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A NEW CONTRACTION-LIKE MULTIVALUED MAPPING ON GEODESIC SPACES

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Abstract. We define a new class of multivalued mappings and prove existence, stability of fixed point sets and convergence results for this class of multivalued hybrid mappings in CAT(κ) spaces which are more general than Hilbert spaces and CAT(0) spaces. We also show with an example that this class is not included in the class of multivalued nonexpansive mappings, however, interestingly, we see that multivalued version of Thianwan iteration is built with this mapping is strong convergent to its fixed point.

1. Introduction and Basic Concepts

Although real-life events and their scientific models are nonlinear in structure, the fixed point theory, which is a precious tool for various branches such as control theory, convex optimization, differential equations and economics, mainly focused on linear spaces like Hilbert spaces and Banach spaces. Therefore, developing the fixed point theory on spaces with nonlinear structures has become valuable. CAT(κ) spaces are non-linear spaces very appropriate for the development of a fixed point theory, due to their convex structure and rich properties similar to Banach and Hilbert spaces.

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Until now, researchers generally worked on CAT(0) spaces for single valued nonexpansive mappings. However, not much is known about multivalued mappings on CAT (κ) spaces.

In this paper, we give the definition of a new class of multivalued mappings on CAT(κ) spaces. We support with an example that this new class is neither a subclass of contraction mappings, nor a subclass of nonexpansive mappings. Moreover, since every Hilbert space and CAT(0) space is a CAT(κ) space for $\kappa > 0$ and any bounded sequence has unique asymptotic center on these spaces, all results are also valid for these spaces.

Many iterative processes to find a fixed point of multivalued mappings have been introduced in metric and Banach spaces. The well-known one is defined by Nadler as generalization of Picard as follows;

$$x_{n+1} \in Tx_n.$$

A multivalued version of Mann and Ishikawa fixed point procedures goes as follows:

$$x_{n+1} \in (1-\zeta_n)x_n + \zeta_n Tx_n$$

and

$$\begin{aligned} x_{n+1} &\in (1-\zeta_n)x_n + \zeta_n Ty_n, \\ y_n &\in (1-\varsigma_n)x_n + \varsigma_n Tx_n, \end{aligned}$$

where $\{\zeta_n\}$ and $\{\varsigma_n\}$ are sequences in $[0,1]$.

In 2009, S. Thianwan [9] introduced two steps iterations as follows;

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)y_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Now we define multivalued version of Thianwan iteration in $CAT(\kappa)$ spaces as follows:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)y_n \oplus \alpha_n u_n, \\ y_n &= (1-\beta_n)x_n \oplus \beta_n v_n, \end{aligned} \tag{1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ with $\sum\limits_{n=0}^{\infty} \beta_n(1-r) = \infty$ and $u_n \in Ty_n$, $v_n \in Tx_n$.

Let (X,d) be a metric space and $K \subseteq X$ non-empty . In this paper we will use following notations:

$C(X)$ stands for the family of all non-empty closed subsets of X, $CC(X)$ for the family of all non-empty closed and convex subsets of X, $KC(X)$ for non-empty, compact and convex subsets of X and $CB(X)$ for the family of all non-empty, closed and convex subsets of X.

Let H be the Hausdorff metric on $CB(X)$, defined by

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\}$$

where $d(x,B) = \inf\{d(x,y); y \in B\}$.

A point p is called fixed point of a multivalued mapping T if $p \in Tp$. The set of all fixed points of T is denoted by $F(T)$.

Let (X,d) be a bounded metric space and let $x, y \in X$ and $K \subseteq X$ non-empty. A geodesic path (or shortly a geodesic) joining x and y is a map $c : [0,t] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(t) = y$ and $d(c(r), c(s)) = |r - s|$ for all $r, s \in [0,t]$. In fact, c is an isometry and $d(c(0), c(t)) = t$. The image of c, $c([0,t])$ is called a geodesic segment from x to y and it is not necessarily unique. If it is unique, then it is denoted by $[x,y]$. We have $z \in [x,y]$ if and only if there exists $t \in [0,1]$ such that $d(z,x) = (1-t)d(x,y)$ and $d(z,y) = td(x,y)$. The point z is denoted by $z = (1-t)x \oplus ty$. For fixed $r > 0$, the space (X,d) is called an r-geodesic space if for every two points $x, y \in X$ with $d(x,y) < r$ there is a geodesic joining x to y. If for every $x, y \in X$, there is a geodesic path then (X,d) is called a geodesic space (respectively, a uniquely geodesic space if that geodesic path is unique for any pair x, y). We call a subset $K \subseteq X$ convex if it contains all geodesic segment joining any pair of points in it.

Definition 1.1. *(see:*[1]*)Let $\kappa \in \mathbb{R}$.*

i) *if $\kappa = 0$, then M^n_κ is the Euclidean space $\mathbb{E}^n$,*
ii) *if $\kappa > 0$, then M^n_κ is obtained from the sphere $\mathbb{S}^n$ by multiplying the distance function by $\frac{1}{\sqrt{\kappa}}$,*
iii) *if $\kappa < 0$, then M^n_κ is obtained from hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by $\frac{1}{\sqrt{-\kappa}}$.*

In a geodesic metric space (X,d), a *geodesic triangle*, denoted by $\Delta(x,y,z)$ consists of three points x, y, z as vertices and the geodesic segments between any pair of these points. That is, $q \in \Delta(x,y,z)$ means that $q \in [x,y] \cup [x,z] \cup [y,z]$. A triangle $\overline{\Delta}(\overline{x}, \overline{y}, \overline{z})$ in M^2_κ is called a *comparison triangle* for the triangle $\Delta(x,y,z)$ such that $d(x,y) = d(\overline{x}, \overline{y}), d(x,z) = d(\overline{x}, \overline{z})$ and $d(y,z) = d(\overline{y}, \overline{z})$ and such a comparison triangle always exists provided that the perimeter $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa($ where $D_\kappa = \frac{\pi}{\sqrt{K}}$ if $\kappa > 0$ and ∞ otherwise) in M^2_κ (Lemma 2.14 in [1]). A point point $\overline{z} \in [\overline{x}, \overline{y}]$ called a *comparison point* for $z \in [x,y]$ if $d(x,z) = d(\overline{x}, \overline{z})$. A geodesic triangle $\Delta(x,y,z)$ in X with perimeter less than $2D_\kappa$ (and given a comparison triangle $\overline{\Delta}(\overline{x},\overline{y},\overline{z})$ for $\Delta(x,y,z)$ in M_κ^2) satisfies $CAT(\kappa)$ inequality if $d(p,q) \leq d(\overline{p},\overline{q})$ for all $p,q \in \Delta(x,y,z)$ where $\overline{p},\overline{q} \in \overline{\Delta}(\overline{x},\overline{y},\overline{z})$ are the comparison points of p,q respectively. The D_κ-geodesic metric space (X,d) is called $CAT(\kappa)$ space if every geodesic triangle in X with perimeter less than $2D_\kappa$ satisfies the $CAT(\kappa)$ inequality.

If for every $x,y,z \in X$, there is an $R \in (0,2]$ satisfying

$$d^2(x,(1-\lambda)y \oplus \lambda z) \leq (1-\lambda)d^2(x,y) + \lambda d^2(x,z) - \frac{R}{2}\lambda(1-\lambda)d^2(y,z),$$

then (X,d) is called $R-$*convex* [6]. Hence, (X,d) is a $CAT(0)$ space if and only if it is a $2-$*convex space.*

Lemma 1.2. *(see:*[7]*) Let $\kappa > 0$ and (X,d) be a $CAT(\kappa)$ space with $diam(X) < \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0,\frac{\pi}{2})$. Then (X,d) is a $R-$convex space for $R = (\pi - 2\varepsilon)\tan(\varepsilon)$.*

Proposition 1.3. *(see:*[1]*) Let X be $CAT(\kappa)$ space. Then any ball of radius smaller than $\frac{\pi}{2\sqrt{\kappa}}$ is convex.*

Proposition 1.4. *(Exercise 2.3(1) in* [1]*) Let $\kappa > 0$ and (X,d) be a $CAT(\kappa)$ space with $diam(X) < \frac{D_\kappa}{2} = \frac{\pi}{2\sqrt{\kappa}}$. Then, for any $x,y,z \in X$ and $t \in [0,1]$, we have*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Let $\{x_n\}$ be a bounded sequence in a $CAT(\kappa)$ space X, $x \in X$ and

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius of $\{x_n\}$ is defined by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}); x \in X.\},$$

the asymptotic radius of $\{x_n\}$ with respect to $K \subseteq X$ is defined by

$$r_K(\{x_n\}) = \inf\{r(x,\{x_n\}); x \in K.\},$$

and the asymptotic center of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}$$

and let $\omega_w(x_n) := \cup A(\{x_n\})$ where the union is taken on all subsequences of $\{x_n\}$.

Definition 1.5. *(see:*[4]*)A sequence $\{x_n\} \subset X$ is said to be $\Delta-$convergent to $x \in X$ if x is the unique asymptotic center of all subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and x is called the $\Delta-$limit of $\{x_n\}$.*

Proposition 1.6. *(see:[4])Let X be a complete $CAT(\kappa)$ space, let $K \subseteq X$ be non-empty, closed and convex, and let $\{x_n\}$ be a sequence in X. If $r_K(\{x_n\}) < \frac{\pi}{2\sqrt{\kappa}}$, then $A_K(\{x_n\})$ consists of exactly one point.*

Lemma 1.7. *If ρ is a real number satisfying $0 \le \rho < 1$ and $(\epsilon_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers such that $lim_{n\to\infty}\epsilon_n = 0$, then for any sequence of positive numbers $(\epsilon_n)_{n\in\mathbb{N}}$ satisfying*

$$a_{n+1} \le \rho a_n + \epsilon_n, n = 1, 2, ...,$$

we have

$$\lim_{n\to\infty} a_n = 0.$$

Definition 1.8. *T is called a* generalized multivalued contractive hybrid mapping *from X to $CB(X)$, for fixed $r \in [0, 1)$, if for all $x, y \in X$ and $u \in Ty$,*

$$a_1(x)\delta(Tx, u) + a_2(x)\delta(Tx, y) \le r[b_1(x)d(x, Ty) + b_2(x)d(x, y)]$$

where $\delta(Tx, z) = \sup\{d(x', z) : x' \in Tx\}$, $a_{1,}a_2 : X \to \mathbb{R}$ and $a_1(x) + a_2(x) \ge 1$ for all $x \in X$ and $b_1, b_2 : X \to [0, 1]$ with $b_1(x) + b_2(x) \le 1$ for all $x, z \in X$.

Note that if Tx contains an unique element for each $x \in K$ (i.e, T it is single valued) and if $a_2(x) \le 0$ for all $x \in K$ then

$$a_1(x)d(Tx, Ty) \le r[b_1(x)d(x, Ty) + b_2(x)d(x, y)] - a_2(x)d(Tx, y)$$

If we take $a_1(x) = 1, a_2(x) = 0, b_1(x) = 0, b_2(x) = 1$ then T is a contraction. If we take $a_1(x) = 1, a_2(x) < 0$, then T is a constant. If $b_1(x) = 0, b_2(x) = 1$ then T is an almost contraction.

However, this new class is not directly comparable with multivalued nonexpansive or multivalued contraction mappings and we support this with an example that there is multivalued mappings which is contained in this class, while it is not nonexpansive.

2. **Main Results**

Theorem 2.1. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \le \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a non-empty closed convex subset of X, and $T : K \to C(K)$ be a generalized multivalued contractive hybrid mapping with $a_1(x) \le 0$ or $a_2(x) \le 0$ for all $x \in K$. Then $F(T) \ne \emptyset$.*

Proof. Let $x_0 \in K$ and chose $x_{n+1} \in Tx_n$. Assume that $A_K\{x_n\} = \{z\}$. Then $z \in K$ by Proposition 1.6 Since T is contractive hybrid, then the following inequality is satisfied for all $n \geq 0$

$$\begin{aligned} a_1(z)\delta(Tz, x_{n+1}) + a_2(z)\delta(Tz, x_n) &\leq r[b_1(z)d(z, Tx_n) + b_2(z)d(z, x_n)] \\ &\leq r[b_1(z)d(z, x_{n+1}) + b_2(z)d(z, x_n)]. \end{aligned}$$

Then since

$$a_1(z)\delta(Tz, x_{n+1}) + a_2(z)\delta(Tz, x_n) \leq r[b_1(z)d(z, x_{n+1}) + b_2(z)d(z, x_n)$$

is satisfied, taking limit superior on both sides we get

$$\limsup_{n\to\infty} \delta(x_{n+1}, Tz) \leq \limsup_{n\to\infty} d(x_{n+1}, z).$$

Since for any $u \in Tz$, $d(x_{n+1}, u) \leq \delta(x_{n+1}, Tz)$ we get

$$\limsup_{n\to\infty} \delta(x_{n+1}, u) \leq \limsup_{n\to\infty} d(x_{n+1}, z).$$

Hence we conclude that $z = u \in Tz = \{u\}$ ∎

Theorem 2.2. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and $T : K \to C(K)$ be a generalized multivalued contractive hybrid mapping with contractive constant r and $F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ be sequences in $(0, 1)$ such that $\sum_{n=0}^{\infty} \alpha_n(1 - r) = \infty$.*

Then a sequence $\{x_n\}$ defined by $x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n u_n$, $u_n \in Tx_n$ is strongly convergent to a fixed point of T.

Proof. Let $p \in F(T)$, then $Tp = \{p\}$. So for all $x \in X$ and $p \in F(T)$, the point p satisfies

$$\begin{aligned} a_1(x)\delta(Tx, p) + a_2(x)\delta(Tx, p) &\leq r[b_1(x)d(x, Tp) + b_2(x)d(x, p)] \\ &\leq rd(x, p) \end{aligned}$$

which implies $\delta(Tx, p) \leq a_1(x)\delta(Tx, p) + a_2(x)\delta(Tx, p) \leq rd(x, p)$.

$$\begin{aligned} d(x_{n+1}, p) &= d((1 - \alpha_n)x_n \oplus \alpha_n u_n, p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(u_n, p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n \delta(Tx_n, p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n r d(x_n, p) \end{aligned}$$

Since

$$\begin{aligned} d(x_{n+1}, p) &\leq (1-\alpha_n)d(x_n, p) + \alpha_n r d(x_n, p) \\ &\leq (1-\alpha_n(1-r))d(x_n, p) \end{aligned}$$

then by Lemma 1.7, $\lim_{n\to\infty} d(x_n, p) = 0$. ∎

Theorem 2.3. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a non-empty closed convex subset of X and $T_1, T_2 : K \to C(K)$ be two generalized multi-valued contractive hybrid mappings with same contraction coefficient $r \in [0, 1)$ Then*

$$H(F(T_1), F(T_2)) \leq \sup_{x\in K} H(T_1x, T_2x)\frac{1}{1-r}.$$

Proof. Let $x_0 \in F(T_1)$. We define the sequence $\{x_n\}$ by $x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n u_n, u_n \in T_2x_n$. Using Theorem 2.2, we obtain $x_n \to p \in F(T_2)$. If we choose $u_0 \in T_2x_0$ such that $d(x_0, u_0) = d(x_0, T_2x_0)$ then

$$\begin{aligned} d(x_0, p) &\leq d(x_0, x_1) + d(x_1, p) \\ &\leq d(x_0, (1-\alpha_0)x_0 \oplus \alpha_0 u_0) + d((1-\alpha_0)x_0 \oplus \alpha_0 u_0, p) \\ &\leq \alpha_0 d(x_0, u_0) + (1-\alpha_0)d(x_0, p) + \alpha_0 d(u_0, p) \\ &\leq \alpha_0 d(x_0, T_2x_0) + (1-\alpha_0)d(x_0, p) + \alpha_0\delta(T_2x_0, p) \\ &\leq \alpha_0 H(T_1x_0, T_2x_0) + (1-\alpha_0)d(x_0, p) + \alpha_0 r d(x_0, p) \\ &\leq \alpha_0 H(T_1x_0, T_2x_0) + (1-\alpha_0(1-r))d(x_0, p) \end{aligned}$$

implies that

$$(1-(1-\alpha_0(1-r)))d(x_0, p) \leq \alpha_0 H(T_1x_0, T_2x_0)$$

so we have

$$\alpha_0(1-r)d(x_0, p) \leq \alpha_0 H(T_1x_0, T_2x_0) \to d(x_0, p) \leq H(T_1x_0, T_2x_0)\frac{1}{1-r}$$

implies

$$d(x_0, p) \leq \sup_{x\in K} H(T_1x, T_2x)\frac{1}{1-r}.$$

So for all $x_0 \in F(T_1)$ we can find $z \in F(T_2)$ and similarly for all $x_0^{'} \in F(T_2)$ we can find $z^{'} \in F(T_1)$.Hence

$$H(F(T_1), F(T_2)) \leq \sup_{x\in K} H(T_1x, T_2x)\frac{1}{1-r}.$$

holds. ∎

Lemma 2.4. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a non-empty closed convex subset of X and $\{T_n\} : K \to K(K)$ be a sequence of generalized multivalued contractive hybrid mapping with same coefficient functions. If $\{T_n\}$ is uniformly convergent to a multivalued map $T : K \to K(K)$, then T is a generalized multivalued contractive hybrid mapping.*

Proof. Since for all for all $n \geq 0$,

$$a_1(x)\delta(T_nx, u_n) + a_2(x)\delta(T_nx, y) \leq r[b_1(x)d(x, T_ny) + b_2(x)d(x, y)]$$

is satisfied,where $u_n \in T_ny$ taking limit for $n \to \infty$ we get

$$a_1(x)\delta(Tx, u) + a_2(x)\delta(Tx, y) \leq r[b_1(x)d(x, Ty) + b_2(x)d(x, y)].$$

where $u \in Ty$ ∎

Theorem 2.5. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a non-empty closed convex subset of X and $\{T_n\} : K \to K(K)$ be a sequence of generalized multivalued contractive hybrid mappings with the same coefficient function. Assume that $a_1(x) \geq 0$ for all $x \in X$, and $a_1(x)$ is bounded. If $\{T_n\}$ is uniformly convergent to a multivalued map $T : K \to K(K)$, then $F(T_n)$ uniformly converges to $F(T)$.*

Proof. By Therorem 2.3,

$$H(F(T_n), F(T)) \leq \sup_{x\in K} H(T_nx, Tx)\frac{1}{1-r}$$

taking limit on n, we get that

$$\lim_{n\to\infty} H(F(T_n), F(T)) = 0$$

∎

Theorem 2.6. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and $T : K \to C(K)$ be a generalized multivalued contractive hybrid mapping with contractive constant r and $F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0, 1)$ such that $\sum_{n=0}^{\infty} \beta_n(1-r) = \infty$ then $\{x_n\}$ sequence defined by (1.1) is strongly convergent to a fixed point of T.*

Proof. Let $p \in F(T)$, then for all $x \in K$, since $\delta(Tx, p) \leq d(x, p)$ for all $x \in K$,we have

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)x_n \oplus \beta_n v_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(v_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n \delta(Tx_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n r d(x_n, p) \\ &= d(x_n, p) \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)y_n \oplus \alpha_n u_n, p) \\ &\leq (1-\alpha_n)d(y_n, p) + \alpha_n d(u_n, p) \\ &\leq (1-\alpha_n)d(y_n, p) + \alpha_n \delta(Ty_n, p) \\ &\leq (1-\alpha_n)d(y_n, p) + \alpha_n r d(y_n, p) \\ &= d(y_n, p) \end{aligned}$$

implies

$$\begin{aligned} d(x_{n+1}, p) &\leq d(y_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n r d(x_n, p) \\ &\leq (1-\beta_n(1-r))d(x_n, p) \end{aligned}$$

then by Lemma 1.7, $\lim_{n\to\infty} d(x_n, p) = 0$. ∎

Theorem 2.7. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and $T : K \to C(K)$ be a generalized multivalued contractive hybrid mapping with contractive constant r and $F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1)$ such that $\sum_{n=0}^{\infty} \beta_n(1-r) = \infty$ then $\{x_n\}$ sequence. Moreover if $Tp = \{p\}$ for all $p \in F(T)$ then $\{x_n\}$ sequence defined by (1.1) converging to fixed point of T is $T-$stable.*

Proof. Let $x_n \to p \in F(T)$, $\{y_n\}$ be arbitrary sequence in K and set $\epsilon_n = d(y_{n+1}, (1-\alpha_n)y'_n \oplus \alpha_n u'_n)$ where $y'_n = (1-\beta_n)y_n \oplus \beta_n v'_n$ and $u'_n \in Ty'_n$, $v'_n \in Ty_n$. We shall show that $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} y_n = p$. Since T is generalized multivalued contractive hybrid mapping,

$$\delta(Tx, p) \leq rd(x, p)$$

Let $\lim_{n\to\infty}\epsilon_n = 0$ then.we get

$$\begin{aligned} d(y'_n, p) &= d((1-\beta_n)y_n \oplus \beta_n v'_n, p) \\ &\leq (1-\beta_n)d(y_n, p) + \beta_n d(v'_n, p) \\ &\leq (1-\beta_n)d(y_n, p) + \beta_n \delta(Ty_n, p) \\ &\leq (1-\beta_n)d(y_n, p) + \beta_n r d(y_n, p) \\ &\leq d(y_n, p) \end{aligned}$$

and so

$$\begin{aligned} d(y_{n+1}, p) &\leq d(y_{n+1}, (1-\alpha_n)y'_n \oplus \alpha_n u'_n) + d(y_{n+1}, (1-\alpha_n)y'_n \oplus \alpha_n u'_n, p) \\ &\leq \epsilon_n + (1-\alpha_n)d(y'_n, p) + \alpha_n d(u'_n, p) \\ &\leq \epsilon_n + (1-\alpha_n)d(y'_n, p) + \alpha_n \delta(Ty'_n, p) \\ &\leq \epsilon_n + d(y'_n, p) \\ &\leq \epsilon_n + (1-\beta_n)d(y_n, p) + \beta_n r d(y_n, p) \\ &= \leq \epsilon_n + (1-\beta_n(1-r))d(y_n, p) \end{aligned}$$

then by Lemma 1.7 $\lim_{n\to\infty} d(y_n, p) = 0$.Let $\lim_{n\to\infty} y_n = p$ then

$$\begin{aligned} \epsilon_n &= d(y_{n+1}, (1-\alpha_n)y'_n \oplus \alpha_n u'_n) \\ &\leq d(y_{n+1}, p) + d(p, (1-\alpha_n)y'_n \oplus \alpha_n u'_n) \\ &\leq d(y_{n+1}, p) + (1-\alpha_n)d(y'_n, p) + \alpha_n d(u'_n, p) \\ &\leq d(y_{n+1}, p) + (1-\alpha_n)d(y'_n, p) + \alpha_n \delta(Ty'_n, p) \\ &\leq d(y_{n+1}, p) + d(y'_n, p) \\ &\leq d(y_{n+1}, p) + d(y_n, p) \end{aligned}$$

aking limit on $n \to \infty$, we get that $\lim_{n\to\infty}\epsilon_n = 0$ ∎

Example 2.8. *Let $X = [0, 5]$ with usual metric and $T : X \to C(X)$ be multivalued mapping defined by*

$$Tx = \begin{cases} [0, \frac{x}{5}], & x \in [0, 3); \\ [\frac{x}{5}, 1], & x \in [3, 5]. \end{cases}$$

We will show that T is a generalized multivalued contractive hybrid mapping with $r = \frac{7}{8}$, $a_1(x) = \frac{3x+4}{2x+3}$, $a_2(x) = \frac{-x-1}{2x+3}$, $b_1(x) = \frac{x+2}{2x+3}$, $b_2(x) = \frac{x+1}{2x+3}$ for all $x \in X$.

Case 1: if $x, y \in [0, 3)$ then for all $u \in Tx$, we have followings; $\delta(Tx, u) \leq \delta(Tx, Ty) = \max\{\frac{x}{5}, \frac{y}{5}\}$, $\delta(Tx, y) = \max\{|0-y|, |\frac{x}{5}-y|\}$, $d(x, Ty) = \min\{|x-0|, |x-\frac{y}{5}|\}$ and $d(x, y) = |x-y|$ satisfied.

Threfore

$$\max\{x, y\} \leq \max\{|0-y|, |\frac{x}{5}-y|\} + \frac{7}{8}[\min\{|x-0|, |x-\frac{y}{5}|\} + |x-y|].$$

Therefore

$$\begin{aligned}
\frac{3x+4}{2x+3}\delta(Tx,u) &\leq \frac{3x+4}{2x+3}\delta(Tx,Ty) \\
&\leq \frac{3x+4}{2x+3}\max\{\frac{x}{5},\frac{y}{5}\} \\
&= \frac{3x/5+4/5}{2x+3}\max\{x,y\} \\
&\leq \frac{x+1}{2x+3}\max\{x,y\} \\
&\leq \frac{x+1}{2x+3}\max\{|0-y|,|\frac{x}{5}-y|\} \\
&\quad +\frac{7}{8}[\frac{x+2}{2x+3}\min\{|x-0|,|x-\frac{y}{5}|\}+\frac{x+1}{2x+3}|x-y|] \\
&= \frac{x+1}{2x+3}\delta(Tx,y)+\frac{7}{8}[\frac{x+2}{2x+3}d(x,Ty) \\
&\quad +\frac{x+1}{2x+3}d(x,y)].
\end{aligned}$$

satisfied for for all $x, y \in [0,3)$ and $u \in Tx$.

Case 2: if $x \in [0,3), y \in [3,5]$ then for all $u \in Tx$ according to the following calculations
$\delta(Tx,u) \leq \delta(Tx,Ty) = 1$, $d(x,Ty) = \min\{|x-1|, |x-\frac{y}{5}|\}$,
$\delta(Tx,y) = y$ and $d(x,y) = |x-y|$

$$4 \leq y + \frac{7}{8}[\min\{|x-0|, |x-\frac{y}{5}|\} + |x-y|].$$

is hold for all $x \in [0,3), y \in [3,5]$. Therefore we have

$$\begin{aligned}
\frac{3x+4}{2x+3}\delta(Tx,u) &\leq \frac{3x+4}{2x+3}\delta(Tx,Ty) \\
&\leq \frac{3x+4}{2x+3} \\
&\leq \frac{x+1}{2x+3}4 \\
&\leq \frac{x+1}{2x+3}y+\frac{7}{8}[\frac{x+2}{2x+3}\min\{|x-1|,|x-\frac{y}{5}|\} \\
&\quad +\frac{x+1}{2x+3}|x-y|] \\
&= \frac{x+1}{2x+3}\delta(Tx,y)+\frac{7}{8}[\frac{x+2}{2x+3}d(x,Ty) \\
&\quad +\frac{x+1}{2x+3}d(x,y)].
\end{aligned}$$

Case 3: if $x,y\in[3,5]$ then for all $u\in Tx$, according to the following calculations
$\delta(Tx,u)\leq\delta(Tx,Ty)=\max\{|\frac{x}{5}-1|,|1-\frac{y}{5}|\}\leq\frac{2}{5}$,
$2<\delta(Tx,y)=\max\{|1-y|,|\frac{x}{5}-y|\}=|\frac{x}{5}-y|$,
$2<d(x,Ty)=\min\{|x-1|,|x-\frac{y}{5}|\}=|x-1|$ and $d(x,y)=|x-y|$,

$$\frac{8}{5}\leq|\frac{x}{5}-y|+\frac{7}{8}[|x-1|+|x-y|].$$

is hold. Therefore we get that

$$\begin{aligned}
\frac{3x+4}{2x+3}\delta(Tx,u) &\leq \frac{3x+4}{2x+3}\delta(Tx,Ty) \\
&= \frac{3x+4}{2x+3}\max\{|\frac{x}{5}-1|,|1-\frac{y}{5}|\} \\
&\leq \frac{x+1}{2x+3}\frac{8}{5} \\
&\leq \frac{x+1}{2x+3}|\frac{x}{5}-y|+\frac{7}{8}[\frac{x+2}{2x+3}\min\{|x-1|,|x-\frac{y}{5}|\} \\
&\quad +\frac{x+1}{2x+3}|x-y|] \\
&= \frac{x+1}{2x+3}\delta(Tx,y)+\frac{7}{8}[\frac{x+2}{2x+3}d(x,Ty) \\
&\quad +\frac{x+1}{2x+3}d(x,y)].
\end{aligned}$$

Hence T is generalized multivalued contractive hybrid mapping. However $T(2.9) = [0, 0.58]$, $T(3) = [0.60, 1]$ and $H(T(2.9), T(3)) = 0.60 > 0.1 = d(2.9, 3), T$ is neither contraction mapping nor nonexpansive mapping.
Now lets show that the multivalued Thianwan iteration scheme produce strongly convergent sequence to $0 \in F(T)$, the fixed point of T.
Let $x_0 \in X$ and $(\alpha_n) = (\frac{1}{2}), (\beta_n) = (\frac{1}{n+1})$.

For $n = 0$, $u_0 \in Tx_0 \subset [0, 1]$ so $y_0 = (1 - 1/1)x_0 + (1/1)u_0 \in [0, 1]$ so $x_1 = \frac{1}{2}y_0 + \frac{1}{2}v_0 \in [0, 1]$ where $v_0 \in Ty_0 \subset [0, \frac{y_0}{5}] \subset [0, 1/2]$.

For $n = 1$ we have that $u_1 \in Tx_1 = [0, \frac{x_1}{5}], y_1 = (1 - 1/2)x_1 + (1/2)u_1 \in [0, \frac{x_1}{2}]$, and $x_2 = \frac{1}{2}y_1 + \frac{1}{2}v_1 \in [0, \frac{x_1}{2}]$ where $v_1 \in Ty_1 \subset [0, \frac{y_0}{5}] \subset [0, \frac{x_1}{10}]$.

For $n = 2$ we have that $u_2 \in Tx_2 = [0, \frac{x_2}{5}], y2 = (1 - 1/3)x_2 + (1/3)u_2 \in [0, \frac{2x_2}{3}]$, and $x_3 = \frac{1}{2}y_2 + \frac{1}{2}v_2 \in [0, \frac{2x_2}{6}] \subset [0, \frac{x_2}{2}]$ where $v_2 \in Ty_2 \subset [0, \frac{y_1}{5}] \subset [0, \frac{2x_2}{15}]$.

Similarly, for all $n \geq 3$, we get that $u_n \in Tx_n = [0, \frac{x_n}{5}], y_n = ((1 - 1/(n+1)))x_2 + (1/(n+1))u_2 \in [0, \frac{nx_2}{n+1}]$, and $x_{n+1} = \frac{1}{2}y_n + \frac{1}{2}v_n \in [0, \frac{nx_2}{2(n+1)}] \subset [0, \frac{x_n}{2}]$ where $v_n \in Ty_n \subset [0, \frac{y_n}{5}] \subset [0, \frac{nx_n}{5(n+1)}]$, that is, $x_{n+1} \in [0, \frac{x_n}{2}] \subset [0, \frac{x_{n-1}}{2^2}] \subset ... \subset [0, \frac{x_0}{2^{n+1}}]$ and taking limit on n gives that $x_n \to 0$.

Hence the sequence (x_n) is convergent to the unique fixed point 0 of T.

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