

**A FIXED POINT THEOREM FOR HYBRID PAIRS
SATISFYING A NEW TYPE OF LIMIT RANGE
PROPERTY IN SYMMETRIC SPACES AND
APPLICATIONS**

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Abstract. A general fixed point theorem for two pairs of hybrid mappings satisfying a new type of limit range property in symmetric spaces is proved, generalizing Theorem 2 and Theorem 3 [7].

As application, we obtain new results for hybrid mappings satisfying contractive conditions of integral type and for mappings satisfying a φ - contractive conditions.

1. INTRODUCTION

Definition 1.1 ([28]). For a nonempty set X , a function $d : X \times X \rightarrow \mathbb{R}_+$ is called a symmetric on X if for all $x, y \in X$:

$(D_1) : d(x, y) = 0$ if and only if $x = y$,

$(D_2) : d(x, y) = d(y, x)$.

The pair (X, d) is called a symmetric space.

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For any $x \in X$ we define an open ball with respect to the corresponding topology τ_d on X , via $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$, where $x \in X$ and $\varepsilon > 0$.

We say that the sequence $\{x_n\}$ in X converges to a point $x \in X$, denoted $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$, if

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0$$

with respect to topology τ_d .

The symmetric d may not be continuous since d is not Hausdorff.

To compensate the missing of continuity of symmetric d , some additional axioms are add.

Definition 1.2. a) (H.E.)[2] $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ and $\lim_{n \rightarrow \infty} d(y_n, x) = 0$ imply $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.

b) (1c)[5] A symmetric d is said to be 1 - continuous, denoted (1c) if $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ implies $\lim_{n \rightarrow \infty} d(x_n, y) = d(x, y)$, whenever $\{x_n\}$ is a sequence in X and $x, y \in X$.

In 2011, Sintunavarat and Kumam [27] introduced the notion of common limit range property for a pair of mappings in metric spaces. Imdad et al. [7] introduced the notion of joint common limit range property for hybrid mappings. The study of fixed points for hybrid pairs of mappings satisfying a new common limit range property is initiated [21]. The study of fixed points for mappings satisfying a contractive condition of integral type is initiated by Branciari in [3]. It is proved in [20] that the study of fixed points for mappings satisfying contractive conditions of integral type is reduced to the study of fixed points for mappings involving altering distances.

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [13], [14] and in other papers. Recently, this method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, Hilbert spaces, ultra - metric spaces, compact metric spaces, in two and three metric spaces, for single - valued mappings, hybrid pairs of mappings and set - valued mappings.

The purpose of this paper is to prove a general fixed point theorem for two pairs of hybrid mappings satisfying new type of common limit range property in symmetric spaces, generalizing the results from [7].

As applications, we obtain new results for hybrid mappings satisfying contractive conditions of integral type and ϕ - contractive conditions.

2. PRELIMINARIES

Let (X, d) be a metric space. We denote by $CL(X)$ the set of all nonempty closed sets of X and by H the Hausdorff - Pompeiu metric, i.e.

$$H(A, B) = \max\{\sup_{a \in A}\{d(a, B)\}, \sup_{b \in B}\{d(b, A)\}\},$$

where $A, B \in CL(X)$ and

$$d(x, A) = \inf_{y \in A}\{d(x, y)\}.$$

Definition 2.1. Let $f : X \rightarrow X$ and $F : X \rightarrow CL(X)$ be.

1) A point $x \in X$ is said to be a coincidence point of f and F if $fx \in Fx$.

The set of all coincidence points of f and F is denoted by $\mathcal{C}(f, F)$.

2) A point $x \in X$ is a common fixed point of f and F if $x = fx \in Fx$.

Definition 2.2 ([26]). $f : X \rightarrow X$ and $F : X \rightarrow CL(X)$ are said to be weakly commuting if $ffx \in Ffx$, for all $x \in X$.

Definition 2.3 ([9], [13]). Let $f : X \rightarrow X$ and $F : X \rightarrow CL(X)$ be. f and F are said to be compatible of type N if $x \in \mathcal{C}(f, F)$ implies $ffx \in Ffx$.

Definition 2.4 ([6]). Let (X, d) be a metric space and Y be a nonempty subset of X with $f : X \rightarrow X$ and $F : Y \rightarrow 2^X$. The pair of hybrid mappings (f, F) is said to be quasi - coincidentally commuting if $fx \in Fx$ for $x \in Y$ implies $fFx \subset Ffx$.

Remark 2.5. If (f, F) is quasi - coincidentally commuting, then (f, F) is compatible of type N because if $fx \in Fx$, then $fFx \subset Ffx$.

Example 2.6 ([22]). Let $X = [0, 1]$ be a metric space with the usual metric $fx = 1 - x$, $Fx = \left[0, \frac{1}{2}\right]$. Then $\mathcal{C}(f, F) = \left\{\frac{1}{2}\right\}$, $fF = \left[\frac{1}{2}, 1\right] \not\subset Ff\frac{1}{2} = \left[0, \frac{1}{2}\right]$ and $ff\frac{1}{2} \in Ff\frac{1}{2}$.

Definition 2.7 ([6]). Let $f : Y \subset X \rightarrow X$ and $F : Y \rightarrow CL(X)$. The mapping f is said to be coincidentally idempotent with respect to F if $fx \in Fx$ with $fx \in Y$ implies $fx = ffx$, that f is idempotent at coincidence point of (f, F) .

Definition 2.8 ([7]). Let $f : Y \subset X \rightarrow X$ and $F : Y \subset X \rightarrow CL(X)$. (f, F) has a common limit in the range property with respect to f , denoted CLR_g - property if there exists a sequence $\{x_n\}$ in Y such that

$$\lim_{n \rightarrow \infty} f x_n = f u \in A = \lim_{n \rightarrow \infty} F x_n$$

for some $u \in Y$ and $A \in CL(X)$.

The following theorem is proved in [7].

Theorem 2.9 ([7]). Let (X, d) be a symmetric space, where d satisfies (1c) and (H.E.), where Y is an arbitrary nonempty subset with $F : Y \rightarrow CL(X)$ and $g : Y \rightarrow X$. Suppose that:

1) the hybrid pair (F, g) enjoys the CLR_g - property, and

2)

(2.1)

$$H(Fx, Fy) < \max \left\{ \begin{array}{l} d(gx, gy), \frac{k}{2} [d(gx, Fx) + d(gy, Fy)], \\ \frac{k}{2} [d(gy, Fx) + d(gx, Fy)] \end{array} \right\},$$

for all $x \neq y \in Y$ and $k \in [0, 2)$.

Then F and g have a coincidence point.

In particular, if $Y \subset X$ and the pair (f, F) is quasi - coincidentally idempotent with respect to F , then F and g have a common fixed point.

We introduce a new type of common limit range property.

Definition 2.10 ([8]). Let (X, d) be a metric space and Y be an arbitrary nonempty subset of X with $F, G : Y \rightarrow CL(X)$ and $f, g : Y \rightarrow X$. Then, the pairs (F, f) and (G, g) are said to have joint common limit range property with respect to T , denoted $JCLR$ - property, if there exist two convergent sequences $\{x_n\}$ and $\{y_n\}$ in X and $A, B \in CL(X)$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} F x_n = A, \lim_{n \rightarrow \infty} G y_n = B \in CB(X), \\ \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} g x_n = t \in A \cap B \cap f(Y) \cap g(Y), \end{aligned}$$

i.e., there exist u and v in Y such that $t \in f(u) \cap g(v) \subset A \cap B$.

Definition 2.11 ([21]). Let (X, d) be a metric space, $A : X \rightarrow CL(X)$ and $S, T : X \rightarrow X$. Then, the pair (A, S) satisfy common range property with respect to T , denoted $CLR_{(A,S)T}$ - property, if there exists a convergent sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} S x_n = \lim_{n \rightarrow \infty} A x_n = t \in D \text{ with } D \in CL(X) \text{ and } t \in S(X) \cap T(X).$$

Example 2.12. Let $X = [0, \infty)$ be a metric space with the usual metric, $Ax = [\frac{1}{4}, 1]$, $Sx = \frac{x^2 + 1}{2}$, $Tx = x + \frac{1}{4}$. Then $S(X) = [\frac{1}{2}, \infty)$, $T(X) = [\frac{1}{4}, \infty)$, $S(X) \cap T(X) = [\frac{1}{2}, \infty)$. Let $\{x_n\}$ be a convergent sequence in X such that $\lim_{n \rightarrow \infty} x_n = 0$. Then

$$\lim_{n \rightarrow \infty} Sx_n = t = \frac{1}{2} \in \left[\frac{1}{4}, 1\right] = \lim_{n \rightarrow \infty} Ax_n = D.$$

Remark 2.13. a) If (A, S) and (B, T) satisfy JCLR - property, then (A, S) and T satisfy $CLR_{(A, S)T}$ - property.

b) For symmetric spaces, the definition is similar.

Definition 2.14 ([10]). An altering distance is a function $\psi : [0, \infty) \rightarrow [0, \infty)$ such that:

- (ψ_1) : ψ is increasing and continuous,
- (ψ_2) : $\psi(t) = 0$ if and only if $t = 0$.

Fixed point problems involving altering distances have been studied in [24] and in other papers.

3. IMPLICIT RELATIONS

Several fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit function [13], [14].

In [15] - [17] and in other papers, the study of fixed points for hybrid mappings satisfying implicit relations is initiated. A general fixed point theorem for pairs of hybrid mappings with common limit range satisfying implicit relation is proved in [4].

A new form of implicit relation is introduced in [1].

Definition 3.1. Let \mathcal{F}_S be the set of all lower semi - continuous functions $F : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ satisfying the following conditions:

- (F_1) : F is nondecreasing in variable t_1 ,
- (F_2) : $F(t, 0, 0, t, t, 0) > 0, \forall t > 0$,
- (F_3) : $F(t, 0, t, 0, 0, t) > 0, \forall t > 0$.

Example 3.2. $F(t_1, \dots, t_6) = t_1 - k \max \left\{ t_2, \frac{k}{2} (t_3 + t_4), \frac{k}{2} (t_5 + t_6) \right\}$, where $k \in [0, 2)$.

Example 3.3. $F(t_1, \dots, t_6) = t_1 - k \max \{ t_2, k (t_3 + t_4), k (t_5 + t_6) \}$, where $k \in [0, 1)$.

Example 3.4. $F(t_1, \dots, t_6) = t_1 - \max \left\{ t_2, t_3, t_4, \frac{t_5 + t_6}{2} \right\}$, where $k \in [0, 1)$.

Example 3.5. $F(t_1, \dots, t_6) = t_1 - at_2 - b \max\{t_3, t_4\} - c \max\{t_3 + t_4, t_5 + t_6\}$, where $a, b, c \geq 0$ and $a + b + 2c < 1$.

Example 3.6. $F(t_1, \dots, t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$, where $\alpha \in (0, 1)$, $a, b \geq 0$ and $a + b < 1$.

Example 3.7. $F(t_1, \dots, t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a, b, c, d, e \geq 0$, $c + d < 1$ and $b + c < 1$.

Example 3.8. $F(t_1, \dots, t_6) = t_1 - at_2 - \frac{b(t_5 + t_6)}{1 + t_3 + t_4}$, where $a, b \geq 0$ and $a + 2b < 1$.

Example 3.9. $F(t_1, \dots, t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, where $a, b, c \geq 0$ and $a + b + c < 1$.

For other examples see [1], [4].

4. MAIN RESULTS

Theorem 4.1. Let (X, d) be a symmetric space, where d satisfies (1c) and (H.E.), Y be an arbitrary subset of X , $A, B : Y \rightarrow CL(X)$ and $S, T : Y \rightarrow Y$. Suppose that

$$(4.1) \quad F \left(\begin{array}{l} \psi(H(Ax, By)), \psi(d(Sx, Ty)), \psi(d(Sx, Ax)), \\ \psi(d(Ty, By)), \psi(d(Sx, By)), \psi(d(Ax, Ty)) \end{array} \right) \leq 0$$

for all $x, y \in X$, some $F \in \mathcal{F}_S$ and ψ is an altering distance.

If (A, S) and T satisfy $CLR_{(A,S)T}$ - property in Y , then

- 1) $\mathcal{C}(A, S) \neq \emptyset$,
- 2) $\mathcal{C}(B, T) \neq \emptyset$.

Moreover, if $Y = X$, (A, S) and (B, T) are compatible of type N and

- a) S is coincidentally idempotent with respect to A , then A and S have a common fixed point,
- b) T is coincidentally idempotent with respect to B , then B and T have a common fixed point,
- c) the hypotheses of a) and b) hold, then A, B, S and T have a common fixed point.

Proof. Since (A, S) and T satisfy $CLR_{(A,S)T}$ - property, there exists a sequence $\{x_n\}$ in Y such that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t \text{ with } D = \lim_{n \rightarrow \infty} Ax_n,$$

$$D \in CL(Y) \text{ and } t \in S(Y) \cap T(Y).$$

Since $t \in T(Y)$, there exists $u \in Y$ such that $t = Tu$.

By (4.1) for $x = x_n$ and $y = u$ we obtain

$$(4.2) \quad F \left(\begin{array}{c} \psi(H(Ax_n, Bu)), \psi(d(Sx_n, Tu)), \psi(d(Sx_n, Ax_n)), \\ \psi(d(Tu, Bu)), \psi(d(Sx_n, Bu)), \psi(d(Ax_n, Tu)) \end{array} \right) \leq 0.$$

Since d satisfies (1c) and (H.E.), letting n tend to infinity in (4.2) we obtain

$$(4.3) \quad F(\psi(H(D, Bu)), 0, 0, \psi(d(t, Bu)), \psi(d(t, Bu)), 0) \leq 0.$$

Since $t \in D$ and ψ is an altering distance, we have $\psi(d(t, Bu)) \leq \psi(H(D, Bu))$. By (F_1) and (4.2) we obtain

$$F(\psi(d(t, Bu)), 0, 0, \psi(d(t, Bu)), \psi(d(t, Bu)), 0) \leq 0,$$

a contradiction of (F_2) if $\psi(d(t, Bu)) > 0$. Hence, $\psi(d(t, Bu)) = 0$, which implies $d(t, Bu) = 0$, i.e. $t = Tu \in Bu$ and $\mathcal{C}(T, B) \neq \emptyset$.

On the other hand, $t \in S(Y)$. Hence, there exists $v \in S(Y)$ such that $t = Sv$.

By (4.1) for $x = v$ and $y = u$ we obtain

$$(4.4) \quad F \left(\begin{array}{c} \psi(H(Av, Bu)), \psi(d(Sv, Tu)), \psi(d(Sv, Av)), \\ \psi(d(Tu, Bu)), \psi(d(Sv, Bu)), \psi(d(Av, Tu)) \end{array} \right) \leq 0.$$

Since $t \in Bu$ and ψ is an altering distance, then $\psi(d(t, Av)) \leq \psi(H(Av, Bu))$. By (4.4) we obtain

$$F(\psi(d(Av, t)), 0, \psi(d(t, Av)), 0, 0, \psi(d(u, Av))) \leq 0,$$

a contradiction of (F_3) if $\psi(d(t, Av)) > 0$. Hence, $\psi(d(t, Av)) = 0$ which implies $t = Sv \in Av$ and $\mathcal{C}(A, S) \neq \emptyset$.

a) If S is coincidentally idempotent with respect to A , then $t = Sv = SSv = St$ and t is a fixed point of S . Since (A, S) is compatible of type N , then $t = Sv = SSv \in SAV \subset ASv = At$ and t is a fixed point of A . Hence t is a fixed point of A and S .

b) If (B, T) is compatible of type N and T is coincidentally idempotent with respect to B , then as in the proof of a) it follows that t is a common fixed point of B and T .

c) If the hypotheses of a) and b) hold, then t is a common fixed point of A, B, S and T . \square

If $\psi(t) = t$ by Theorem 4.1 we obtain

Theorem 4.2. *Let (X, d) be a symmetric space, where d satisfies (1c) and (H.E.), Y an arbitrary subset of X , $A, B : Y \rightarrow CL(X)$ and $S, T : Y \rightarrow Y$ such that*

$$(4.5) \quad F \left(\begin{array}{l} H(Ax, By), d(Sx, Ty), d(Sx, Ax), \\ d(Ty, By), d(Sx, By), d(Ax, Ty) \end{array} \right) \leq 0$$

for all $x, y \in X$ and some $F \in \mathcal{F}_S$.

If (A, S) and T satisfy $CLR_{(A,S)T}$ - property in Y , then

$$1) \quad \mathcal{C}(A, S) \neq \emptyset,$$

$$2) \quad \mathcal{C}(B, T) \neq \emptyset.$$

Moreover, if (A, S) and (B, T) are compatible of type N and

a) S is coincidentally idempotent with respect to A , then A and S have a common fixed point,

b) T is coincidentally idempotent with respect to B , then B and T have a common fixed point,

c) the hypotheses of a) and b) hold, then A, B, S and T have a common fixed point.

If $A = B$ and $S = T$ by Theorem 4.2 we obtain

Theorem 4.3. *Let (X, d) be a symmetric space, where d satisfies (1c) and (H.E.), Y an arbitrary subset of X , $A : Y \rightarrow CL(X)$ and $S : Y \rightarrow Y$ such that*

$$(4.6) \quad F \left(\begin{array}{l} H(Ax, Ay), d(Sx, Sy), d(Sx, Ax), \\ d(Sy, Ay), d(Sx, Ay), d(Ax, Sy) \end{array} \right) \leq 0$$

for all $x \neq y$ and some $F \in \mathcal{F}_S$.

If (A, S) satisfy CLR_S - property, then A and S have a coincidence point. If A and S are compatible of type N and S is coincidentally idempotent with respect to A , then A and S have a common fixed point.

Remark 4.4. 1) *By Theorem 4.3, Example 3.2 and Remark 2.5 we obtain an Corollary which generalize Theorem 2.9.*

2) *By Theorem 4.3 and Examples 3.3 - 3.9 we obtain new particular results.*

5. APPLICATIONS

5.1. Coincidence and common fixed points for hybrid mappings satisfying implicit relations of integral type. In [3], Branciari established the following theorem, which opened the way to the study of fixed points for mappings satisfying a contractive/extensive condition of integral type.

Theorem 5.1 ([3]). *Let (X, d) be a metric space, $c \in (0, 1)$ and $f : X \rightarrow X$ a mapping such that for all $x, y \in X$*

$$\int_0^{d(fx, fy)} h(t) dt \leq c \int_0^{d(x, y)} h(t) dt,$$

where $h : [0, \infty) \rightarrow [0, \infty)$ is a Lebesgue measurable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, \infty)$ such that for $\varepsilon > 0$, $\int_0^\varepsilon h(t) dt > 0$. Then, f has a unique fixed point $z \in X$ such that for all $x \in X$, $z = \lim_{n \rightarrow \infty} f^n x$.

Some fixed point results for mappings satisfying contractive conditions of integral type in symmetric spaces are proved in [17], [23] and in other papers.

Some fixed point results for multivalued mappings using altering distance are proved in [18], [21] and in other papers.

Lemma 5.2 ([20]). *Let $h : [0, \infty) \rightarrow [0, \infty)$ be as in Theorem 5.1. Then $\psi(t) = \int_0^t h(x) dx$ is an altering distance.*

Theorem 5.3. *Let (X, d) be a symmetric space, where d satisfies (1c) and (H.E.) and Y is an arbitrary nonempty subset of X , $A, B : Y \rightarrow CL(X)$ and $S, T : Y \rightarrow X$. Suppose that*

$$(5.1) \quad F \left(\begin{array}{l} \int_0^{H(Ax, By)} h(t) dt, \int_0^{d(Sx, Ty)} h(t) dt, \int_0^{d(Sx, Ax)} h(t) dt, \\ \int_0^{d(Ty, By)} h(t) dt, \int_0^{d(Sx, By)} h(t) dt, \int_0^{d(Ax, Ty)} h(t) dt \end{array} \right) \leq 0$$

for all $x, y \in Y$, $h(t)$ as in Theorem 5.1 and some $F \in \mathcal{F}_S$.

If (A, S) and T satisfy $CLR_{(A, S)T}$ -property in Y , then

- 1) $\mathcal{C}(A, S) \neq \emptyset$,
- 2) $\mathcal{C}(B, T) \neq \emptyset$.

Moreover, if $X = Y$, (A, S) and (B, T) are compatible of type N and

- a) *S is coincidentally idempotent with respect to A , then A and S have a common fixed point,*
- b) *T is coincidentally idempotent with respect to B , then B and T have a common fixed point,*

c) the hypotheses of a) and b) hold, then A, B, S and T have a common fixed point.

Proof. Let $\psi(t)$ as in Lemma 5.2. Then

$$\begin{aligned}\psi(H(Ax, By)) &= \int_0^{H(Ax, By)} h(t) dt, \psi(d(Sx, Ty)) = \int_0^{d(Sx, Ty)} h(t) dt, \\ \psi(d(Sx, Ax)) &= \int_0^{d(Sx, Ax)} h(t) dt, \psi(d(Ty, By)) = \int_0^{d(Ty, By)} h(t) dt, \\ \psi(d(Sx, By)) &= \int_0^{d(Sx, By)} h(t) dt, \psi(d(Ax, Ty)) = \int_0^{d(Ax, Ty)} h(t) dt.\end{aligned}$$

Then, by (5.1) we obtain

$$F \left(\begin{array}{c} \psi(H(Ax, By)), \psi(d(Sx, Ty)), \psi(d(Sx, Ax)), \\ \psi(d(Ty, By)), \psi(d(Sx, By)), \psi(d(Ax, Ty)) \end{array} \right) \leq 0,$$

which is inequality (4.1). Hence, the conditions of Theorem 4.1 are satisfied and the proof of Theorem 5.3 follows by Theorem 4.1. \square

By Theorem 5.3 and Example 3.2 we obtain

Theorem 5.4. Let (X, d) be a symmetric space, where d satisfies (1c) and (H.E.) and Y is an arbitrary nonempty subset of X , $A, B : Y \rightarrow CL(X)$ and $S, T : Y \rightarrow X$. Suppose that

$$\int_0^{H(Ax, By)} h(t) dt \leq \max \left\{ \begin{array}{l} \int_0^{d(Sx, Ty)} h(t) dt, \\ \frac{k}{2} \left[\int_0^{d(Sx, Ax)} h(t) dt + \int_0^{d(Ty, By)} h(t) dt \right], \\ \frac{k}{2} \left| \int_0^{d(Sx, By)} h(t) dt + \int_0^{d(Ax, Ty)} h(t) dt \right| \end{array} \right\}$$

where $h(t)$ is as in Theorem 5.1 and $k \in [0, 2)$.

If (A, S) and T satisfy $CLR_{(A, S)T}$ - property, then

- 1) $\mathcal{C}(A, S) \neq \emptyset$,
- 2) $\mathcal{C}(B, T) \neq \emptyset$.

Moreover, if $X = Y$ and (A, S) and (B, T) are compatible of type N and

- a) S is coincidentally idempotent with respect to A , then A and S have a common fixed point,
- b) T is coincidentally idempotent with respect to B , then B and T have a common fixed point,
- c) the hypotheses of a) and b) are true, then A, B, S and T have a common fixed point.

If $A = B$ and $S = T$ by Theorem 4.2 we obtain

Corollary 5.5. Let (X, d) be a symmetric space, where d satisfies (1c) and (H.E.) and Y is an arbitrary nonempty subset of X , $A : Y \rightarrow$

$CL(X)$ and $S : Y \rightarrow X$. Suppose that

$$\int_0^{H(Ax, Ay)} h(t) dt \leq \max \left\{ \begin{array}{l} \int_0^{d(Sx, Sy)} h(t) dt, \\ \frac{k}{2} \left[\int_0^{d(Sx, Ax)} h(t) dt + \int_0^{d(Sy, Ay)} h(t) dt \right], \\ \frac{k}{2} \left[\int_0^{d(Sx, Ay)} h(t) dt + \int_0^{d(Ax, Sy)} h(t) dt \right] \end{array} \right\}$$

for all $x \neq y$, $k \in [0, 2)$ and $h(t)$ is as in Theorem 5.1.

If A and S satisfy CLR_S - property, then A and S have a coincidence point.

Moreover, if $Y = X$, A and S are compatible of type N and S is coincidentally idempotent with respect to A , then A and S have a common fixed point.

Remark 5.6. By Theorem 5.4 and Examples 3.3 - 3.9 we obtain new particular results.

5.2. Coincidence and common fixed points for φ - hybrid mappings. As in [11], let Φ be the set of all nondecreasing upper semi-continuous functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that:

- 1) $\varphi(t) < t$ for all $t > 0$,
- 2) $\varphi(0) = 0$.

The following functions $F \in \mathcal{F}_S$.

Example 5.7. $F(t_1, \dots, t_6) = t_1 - \varphi \left(\max \left\{ t_2, \frac{k}{2} (t_3 + t_4), \frac{k}{2} (t_5 + t_6) \right\} \right)$, where $k \in [0, 2]$.

Example 5.8. $F(t_1, \dots, t_6) = t_1 - \varphi \left(\max \left\{ t_2, t_3, t_4, \frac{t_5 + t_6}{2} \right\} \right)$.

Example 5.9. $F(t_1, \dots, t_6) = t_1 - \varphi(\max\{t_2, t_3, t_4, t_5, t_6\})$.

Example 5.10. $F(t_1, \dots, t_6) = t_1 - \varphi(\max\{t_2, \sqrt{t_3 t_4}, \sqrt{t_3 t_5}, \sqrt{t_4 t_6}, \sqrt{t_5 t_6}\})$.

Example 5.11. $F(t_1, \dots, t_6) = t_1 - \varphi(at_2 + bt_3 + ct_4 + dt_5 + et_6)$, where $a, b, c, d, e \geq 0$ and $a + b + c + d + e \leq 1$.

Example 5.12. $F(t_1, \dots, t_6) = t_1 - \varphi \left(at_2 + \frac{b\sqrt{t_5 t_6}}{1 + t_3 + t_4} \right)$, where $a, b \geq 0$ and $a + b \leq 1$.

Example 5.13. $F(t_1, \dots, t_6) = t_1 - \varphi \left(at_2 + b \max \left\{ \frac{2t_4 + t_5}{3}, \frac{2t_4 + t_6}{3} \right\} \right)$, where $a, b \geq 0$ and $a + b \leq 1$.

By Theorem 4.2 and Example 5.7 we obtain

Theorem 5.14. *Let (X, d) be a symmetric space, where d satisfies (1c) and (H.E.) and Y is an arbitrary nonempty subset of X , $A, B : Y \rightarrow CL(X)$ and $S, T : Y \rightarrow X$. Suppose that*

$$H(Ax, By) \leq \varphi \left(\max \left\{ \begin{array}{l} d(Sx, Ty), \\ \frac{k}{2} [d(Sx, Ax) + d(Ty, By)], \\ \frac{k}{2} [d(Sx, By) + d(Ax, Ty)] \end{array} \right\} \right),$$

where $\varphi \in \Phi$.

If (A, S) and T satisfy $CLR_{(A,S)T}$ - property, then

- 1) $\mathcal{C}(A, S) \neq \emptyset$,
- 2) $\mathcal{C}(B, T) \neq \emptyset$.

Moreover, if (A, S) and (B, T) are compatible of type N and

- a) if S is coincidentally idempotent with respect to A , then A and S have a common fixed point,
- b) if T is coincidentally idempotent with respect to B , then B and T have a common fixed point,
- c) if the hypotheses of a) and b) are true, then A, B, S and T have a common fixed point.

If $A = B$ and $S = T$ we obtain a generalization of Theorem 4.3.

Corollary 5.15. *Let (X, d) be a symmetric space, where d satisfies (1c) and (H.E.) and Y is an arbitrary nonempty subset of X , $A : Y \rightarrow CL(X)$ and $S : Y \rightarrow X$. Suppose that*

$$H(Ax, Ay) \leq \varphi \left(\max \left\{ \begin{array}{l} d(Sx, Sy), \frac{k}{2} [d(Sx, Ax) + d(Sy, Ay)], \\ \frac{k}{2} [d(Sx, Ay) + d(Ax, Sy)] \end{array} \right\} \right)$$

for all $x \neq y$, $k \in [0, 2)$ and $\varphi \in \Phi$.

If A and S satisfy CLR_S - property, then A and S have a coincidence point.

Moreover, if $Y = X$, A and S are compatible of type N and S is coincidentally idempotent with respect to A , then A and S have a common fixed point.

Remark 5.16. *By Theorem 4.2 and Examples 5.8 - 5.13 we obtain new particular results.*

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