if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \text{for all } X, \ Y \in \chi(M). \tag{2.6}$$

Further, for Sasakian manifold with structure (ϕ, ξ, η, g), the following relations hold:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \text{for all } X, Y \in \chi(M), \tag{2.7}$$

$$(\nabla_X \eta)Y = g(X, \phi Y), \tag{2.8}$$
$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.9}$$
$$S(X,\xi) = (n-1)\eta(X), \tag{2.10}$$
$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.11}$$

$$Q\xi = (n-1)\xi, \tag{2.12}$$

where S and Q are the Ricci tensor and the Ricci operator, respectively.

3. Some properties of Sasakian manifold admitting general connection

For a Sasakian manifold, the relation (1.5) reduces to

$$\nabla^G_X Y = \nabla_X Y + \lambda [g(X,\phi Y)\xi + \eta(Y)\phi X] + \mu\eta(X)\phi Y. \tag{3.1}$$

Putting $Y = \xi$ in (1.5)

$$\nabla^G_X \xi = -\phi X + \lambda\phi X. \tag{3.2}$$

Now, in view of (2.5), (2.6) and (3.1) we get the following

$$\begin{aligned} & \nabla^G_X \eta(Y) \\ = {} & \nabla^G_X g(Y,\xi) \\ = {} & \eta(\nabla_X Y) + \lambda g(X,\phi Y) - g(Y,\phi X) + \lambda g(Y,\phi X), \end{aligned} \tag{3.3}$$

$$\begin{aligned} & \nabla^G_X(\phi Y) \\ = {} & \nabla_X(\phi Y) - \lambda g(\phi X,\phi Y)\xi - \mu\eta(X)Y + \mu\eta(X)\eta(Y)\xi, \end{aligned} \tag{3.4}$$

$$\begin{aligned} & \nabla^G_X g(Y,\phi Z) \\ = {} & g(\nabla_X Y,\phi Z) + \mu\eta(X) g(\phi Y,\phi Z) + g(Y,\nabla_X(\phi Z)) \\ & -\mu\eta(X) g(Y,Z) + \mu\eta(X)\eta(Y)\eta(Z). \end{aligned} \tag{3.5}$$

Now, we know that

$$R^G(X,Y)Z = \nabla^G_X\nabla^G_Y Z - \nabla^G_Y\nabla^G_X Z - \nabla^G_{[X,Y]}Z. \tag{3.6}$$

By using (3.1), (3.2), (3.3), (3.4) and (3.5), we obtain the following

$$\begin{aligned} & \nabla^G_X\nabla^G_Y Z \\ = {} & \nabla_X\nabla_Y Z + \lambda g(X,\phi\nabla_Y Z)\xi + \lambda\eta(\nabla_Y Z)\phi X + \mu\eta(X)\phi\nabla_Y Z \\ & +\lambda g(\nabla_X Y,\phi Z)\xi + \lambda\mu\eta(X) g(\phi Y,\phi Z)\xi + \lambda g(Y,\nabla_X(\phi Z))\xi \\ & -\lambda\mu\eta(X) g(Y,Z)\xi + \lambda\mu\eta(X)\eta(Y)\eta(Z)\xi - \lambda g(Y,\phi Z)\phi X \\ & +\lambda^2 g(Y,\phi Z)\phi X + \lambda\eta(\nabla_X Z)\phi Y + \lambda^2 g(X,\phi Z)\phi Y \\ & +\lambda^2 g(Z,\phi X)\phi Y + \lambda\eta(Z)\nabla_X(\phi Y) - \lambda^2\eta(Z) g(\phi X,\phi Y)\xi \\ & -\lambda\mu\eta(Z)\eta(X)Y + \lambda\mu\eta(Z)\eta(X)\eta(Y)\xi + \mu\eta(\nabla_X Y)\phi Z \\ & +\lambda\mu g(X,\phi Y)\phi Z - \mu g(Y,\phi X)\phi Z + \lambda\mu g(Y,\phi X)\phi Z \\ & +\mu\eta(Y)\nabla_X(\phi Z) - \lambda\mu\eta(Y) g(\phi X,\phi Z)\xi - \lambda g(Z,\phi X)\phi Y \\ & -\mu^2\eta(Y)\eta(X)Z + \mu^2\eta(Y)\eta(X)\eta(Z)\xi, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} & \nabla^G_{[X,Y]}Z \\ = {} & \nabla_{[X,Y]}Z + \lambda g(\nabla_X Y,\phi Z)\xi - \lambda g(\nabla_Y X,\phi Z)\xi + \lambda\eta(Z)\phi\nabla_X Y \\ & -\lambda\eta(Z)\phi\nabla_Y X + \mu\eta(\nabla_X Y)\phi Z - \mu\eta(\nabla_Y X)\phi Z. \end{aligned} \tag{3.8}$$

By interchanging X and Y in (3.7)

$$\begin{aligned}
& \nabla_Y^G \nabla_X^G Z \\
= \; & \nabla_Y \nabla_X Z + \lambda g\left(Y, \phi \nabla_X Z\right) \xi + \lambda \eta\left(\nabla_X Z\right) \phi Y + \mu \eta\left(Y\right) \phi \nabla_X Z \\
& + \lambda g\left(\nabla_Y X, \phi Z\right) \xi + AB\eta\left(Y\right) g\left(\phi X, \phi Z\right) \xi + \lambda g\left(X, \nabla_Y\left(\phi Z\right)\right) \xi \\
& - \lambda \mu \eta\left(Y\right) g\left(X, Z\right) \xi + \lambda \mu \eta\left(Y\right) \eta\left(X\right) \eta\left(Z\right) \xi - \lambda g\left(X, \phi Z\right) \phi Y \\
& + \lambda^2 g\left(X, \phi Z\right) \phi Y + \lambda \eta\left(\nabla_Y Z\right) \phi X + \lambda^2 g\left(Y, \phi Z\right) \phi X \\
& - \lambda g\left(Z, \phi Y\right) \phi X + \lambda^2 g\left(Z, \phi Y\right) \phi X + \lambda \eta\left(Z\right) \nabla_Y\left(\phi X\right) \\
& - \lambda^2 \eta\left(Z\right) g\left(\phi Y, \phi X\right) \xi - \lambda \mu \eta\left(Z\right) \eta\left(Y\right) X + \lambda \mu \eta\left(Z\right) \eta\left(Y\right) \eta\left(X\right) \xi \\
& + \mu \eta\left(\nabla_Y X\right) \phi Z + \lambda \mu g\left(Y, \phi X\right) \phi Z - \mu g\left(X, \phi Y\right) \phi Z \\
& + \lambda \mu g\left(X, \phi Y\right) \phi Z + \mu \eta\left(X\right) \nabla_Y\left(\phi Z\right) - \lambda \mu \eta\left(X\right) g\left(\phi Y, \phi Z\right) \xi \\
& - \mu^2 \eta\left(X\right) \eta\left(Y\right) Z + \mu^2 \eta\left(X\right) \eta\left(Y\right) \eta\left(Z\right) \xi,
\end{aligned} \tag{3.9}$$

In consequence of (3.7), (3.8), (3.9) and (3.6) we get

$$\begin{aligned}
& R^G\left(X, Y\right) Z \\
= \; & R\left(X, Y\right) Z + \left(\lambda^2 - 2\lambda\right)\left[g\left(Z, \phi X\right) \phi Y + g\left(Y, \phi Z\right) \phi X\right] \\
& - 2\mu g\left(Y, \phi X\right) \phi Z \\
& + \left(\lambda - \lambda\mu + \mu\right)\left[g\left(X, Z\right) \eta\left(Y\right) \xi - \eta\left(X\right) g\left(Y, Z\right) \xi\right] \\
& + \left(\lambda - \lambda\mu + \mu\right)\left[\eta\left(X\right) \eta\left(Z\right) Y - \eta\left(Y\right) \eta\left(Z\right) X\right].
\end{aligned} \tag{3.10}$$

Subsequently, one can easily obtain the following:

$$S^G\left(Y, Z\right) = S\left(Y, Z\right) - \overline{A} g\left(Y, Z\right) + \overline{B} \eta\left(Y\right) \eta\left(Z\right), \tag{3.11}$$

$$S^G\left(Y, \xi\right) = -\left(n-1\right) \overline{C} \eta\left(Y\right), \tag{3.12}$$

$$S^G\left(\xi, Z\right) = -\left(n-1\right) \overline{C} \eta\left(Z\right), \tag{3.13}$$

$$Q^G Y = QY - \overline{A} Y + \overline{B} \eta\left(Y\right) \xi, \tag{3.14}$$

$$Q^G \xi = -\left(n-1\right) \overline{C} \xi, \tag{3.15}$$

$$r^G = r - \overline{A} n + \overline{B}, \tag{3.16}$$

$$R^G\left(X, Y\right) \xi = \overline{C}\left[\eta\left(X\right) Y - \eta\left(Y\right) X\right], \tag{3.17}$$

$$R^G\left(\xi, Y\right) Z = \overline{C}\left[\eta\left(Z\right) Y - g\left(Y, Z\right) \xi\right], \tag{3.18}$$

$$R^G\left(X, \xi\right) Z = \overline{C}\left[g\left(X, Z\right) \xi - \eta\left(Z\right) X\right], \tag{3.19}$$

where S^G, Q^G, r^G are the Ricci tensor, Ricci operator and scalar curvature, respectively, with respect to the general connection and

$$\overline{A} = \left(\lambda^2 - \lambda - \mu - \lambda\mu\right), \tag{3.20}$$

$$\overline{B} = \left[\lambda^2 + (n-2)\lambda\mu - n(\lambda+\mu)\right], \tag{3.21}$$

$$\overline{C} = (\lambda - \lambda\mu + \mu - 1). \tag{3.22}$$

Therefore for quarter-symmetric metric connection

$$\overline{A} = 1; \overline{B} = n; \overline{C} = -2, \tag{3.23}$$

for Tanaka Webster connection

$$\overline{A} = 2; \overline{B} = 3 - n; \overline{C} = 0, \tag{3.24}$$

for Zamkovoy connection

$$\overline{A} = -2; \overline{B} = -1 - n; \overline{C} = 0, \tag{3.25}$$

and for Schouten-Van Kampen connection

$$\overline{A} = 0; \overline{B} = 1 - n; \overline{C} = 0 \tag{3.26}$$

respectively. Thus, we can state the following:

Proposition 3.1. *Let M be an n-dimensional Sasakian manifold admitting general connection ∇^G. Then*
(i) The curvature tensor R^G of ∇^G is given by (3.10),
(ii) The Ricci tensor S^G of ∇^G is given by (3.11),
(iii) The scalar curvature r^G of ∇^G is given by (3.16),
(iv) The Ricci tensor S^G of ∇^G is symmetric.

Now if we assume that the Sasakian manifold is Ricci flat with respect to the general connection, then from (3.11), we get

$$S(Y,Z) = \overline{A}g(Y,Z) - \overline{B}\eta(Y)\eta(Z).$$

This leads to the following

Theorem 3.2. *The manifold M^n is Ricci flat with respect to the general connection if and only if M^n is an η-Einstein manifold.*

4. Almost pseudo symmetric Sasakian manifold with respect to general connection

In 2007, Chaki and Kawaguchi [1], studied the notion of almost pseudo symmetric manifolds, which is defined as follows:

$$\begin{aligned}
&(\nabla_X R)(U,V)W \\
=\ & [A(X)+B(X)]R(U,V)W + A(U)R(X,V)W \\
&+A(V)R(U,X)W + A(W)R(U,V)X + g(R(U,V)W,X)P,
\end{aligned} \tag{4.1}$$

for all vectors fields $U, V, W \in \chi(M)$, where A, B are non zero one forms defined by

$$A(X) = g(X,P);\, B(X) = g(X,Q). \tag{4.2}$$

Here P and Q are associated vector fields corresponding to the one forms A and B.

In analogy with the definition of $(APS)_n$, a Sasakian manifold is said to be almost pseudo symmetric with respect to general connection (which will be denoted by $\left[(APS)_n, \nabla^G\right]$), if the curvature tensor R^G satisfies the following condition

$$\begin{aligned}
&\left(\nabla^G_X R^G\right)(U,V)W \\
=\ & [A(X)+B(X)]R^G(U,V)W + A(U)R^G(X,V)W \\
&+A(V)R^G(U,X)W + A(W)R^G(U,V)X + g\left(R^G(U,V)W,X\right)P.
\end{aligned} \tag{4.3}$$

Contracting (4.3), we obtain

$$\begin{aligned}
&\left(\nabla^G_X S^G\right)(V,W) \\
=\ & [A(X)+B(X)]S^G(V,W) + A\left(R^G(X,V)W\right) \\
&+A(V)S^G(X,W) + A(W)S^G(V,X) + A\left(R^G(X,W)V\right),
\end{aligned} \tag{4.4}$$

which yields for $W = \xi$

$$\begin{aligned}
&\left(\nabla^G_X S^G\right)(V,\xi) \\
=\ & [A(X)+B(X)]S^G(V,\xi) + A\left(R^G(X,V)\xi\right) \\
&+A(V)S^G(X,\xi) + A(\xi)S^G(V,X) + A\left(R^G(X,\xi)V\right).
\end{aligned} \tag{4.5}$$

Using (2.3), (3.12), we get the following

$$\left(\nabla^G_X S^G\right)(V,\xi) = (1-\lambda)\left[S(V,\phi X) + \overline{C}(n-1)g(V,\phi X) - \overline{A}g(V,\phi X)\right], \tag{4.6}$$

Taking account of (3.12), (3.17), (3.19) and (4.6) we get from (4.5)

$$
\begin{aligned}
& (1-\lambda)\left[S(V,\phi X)+\overline{C}(n-1)\,g(V,\phi X)\right] \\
& -(1-\lambda)\,\overline{A}g(V,\phi X) \\
= \; & -(n-1)\,\overline{C}\,[A(X)+B(X)]\,\eta(V) \\
& +\overline{C}\,[\eta(X)\,A(V)-\eta(V)\,A(X)] \\
& -(n-1)\,\overline{C}A(V)\,\eta(X)+A(\xi)\,S^{G}(V,X) \\
& +\overline{C}\,[g(X,V)\,A(\xi)-\eta(V)\,A(X)], \qquad (4.7)
\end{aligned}
$$

which yields for $X=V=\xi$

$$0=\overline{C}\,[3A(\xi)+B(\xi)] \qquad (4.8)$$

Proposition 4.1. *Let M^n be an almost pseudo symmetric Sasakian manifold with respect to the general connection ∇^G. Then $3P+Q$ is perpendicular to the characteristic vector field ξ,where P and Q are associated vector fields corresponding to the one forms A and B.*

Replacing V by ξ, in (4.4) and using (4.6), (3.12), (3.17) and (3.19) we obtain

$$
\begin{aligned}
& (1-\lambda)\left[S(\phi X,W)+\overline{C}(n-1)\,g(\phi X,W)\right] \\
& -(1-\lambda)\,\overline{A}g(\phi X,W) \\
= \; & -(n-1)\,\overline{C}\,[A(X)+B(X)]\,\eta(W) \\
& +\overline{C}\,[g(X,W)\,A(\xi)-\eta(W)\,A(X)] \\
& +A(\xi)\,S^{G}(X,W)-(n-1)\,\overline{C}A(W)\,\eta(X) \\
& +\overline{C}\,[\eta(X)\,A(W)-\eta(W)\,A(X)]. \qquad (4.9)
\end{aligned}
$$

Further replacing X by ξ in (4.9) and using (2.1), and (3.19) we obtain

$$
\begin{aligned}
& 0=-2(n-1)\,\overline{C}A(\xi)\,\eta(W)-(n-1)\,\overline{C}B(\xi)\,\eta(W) \\
& +\overline{C}\,[A(W)-\eta(W)\,A(\xi)]-(n-1)\,\overline{C}A(W). \qquad (4.10)
\end{aligned}
$$

Again replacing W by ξ in (4.9) and using (2.1) and (3.19) we obtain

$$
\begin{aligned}
& 0=-(n-1)\,\overline{C}\,[A(X)+B(X)] \\
& +2\overline{C}\,[\eta(X)\,A(\xi)-A(X)]-2(n-1)\,\overline{C}A(\xi)\,\eta(X). \qquad (4.11)
\end{aligned}
$$

Replacing W by X in (4.10)

$$
\begin{aligned}
& 0=-2(n-1)\,\overline{C}A(\xi)\,\eta(X)-(n-1)\,\overline{C}B(\xi)\,\eta(X) \\
& +\overline{C}\,[A(X)-\eta(X)\,A(\xi)]-(n-1)\,\overline{C}A(X). \qquad (4.12)
\end{aligned}
$$

Now adding (4.12) and (4.11) and using (4.8), we obtain

$$
\begin{aligned}
0 &= -(n-2)\overline{C}A(\xi)\eta(X) \\
&\quad -(2n-1)\overline{C}A(X)-(n-1)\overline{C}B(X). \qquad (4.13)
\end{aligned}
$$

Further, adding (4.12) and (4.13) and using (4.8), we obtain

$$\overline{C}[3A(X)+B(X)]=0.$$

Thus, we can state following

Theorem 4.2. *There does not exist almost pseudo symmetric Sasakian manifold admitting a general connection, unless* $3A+B$ *vanishes everywhere, provided that* $\overline{C}\neq 0$.

Corollary 4.3. *There do not exist* $[(APS)_n,\nabla^T]$, $[(APS)_n,\nabla^s]$, $[(APS)_n,\nabla^z]$.

Corollary 4.4. *Let* M^n *be an almost pseudo symmetric Sasakian manifold with respect to the general connection* ∇^G*, then* $3\parallel P\parallel=\parallel Q\parallel$*, provided that* $\overline{C}\neq 0$.

5. Almost pseudo Ricci symmetric Sasakian manifold with respect to general connection

A non flat n-dimensional Sasakian manifold M^n $(n\geq 2)$ is said to be an almost pseudo Ricci symmetric with respect to general connection briefly ($[(APRS)_n,\nabla^G]$) if the Ricci tensor S^G satisfies the following condition

$$\left(\nabla^G_X S\right)(U,V)=[\alpha(X)+\beta(X)]S^G(U,V)+\alpha(U)S^G(X,V)+\alpha(V)S^G(X,U). \qquad (5.1)$$

for all vectors fields $U,V\in\chi(M)$, where α, β are non zero one forms defined by

$$\alpha(X)=g(X,\rho);\beta(X)=g(X,\sigma), \qquad (5.2)$$

where ρ and σ are associated vector fields corresponding to the one forms α and β.

Replacing V by ξ in (5.1) and using (3.19) and (4.6)

$$
\begin{aligned}
&(1-\lambda)\left[S(U,\phi X)+\overline{C}(n-1)g(U,\phi X)-\overline{A}g(U,\phi X)\right] \\
&= -(n-1)\overline{C}[\alpha(X)+\beta(X)]\eta(U) \\
&\quad -(n-1)\overline{C}\alpha(U)\eta(X)+\alpha(\xi)S^G(X,U). \qquad (5.3)
\end{aligned}
$$

By putting $X = U = \xi$ in (5.3) and using (3.19) we get

$$\overline{C}\left[3\alpha\left(\xi\right) + \beta\left(\xi\right)\right] = 0 \tag{5.4}$$

By replacing X by ξ in (5.3) and using (3.19), (2.1) we obtain

$$\begin{aligned} & 0 \\ = \; & -\left(n-1\right)\overline{C}\left[\alpha\left(\xi\right) + \beta\left(\xi\right)\right]\eta\left(U\right) \\ & -\left(n-1\right)\overline{C}\alpha\left(U\right) - \left(n-1\right)\overline{C}\alpha\left(\xi\right)\eta\left(U\right). \end{aligned} \tag{5.5}$$

By replacing U by ξ in (5.3) and using (3.19), (2.1) we obtain

$$\begin{aligned} & 0 \\ = \; & -\left(n-1\right)\overline{C}\left[\alpha\left(X\right) + \beta\left(X\right)\right] \\ & -\left(n-1\right)\overline{C}\alpha\left(\xi\right)\eta\left(X\right) - \left(n-1\right)\overline{C}\alpha\left(\xi\right)\eta\left(X\right). \end{aligned} \tag{5.6}$$

Replacing U by X in (5.5)

$$\begin{aligned} & 0 \\ = \; & -\left(n-1\right)\overline{C}\left[\alpha\left(\xi\right) + \beta\left(\xi\right)\right]\eta\left(X\right) \\ & -\left(n-1\right)\overline{C}\alpha\left(X\right) - \left(n-1\right)\overline{C}\alpha\left(\xi\right)\eta\left(X\right). \end{aligned} \tag{5.7}$$

Adding (5.6) and (5.7) and using (5.4)

$$\begin{aligned} 0 \; = \; & -\left(n-1\right)\overline{C}\left[\alpha\left(X\right) + \beta\left(X\right)\right] \\ & -\left(n-1\right)\overline{C}\alpha\left(X\right) - \left(n-1\right)\overline{C}\alpha\left(\xi\right)\eta\left(X\right). \end{aligned} \tag{5.8}$$

Further adding (5.7) and (5.8) and using (5.4) we obtain

$$\overline{C}\left[3\alpha\left(X\right) + \beta\left(X\right)\right] = 0. \tag{5.9}$$

Thus we can state the following

Theorem 5.1. *There exists no almost pseudo Ricci symmetric Sasakian manifold admitting a general connection, unless* $3A + B = 0$ *vanishes everywhere, provided that* $\overline{C} \neq 0$.

Corollary 5.2. *There do not exist* $[(APRS)_n, \nabla^T]$, $[(APRS)_n, \nabla^s]$, $[(APRS)_n, \nabla^z]$

Corollary 5.3. *Let* M^n *be an almost pseudo Ricci symmetric Sasakian manifold with respect to the general connection* ∇^G. *Then* $3 \parallel \rho \parallel = \parallel \sigma \parallel$, *provided that* $\overline{C} \neq 0$.

6. Almost pseudo symmetric Sasakian manifold with cyclic parallel Ricci tensor admitting the general connection

A non flat n-dimensional almost pseudo Ricci symmetric Sasakian manifold M^n $(n \geq 2)$ is said to be cyclic parallel Ricci tensor if

$$(\nabla_X S)(U,V) + (\nabla_U S)(V,X) + (\nabla_V S)(X,U) = 0.$$

Therefore, an almost pseudo Ricci symmetric Sasakian manifold has a cyclic parallel Ricci tensor with respect to general connection if

$$\left(\nabla^G_X S\right)(U,V) + \left(\nabla^G_U S\right)(V,X) + \left(\nabla^G_V S\right)(X,U) = 0. \tag{6.1}$$

Using (5.1) in (6.1), we obtain

$$\begin{aligned} 0 &= 3\alpha(X) S^G(U,V) + 3\alpha(U) S^G(X,V) + 3\alpha(V) S^G(X,U) \\ &\quad +\beta(X) S^G(U,V) + \beta(U) S^G(V,X) + \beta(V) S^G(X,U) \end{aligned} \tag{6.2}$$

Setting $V = \xi$ in (6.2) and using (3.19) we obtain

$$\begin{aligned} &0 = -3(n-1)\overline{C}\alpha(X)\eta(U) - 3(n-1)\overline{C}\alpha(U)\eta(X) \\ &+3\eta(\rho) S^G(X,U) - (n-1)\overline{C}\beta(X)\eta(U) \\ &-(n-1)\overline{C}\beta(U)\eta(X) + \eta(\sigma) S^G(X,U). \end{aligned} \tag{6.3}$$

Replacing U by ξ in (6.3) and using (2.1), (4.6), (4.7) and (3.12) we obtain

$$\begin{aligned} &0 = -3(n-1)\overline{C}\alpha(X) - 3(n-1)\overline{C}\eta(P_1)\eta(X) \\ &-3(n-1)\overline{C}\eta(P_1)\eta(X) - (n-1)\overline{C}\beta(X) \\ &-(n-1)\overline{C}\eta(Q_1)\eta(X) - (n-1)\overline{C}\eta(Q_1)\eta(X). \end{aligned} \tag{6.4}$$

Putting $X = \xi$ in (6.4) and by the help of (2.1), (4.6), (4.7) and (3.12) we obtain

$$0 = \overline{C}\left[3\eta(\rho) + \eta(\sigma)\right]. \tag{6.5}$$

Now by the help of (6.4) and (6.5) we get

$$0 = \overline{C}\left[3\alpha(X) + \beta(X)\right].$$

Theorem 6.1. *An almost pseudo symmetric Sasakian manifold has a cyclic parallel Ricci tensor admitting general connection if and only if* $3\alpha(X) + \beta(X) = 0$, *provided that* $\overline{C} \neq 0$ *for any vector field* X *on* M^n.

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