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FUZZY PRE γ-CONTINUOUS AND ALMOST PRE γ-CONTINUOUS FUNCTIONS

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Abstract. In this paper we first introduce a new type of fuzzy open-like set, viz., fuzzy pre-γ-open set, the collection of which is strictly larger than that of fuzzy open set. Afterwards, two new types of fuzzy continuous-like functions, viz., fuzzy pre-γ-continuous and fuzzy almost pre-γ-continuous functions are introduced and studied. It is shown that fuzzy almost pre-γ-continuous function is fuzzy pre-γ-continuous and the converse is true only in fuzzy pre-γ-regular space.

1. Introduction

In [12], L.A. Zadeh introduced fuzzy set. Afterwards many mathematicians have engaged themselves to introduce different types of fuzzy sets. In this context we have to mention [4, 5]. In [3], fuzzy γ-open set is introduced and studied. Taking this definition as a basic tool, here we introduce fuzzy pre-γ-open set which lies in between fuzzy open and fuzzy preopen set [9]. Also it is shown that fuzzy almost pre-γ-continuous function is fuzzy almost continuous [8] but not conversely.

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2. **Preliminaries**

Throughout this paper, (X, τ) or simply by X we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [6]. In 1965, L.A. Zadeh introduced fuzzy set [12] A which is a function from a non-empty set X into the closed interval $I = [0, 1]$, i.e., $A \in I^X$. The support [12] of a fuzzy set A, denoted by $suppA$ or A_0 and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [12] of a fuzzy set A in an fts X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [12] while AqB means A is quasi-coincident (q-coincident, for short) [10] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For a fuzzy set A, clA and $intA$ will stand for fuzzy closure [6] and fuzzy interior [6] respectively.

A fuzzy set A in an fts (X, τ) is called fuzzy regular open [1] (resp., fuzzy semiopen [1], fuzzy preopen [9], fuzzy γ-open [3]) if $A = intclA$ (resp., $A \leq clintA$, $A \leq intclA$, $A \leq (intclA) \bigcup (clintA)$). The complement of a fuzzy semiopen (resp., fuzzy preopen, fuzzy γ-open) set is called fuzzy semiclosed [1] (resp., fuzzy preclosed [9], fuzzy γ-closed [3]). The union of all fuzzy γ-open sets contained in a fuzzy set A is called fuzzy γ-interior [3] of A, denoted by $\gamma intA$. The intersection of all fuzzy γ-closed (resp., fuzzy semiclosed, fuzzy preclosed) sets containing a fuzzy set A in an fts X is called fuzzy γ-closure [3] (resp., fuzzy semiclosure [1], fuzzy preclosure [9]) of A, denoted by γclA [3] (resp., $sclA$ [1], $pclA$ [9]). A fuzzy set A in X is called a fuzzy neighbourhood (nbd, for short) [10] of a fuzzy point x_t if there exists a fuzzy open set G in X such that $x_t \in G \leq A$. If, in addition, A is fuzzy open, then A is called fuzzy open nbd of x_t. A fuzzy set A is said to be a fuzzy quasi neighbourhood (q-nbd, for short) of a fuzzy point x_t in an fts X if there is a fuzzy open set U in X such that $x_t qU \leq A$. If, in addition, A is fuzzy open (resp., fuzzy γ-open), then A is called a fuzzy open [10] (resp., fuzzy γ-open [3]) q-nbd of x_t. For a fuzzy set A and a fuzzy point x_α in an fts X, $x_\alpha \in \gamma clA$ iff every fuzzy γ-open q-nbd U of x_α, UqA [3]. The collection of all fuzzy γ-open (resp., fuzzy preopen, fuzzy semiopen) sets in an fts X is denoted by $F\gamma O(X)$ (resp., $FSO(X)$, $FPO(X)$) and that of fuzzy

γ-closed (resp., fuzzy semiclosed, fuzzy preclosed) sets in an fts X is denoted by $F\gamma C(X)$ (resp., $FSC(X)$, $FPC(X)$).

3. **Fuzzy Pre-γ-Open Set : Some Properties**

In this section we first introduce a new type of fuzzy open-like set, viz., fuzzy pre-γ-open set. Then fuzzy pre-γ-closure operator is introduced which is an idempotent operator. Some important properties of fuzzy pre-γ-open set are established here. Lastly, mutual relationship of this newly defined fuzzy set with the sets defined in [4, 5] are established.

Definition 3.1. A fuzzy set A in an fts (X, τ) is called fuzzy pre-γ-open if $A \leq int(\gamma clA)$.
The complement of fuzzy pre-γ-open set is called fuzzy pre-γ-closed set.
The collection of fuzzy pre-γ-open (resp., fuzzy pre-γ-closed) sets in an fts (X, τ) is denoted by $FP\gamma O(X)$ (resp., $FP\gamma C(X)$).
The union (resp., intersection) of all fuzzy pre-γ-open (resp., fuzzy pre-γ-closed) sets contained in (containing) a fuzzy set A is called fuzzy pre-γ-interior (resp., fuzzy pre-γ-closure) of A, denoted by $p\gamma intA$ (resp., $p\gamma clA$).

Definition 3.2. A fuzzy set A in an fts (X, τ) is called fuzzy pre-γ-neighbourhood (fuzzy pre-γ-nbd, for short) of a fuzzy point x_α if there exists a fuzzy pre-γ-open set U in X such that $x_\alpha \leq U \leq A$. If, in addition, A is fuzzy pre-γ-open, then A is called fuzzy pre-γ-open nbd of x_α.

Definition 3.3. A fuzzy set A in an fts (X, τ) is called fuzzy pre-γ quasi neighbourhood (fuzzy pre-γ-q-nbd, for short) of a fuzzy point x_α if there exists a fuzzy pre-γ-open set U in X such that $x_\alpha qU \leq A$. If, in addition, A is fuzzy pre-γ-open, then A is called fuzzy pre-γ-open q-nbd of x_α.

Remark 3.4. It is clear from definitions that fuzzy pre-γ-open set is fuzzy preopen as $\gamma clA \leq clA$ for any fuzzy set A in an fts X. So we have the following diagram :
fuzzy open set $\Rightarrow$ fuzzy pre-γ-open set $\Rightarrow$ fuzzy preopen set
But the converses are not true, as it seen from the following examples.

Example 3.5. Fuzzy preopen set $\not\Rightarrow$ fuzzy pre-γ-open set
Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.55$. Then (X,τ) is an fts. Consider the fuzzy set C, defined by $C(a) = 0.5, C(b) = 0.7$. Now $intclC = 1_X > C \Rightarrow C \in FPO(X)$. But $int\gamma clC = intC = B \not\geq C \Rightarrow C \notin FP\gamma O(X)$.

Example 3.6. Fuzzy pre-γ-open set $\not\Rightarrow$ fuzzy open set
Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0, B(a) = 0.7, B(b) = 0$. Then (X,τ) is an fts. Consider the fuzzy set C defined by $C(a) = 0.8, C(b) = 0.5$. The clearly $C \in FP\gamma O(X)$ as $\gamma clC = 1_X$. But $C \notin \tau$.

Theorem 3.7. Let $A, B \in I^X$. Then the following statements hold :
(a) $int(\gamma cl0_X) = 0_X$,
(b) $int(\gamma cl1_X) = 1_X$,
(c) $A \leq B \Rightarrow int(\gamma clA) \leq int(\gamma clB)$,
(d) $A, B \in FP\gamma O(X) \Rightarrow A \bigcup B \in FP\gamma O(X)$.
Proof. (a), (b) and (c) are obvious.
(d) Let $A, B \in FP\gamma O(X)$. Then $int(\gamma cl(A \bigcup B)) = int(\gamma clA \bigcup \gamma clB) \geq int(\gamma clA) \bigcup int(\gamma clB) \geq A \bigcup B$ (as $A, B \in FP\gamma O(X)) \Rightarrow A \bigcup B \in FP\gamma O(X)$.

Result 3.8. From Theorem 3.7 (d), we can conclude that intersection of any two fuzzy pre-γ-closed sets is also so. But intersection of two fuzzy pre-γ-open sets need not be so, as it seen from the following example.

Example 3.9. Consider Example 3.6. Consider two fuzzy sets C and D defined by $C(a) = 0.8, C(b) = 0.5, D(a) = 0.5, D(b) = 0.7$. Then $\gamma clC = \gamma clD = 1_X \Rightarrow C, D \in FP\gamma O(X)$. Let $E = C \bigcap D$. Then $E(a) = E(b) = 0.5$. Now $int(\gamma clE) = intE = A \not\geq E \Rightarrow E \notin FP\gamma O(X)$.

Theorem 3.10. Let A be any fuzzy set in an fts X and x_α, any fuzzy point X. Then $x_\alpha \in p\gamma clA$ iff every fuzzy pre-γ-open q-nbd U of x_α, UqA.
Proof. Let $x_\alpha \in p\gamma clA$ for any fuzzy set A in an fts (X,τ). Let $U \in FP\gamma O(X)$ with $x_\alpha qU$. Then $U(x) + \alpha > 1 \Rightarrow x_\alpha \not\leq 1_X \setminus U \in FP\gamma C(X)$. Then by definition, $A \not\leq 1_X \setminus U \Rightarrow$ there exists $y \in X$

such that $A(y) > 1 - U(y) \Rightarrow A(y) + U(y) > 1 \Rightarrow UqA$.
Conversely, let the given condition hold. Let $U \in FP\gamma C(X)$ with $A \leq U$ ... (1). We have to show that $x_\alpha \in U$, i.e., $U(x) \geq \alpha$. If possible, let $U(x) < \alpha$. Then $1 - U(x) > 1 - \alpha \Rightarrow x_\alpha q(1_X \setminus U)$ where $1_X \setminus U \in FP\gamma O(X)$. By hypothesis, $(1_X \setminus U)qA \Rightarrow$ there exists $y \in X$ such that $1 - U(y) + A(y) > 1 \Rightarrow A(y) > U(y)$, contradicts (1).

Theorem 3.11. $p\gamma cl(p\gamma clA) = p\gamma clA$ for any fuzzy set A in an fts (X, τ).
Proof. Let $A \in I^X$. Then $A \leq p\gamma clA \Rightarrow p\gamma clA \leq p\gamma cl(p\gamma clA)$ ... (1). Conversely, let $x_\alpha \in p\gamma cl(p\gamma clA)$. If possible, let $x_\alpha \notin p\gamma clA$. Then there exists $U \in FP\gamma O(X)$, $x_\alpha qU$, $U \not{q} A$ ... (2). But as $x_\alpha \in p\gamma cl(p\gamma clA)$, $Uq(p\gamma clA) \Rightarrow$ there exists $y \in X$ such that $U(y) + (p\gamma clA)(y) > 1 \Rightarrow U(y) + t > 1$ where $t = (p\gamma clA)(y)$. Then $y_t \in p\gamma clA$ and $y_t qU$ where $U \in FP\gamma O(X)$. Then by definition, UqA, contradicts (2). So $p\gamma cl(p\gamma clA) \leq p\gamma clA$ ... (3). Combining (1) and (3), we get the result.

Let us recall the following two definitions from [4, 5] for ready references.

Definition 3.12 [5]. A fuzzy set A in an fts (X, τ) is called fuzzy s^*-open if $A \leq int(sclA)$.

Definition 3.13 [4]. A fuzzy set A in an fts (X, τ) is called fuzzy p^*-open if $A \leq int(pclA)$.
Note 3.14. It is clear from definitions that fuzzy pre-γ-open set is fuzzy s^*-open as well as fuzzy p^*-open set. But the converses are not true, in general, as it seen from the following examples.

Example 3.15. Fuzzy s^*-open set $\not\Rightarrow$ fuzzy pre-γ-open set
Consider Example 3.5 and the fuzzy set D, defined by $D(a) = 0.5, D(b) = 0.46$. Here $D \in F\gamma C(X)$. So $int(\gamma clD) = intD = A \not\geq D \Rightarrow D \notin FP\gamma O(X)$. Now $sclD = B$. So $int(sclD) = intB = B \geq D \Rightarrow D$ is fuzzy s^*-open set in X.

Example 3.16. Fuzzy p^*-open set $\not\Rightarrow$ fuzzy pre-γ-open set
Consider Example 3.5 and the fuzzy set U, defined by $U(a) = U(b) = 0.5$. Here $pclU = 1_X \setminus A$ and so $int(pclU) = int(1_X \setminus A) = B \geq U \Rightarrow U$

is fuzzy p^*-open set in X. Now $\gamma clU = U$. So $int(\gamma clU) = intU = A \not\geq U \Rightarrow U \notin FP\gamma O(X)$.

4. **Fuzzy Pre-γ-Continuous Function : Some Characterizations**

In this section we first introduce a new type of fuzzy continuous-like function and then characterize it and shows that it lies in between fuzzy continuous function [6] and fuzzy almost continuous function [8].

Definition 4.1. A function $f : X \to Y$ is said to be fuzzy pre-γ-continuous if for each fuzzy point x_α in X and every fuzzy nbd V of $f(x_\alpha)$ in Y, $\gamma cl(f^{-1}(V))$ is a fuzzy nbd of x_α in X.

Theorem 4.2. For a function $f : X \to Y$, the following statements are equivalent :
(a) f is fuzzy pre-γ-continuous,
(b) $f^{-1}(B) \leq int(\gamma cl(f^{-1}(B)))$, for all fuzzy open set B of Y,
(c) $f(clA) \leq cl(f(A))$, for all $A \in F\gamma O(X)$.

Proof (a) $\Rightarrow$ (b). Let B be any fuzzy open set in Y and $x_\alpha \in f^{-1}(B)$. Then $f(x_\alpha) \leq B \Rightarrow B$ is a fuzzy nbd of $f(x_\alpha)$. By (a), $\gamma cl(f^{-1}(B))$ is a fuzzy nbd of x_α in X and so there exists a fuzzy open set U in X such that $x_\alpha \leq U \leq \gamma cl(f^{-1}(B)) \Rightarrow x_\alpha \leq U = intU \leq int(\gamma cl(f^{-1}(B))) \Rightarrow x_\alpha \leq int(\gamma cl(f^{-1}(B)))$. Hence $f^{-1}(B) \leq int(\gamma cl(f^{-1}(B)))$.
(b) $\Rightarrow$ (a). Let x_α be a fuzzy point in X and B be a fuzzy nbd of $f(x_\alpha)$ in Y. Then $x_\alpha \leq f^{-1}(B) \leq int(\gamma cl(f^{-1}(B)))$ (by (b)) $\leq \gamma cl(f^{-1}(B)) \Rightarrow \gamma cl(f^{-1}(B))$ is a fuzzy nbd of x_α in X.
(b) $\Rightarrow$ (c). Let $A \in F\gamma O(X)$. Then $1_Y \setminus cl(f(A))$ is a fuzzy open set in Y. By (b), $f^{-1}(1_Y \setminus cl(f(A))) \leq int(\gamma cl(f^{-1}(1_Y \setminus cl(f(A))))) = int(\gamma cl(1_X \setminus f^{-1}(cl(f(A))))) \leq int(\gamma cl(1_X \setminus f^{-1}(f(A)))) \leq int(\gamma cl(1_X \setminus A)) = 1_X \setminus cl(\gamma intA) = 1_X \setminus clA$. Then $1_X \setminus f^{-1}(cl(f(A))) \leq 1_X \setminus clA \Rightarrow clA \leq f^{-1}(cl(f(A))) \Rightarrow f(clA) \leq cl(f(A))$.
(c) $\Rightarrow$ (b). Let B be any fuzzy open set in Y. Then $\gamma int(f^{-1}(1_Y \setminus B)) \in F\gamma O(X)$. By (c), $f(cl(\gamma int(f^{-1}(1_Y \setminus B)))) \leq cl(f(\gamma int(f^{-1}(1_Y \setminus B)))) \leq cl(f(f^{-1}(1_Y \setminus B))) \leq cl(1_Y \setminus B) = 1_Y \setminus B \Rightarrow f^{-1}(B) = 1_X \setminus f^{-1}(1_Y \setminus B) \leq 1_X \setminus cl(\gamma int(f^{-1}(1_Y \setminus B))) = 1_X \setminus cl(\gamma int(1_X \setminus f^{-1}(B))) = int(\gamma cl(f^{-1}(B)))$.

Note 4.3. It is clear from Theorem 4.2 that the inverse image under fuzzy pre-γ-continuous function of any fuzzy open set is fuzzy pre-γ-open set in X.

Theorem 4.4. For a function $f : X \to Y$, the following statements are equivalent :
(a) f is fuzzy pre-γ-continuous,
(b) $f^{-1}(B) \in FP\gamma O(X)$, for all fuzzy open set B of Y,
(c) for each fuzzy point x_α in X and each fuzzy open nbd V of $f(x_\alpha)$ in Y, there exists $U \in FP\gamma O(X)$ containing x_α such that $f(U) \leq V$,
(d) $f^{-1}(F) \in FP\gamma C(X)$, for all fuzzy closed sets F in Y,
(e) for each fuzzy point x_α in X, the inverse image under f of every fuzzy nbd of $f(x_\alpha)$ is a fuzzy pre-γ-nbd of x_α in X,
(f) $f(p\gamma clA) \leq cl(f(A))$, for all $A \in I^X$,
(g) $p\gamma cl(f^{-1}(B)) \leq f^{-1}(clB)$, for all $B \in I^Y$,
(h) $f^{-1}(intB) \leq p\gamma int(f^{-1}(B))$, for all $B \in I^Y$,
(i) for every basic open fuzzy set V in Y, $f^{-1}(V) \in FP\gamma O(X)$.

Proof (a) $\Leftrightarrow$ (b). Follows from Theorem 4.2 (a) $\Leftrightarrow$ (b).
(b) $\Rightarrow$ (c). Let x_α be a fuzzy point in X and V be a fuzzy open nbd of $f(x_\alpha)$ in Y. By (b), $f^{-1}(V) \in FP\gamma O(X)$ ... (1) . Now $f(x_\alpha) \leq V \Rightarrow x_\alpha \in f^{-1}(V)$ ($= U$, say). Then $x_\alpha \in U$ and by (1), $U(= f^{-1}(V)) \in FP\gamma O(X)$ and $f(U) = f(f^{-1}(V)) \leq V$.
(c) $\Rightarrow$ (b). Let V be a fuzzy open set in Y and let $x_\alpha \leq f^{-1}(V)$. Then $f(x_\alpha) \leq V \Rightarrow V$ is a fuzzy open nbd of $f(x_\alpha)$. By (c), there exists $U \in FP\gamma O(X)$ containing x_α such that $f(U) \leq V$. Then $x_\alpha \leq U \leq f^{-1}(V)$. Now $U \leq int(\gamma clU)$. Then $U \leq int(\gamma clU) \leq int(\gamma cl(f^{-1}(V))) \Rightarrow x_\alpha \leq U \leq int(\gamma cl(f^{-1}(V))) \Rightarrow f^{-1}(V) \leq int(\gamma cl(f^{-1}(V)))$.
(b) $\Leftrightarrow$ (d). Obvious.
(b) $\Rightarrow$ (e). Let W be a fuzzy nbd of $f(x_\alpha)$. Then there exists a fuzzy open set V in Y such that $f(x_\alpha) \leq V \leq W \Rightarrow V$ is a fuzzy open nbd of $f(x_\alpha)$ in Y. Then by (b), $f^{-1}(V) \in FP\gamma O(X)$ and $x_\alpha \leq f^{-1}(V) \leq f^{-1}(W) \Rightarrow f^{-1}(W)$ is a fuzzy pre-γ-nbd of x_α in X.
(e) $\Rightarrow$ (b). Let V be a fuzzy open set in Y and $x_\alpha \leq f^{-1}(V)$. Then $f(x_\alpha) \leq V \Rightarrow V$ is a fuzzy open nbd of $f(x_\alpha)$. By (e), $f^{-1}(V)$ is a fuzzy pre-γ-nbd of x_α in X. Then there exists $U \in FP\gamma O(X)$ containing x_α such that $U \leq f^{-1}(V) \Rightarrow x_\alpha \leq U \leq int(\gamma clU) \leq int(\gamma cl(f^{-1}(V))) \Rightarrow f^{-1}(V) \leq int(\gamma cl(f^{-1}(V)))$.
(d) $\Rightarrow$ (f). Let $A \in I^X$. Then $cl(f(A))$ is a fuzzy closed set in

Y. By (d), $f^{-1}(cl(f(A))) \in FP\gamma C(X)$ containing A. Therefore, $p\gamma clA \leq p\gamma cl(f^{-1}(cl(f(A)))) = f^{-1}(cl(f(A))) \Rightarrow f(p\gamma clA) \leq cl(f(A))$.
(f) $\Rightarrow$ (d). Let B be a fuzzy closed set in Y. Then $f^{-1}(B) \in I^X$. By (f), $f(p\gamma cl(f^{-1}(B))) \leq cl(f(f^{-1}(B))) \leq clB = B \Rightarrow p\gamma cl(f^{-1}(B)) \leq f^{-1}(B) \Rightarrow f^{-1}(B) \in FP\gamma C(X)$.
(f) $\Rightarrow$ (g). Let $B \in I^Y$. Then $f^{-1}(B) \in I^X$. By (f), $f(p\gamma cl(f^{-1}(B))) \leq cl(f(f^{-1}(B))) \leq clB \Rightarrow p\gamma cl(f^{-1}(B)) \leq f^{-1}(clB)$.
(g) $\Rightarrow$ (f). Let $A \in I^X$. Let $B = f(A)$. Then $B \in I^Y$. By (g), $p\gamma clA = p\gamma cl(f^{-1}(B)) \leq f^{-1}(clB) = f^{-1}(cl(f(A))) \Rightarrow f(p\gamma clA) \leq cl(f(A))$.
(b) $\Rightarrow$ (h). Let $B \in I^Y$. Then $intB$ is a fuzzy open set in Y. By (b), $f^{-1}(intB) \leq int(\gamma cl(f^{-1}(intB))) \Rightarrow f^{-1}(intB) \in FP\gamma O(X) \Rightarrow f^{-1}(intB) = p\gamma int(f^{-1}(intB)) \leq p\gamma int(f^{-1}(B))$.
(h) $\Rightarrow$ (b). Let A be any fuzzy open set in Y. Then $f^{-1}(A) = f^{-1}(intA) \leq p\gamma int(f^{-1}(A))$ (by (h)) $\Rightarrow f^{-1}(A) \in FP\gamma O(X)$.
(b) $\Rightarrow$ (i). Obvious.
(i) $\Rightarrow$ (b). Let W be any fuzzy open set in Y. Then there exists a collection $\{W_\alpha : \alpha \in \Lambda\}$ of fuzzy basic open sets in Y such that $W = \bigcup\limits_{\alpha\in\Lambda} W_\alpha$. Now $f^{-1}(W) = f^{-1}(\bigcup\limits_{\alpha\in\Lambda} W_\alpha) = \bigcup\limits_{\alpha\in\Lambda} f^{-1}(W_\alpha) \in FP\gamma O(X)$ (by (i) and by Theorem 3.7 (d)). Hence (b) follows.

Theorem 4.5. A function $f : X \to Y$ is fuzzy pre-γ-continuous iff for each fuzzy point x_α in X and each fuzzy open q-nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy pre-γ-open set W in X with $x_\alpha qW$ such that $f(W) \leq V$.

Proof. Let f be fuzzy pre-γ-continuous function and x_α be a fuzzy point in X and V be a fuzzy open set in Y with $f(x_\alpha)qV$. Let $f(x) = y$. Then $V(y) + \alpha > 1 \Rightarrow V(y) > 1 - \alpha \Rightarrow V(y) > \beta > 1 - \alpha$, for some real number β. Then V is a fuzzy open nbd of y_β. By Theorem 4.4 (a)$\Rightarrow$(c), there exists $W \in FP\gamma O(X)$ containing x_β, i.e., $W(x) \geq \beta$ such that $f(W) \leq V$. Then $W(x) \geq \beta > 1 - \alpha \Rightarrow x_\alpha qW$ and $f(W) \leq V$.
Conversely, let the given condition hold and let V be a fuzzy open set in Y. Put $W = f^{-1}(V)$. If $W = 0_X$, then we are done. Suppose $W \neq 0_X$. Then for any $x \in W_0$, let $y = f(x)$. Then $W(x) = [f^{-1}(V)](x) = V(f(x)) = V(y)$. Let us choose $m \in \mathcal{N}$ where $\mathcal{N}$ is the set of all natural numbers such that $1/m \leq W(x)$. Put $\alpha_n = 1 + 1/n - W(x)$, for all $n \in \mathcal{N}$. Then for $n \in \mathcal{N}$ and $n \geq m$,

$1/n \leq 1/m \Rightarrow 1 + 1/n \leq 1 + 1/m \Rightarrow \alpha_n = 1 + 1/n - W(x) \leq 1 + 1/m - W(x) \leq 1$. Again $\alpha_n > 0$, for all $n \in \mathcal{N} \Rightarrow 0 < \alpha_n \leq 1$ so that $V(y) + \alpha_n > 1 \Rightarrow y_{\alpha_n} q V \Rightarrow V$ is a fuzzy open q-nbd of y_{α_n}. By the given condition, there exists $U_n^x \in FP\gamma O(X)$ such that $x_{\alpha_n} q U_n^x$ and $f(U_n^x) \leq V$, for all $n \geq m$. Let $U^x = \bigcup\{U_n^x : n \in \mathcal{N}, n \geq m\}$. Then $U^x \in FP\gamma O(X)$ (by Theorem 3.7 (d)) and $f(U^x) \leq V$. Again $n \geq m \Rightarrow U_n^x(x) + \alpha_n > 1 \Rightarrow U_n^x(x) + 1 + 1/n - W(x) > 1 \Rightarrow U_n^x(x) > W(x) - 1/n \Rightarrow U_n^x(x) \geq W(x)$, for each $x \in W_0$. Then $W \leq U_n^x$, for all $n \geq m$ and for all $x \in W_0 \Rightarrow W \leq U^x$, for all $x \in W_0 \Rightarrow W \leq \bigcup_{x \in W_0} U^x = U$ (say) ... (1) and $f(U^x) \leq V$, for all $x \in W_0 \Rightarrow f(U) \leq V \Rightarrow U \leq f^{-1}(f(U)) \leq f^{-1}(V) = W$ ... (2). By (1) and (2), $U = W = f^{-1}(V) \Rightarrow f^{-1}(V) \in FP\gamma O(X)$. Hence by Theorem 4.2, f is fuzzy pre-γ-continuous function.

Remark 4.6. If $f : X \to Y$ is fuzzy pre-γ-continuous function, then the inverse image of every fuzzy regular open set is fuzzy pre-γ-open set in X. But the converse may not be true, as it seen from the following example.

Example 4.7. Let $X = \{a\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = 0.3, B(a) = 0.7$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Here 0_X and 1_X are the only fuzzy regular open sets in (X, τ_2) and so clearly the inverse image of fuzzy regular open sets in (X, τ_2) under i is fuzzy pre-γ-open set is (X, τ_1). Now $B \in \tau_2$, $i^{-1}(B) = B$. But $int_{\tau_1}(\gamma cl_{\tau_1} B) = int_{\tau_1} B = A < B$ as every fuzzy set in (X, τ_1) is fuzzy γ-open as well as fuzzy γ-closed set in (X, τ_1). So $B \not\leq int(\gamma cl B) \Rightarrow B \notin FP\gamma O(X) \Rightarrow i$ is not fuzzy pre-γ-continuous function.

Remark 4.8. The inverse image of a fuzzy γ-open set under fuzzy pre-γ-continuous function may not be fuzzy γ-open set follows from the following example.

Example 4.9. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = 0.4, A(b) = 0.7, B(a) = 0.5, B(b) = 0.8$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $B \in \tau_2$, $i^{-1}(B) = B \leq int_{\tau_1}(\gamma cl_{\tau_1} B) = 1_X \Rightarrow i$ is fuzzy pre-γ-continuous function. Now consider the fuzzy set C

defined by $C(a) = 0.6, C(b) = 0.2$. Then $(cl_{\tau_2} int_{\tau_2} C) \bigcup (int_{\tau_2} cl_{\tau_2} C) = 0_X \bigcup 1_X = 1_X \Rightarrow C \in FP\gamma O(X, \tau_2)$. Now $i^{-1}(C) = C$. Again $(cl_{\tau_1} int_{\tau_1} C) \bigcup (int_{\tau_1} cl_{\tau_1} C) = 0_X \bigcup 0_X = 0_X \not\geq C \Rightarrow C \notin F\gamma O(X, \tau_1)$.

Let us now recall the following definitions from [4, 5, 6, 8] for ready references.

Definition 4.10 [6]. A function $f : X \to Y$ is called fuzzy continuous function if $f^{-1}(V)$ is fuzzy open set in X for every fuzzy open set V in Y.

Definition 4.11. A function $f : X \to Y$ is said to be fuzzy almost continuous [8] (resp., fuzzy almost s-continuous [5], fuzzy almost p-continuous [4]) if $f^{-1}(V) \leq int(cl(f^{-1}(V)))$ (resp., $f^{-1}(V) \leq int(scl(f^{-1}(V)))$, $f^{-1}(V) \leq int(pcl(f^{-1}(V)))$.

Remark 4.12. It is clear from definitions that fuzzy continuous function $\Rightarrow$ fuzzy pre-γ-continuous function $\Rightarrow$ fuzzy almost continuous function, fuzzy almost s-continuous function, fuzzy almost p-continuous function. But the reverse implications are not true, in general, follow from the following examples.

Example 4.13. Fuzzy pre-γ-continuous function $\not\Rightarrow$ fuzzy continuous function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Since fuzzy γ-closure of every fuzzy set in (X, τ_1) other than 0_X is 1_X, clearly i is fuzzy pre-γ-continuous function. Now $A \in \tau_2$, $i^{-1}(A) = A \notin \tau_1 \Rightarrow i$ is not fuzzy continuous function.

Example 4.14. Fuzzy almost continuous function $\not\Rightarrow$ fuzzy pre-γ-continuous function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = 0.4, A(b) = 0.7, B(a) = 0.6, B(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $B \in \tau_2$, $i^{-1}(B) = B \leq int_{\tau_1} cl_{\tau_1} B = 1_X \Rightarrow i$ is fuzzy almost continuous function. But $int_{\tau_1} \gamma cl_{\tau_1} B = int_{\tau_1} B = 0_X \not\geq B \Rightarrow i$ is not fuzzy pre-γ-continuous function.

Example 4.15. Fuzzy almost s-continuous function $\not\Rightarrow$ fuzzy pre-γ-continuous function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X, D\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.55, D(a) = 0.5, D(b) = 0.46$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Here $D \in \tau_2$, $i^{-1}(D) = D$. Now $int_{\tau_1} scl_{\tau_1} D = int_{\tau_1} B = B \geq D \Rightarrow i$ is fuzzy almost s-continuous function. But $\gamma cl_{\tau_1} D = D \Rightarrow int_{\tau_1} \gamma cl_{\tau_1} D = int_{\tau_1} D = A \not\geq D \Rightarrow i$ is not fuzzy pre-γ-continuous function.

Example 4.16. Fuzzy almost p-continuous function $\not\Rightarrow$ fuzzy pre-γ-continuous function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X, C\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.55, C(a) = C(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. $C \in \tau_2$, $i^{-1}(C) = C$. Now $int_{\tau_1} pcl_{\tau_1} C = int_{\tau_1}(1_X \setminus A) = B \geq C \Rightarrow i$ is fuzzy almost p-continuous function. But $\gamma cl_{\tau_1} C = C \Rightarrow int_{\tau_1} \gamma cl_{\tau_1} C = int_{\tau_1} C = A \not\geq C \Rightarrow i$ is not fuzzy pre-γ-continuous function.

Remark 4.17. Composition of two fuzzy pre-γ-continuous functions may not be so, as it seen from the following example.

Example 4.18. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5, B(b) = 0.3$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider the identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$ and $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly i_1 and i_2 are fuzzy pre-γ-continuous functions. Let $i_3 = i_2 \circ i_1 : (X, \tau_1) \to (X, \tau_3)$. Now $B \in \tau_3$, $i^{-1}(B) = B$. Again $\gamma cl_{\tau_1} B = B$ and so $B \not\leq int_{\tau_1}(\gamma cl_{\tau_1} B) = 0_X \Rightarrow B \notin FP\gamma O(X, \tau_1) \Rightarrow i_3$ is not fuzzy pre-γ-continuous function.

Theorem 4.19. If $f : X \to Y$ is fuzzy pre-γ-continuous function and $g : Y \to Z$ is fuzzy continuous function, then $g \circ f : X \to Z$ is fuzzy pre-γ-continuous function.
Proof. Obvious.

Lemma 4.20 [2]. Let Z, X, Y be fts's and $f_1 : Z \to X$ and $f_2 : Z \to Y$ be functions. Let $f : Z \to X \times Y$ be defined by

$f(z) = (f_1(z), f_2(z))$ for $z \in Z$, where $X \times Y$ is provided with the product fuzzy topology. Then if B, U_1, U_2 are fuzzy sets in Z, X, Y respectively such that $f(B) \leq U_1 \times U_2$, then $f_1(B) \leq U_1$ and $f_2(B) \leq U_2$.

Theorem 4.21. Let Z, X, Y be fts's. For any functions $f_1 : Z \to X, f_2 : Z \to Y$, if $f : Z \to X \times Y$, defined by $f(x) = (f_1(x), f_2(x))$, for all $x \in Z$, is fuzzy pre-γ-continuous, so are f_1 and f_2.

Proof. Let U_1 be any fuzzy open q-nbd of $f_1(x_\alpha)$ in X for any fuzzy point x_α in Z. Then $U_1 \times 1_Y$ is a fuzzy open q-nbd of $f(x_\alpha)$, i.e., $(f(x))_\alpha$ in $X \times Y$. Since f is fuzzy pre-γ-continuous function, there exists $V \in FP\gamma O(Z)$ with $x_\alpha q V$ such that $f(V) \leq U_1 \times 1_Y$. By Lemma 4.17, $f_1(V) \leq U_1$, $f_2(V) \leq 1_Y$. Consequently, f_1 is fuzzy preγ-continuous function.
Similarly, f_2 is fuzzy pre-γ-continuous function.

Lemma 4.22 [1]. Let X, Y be fts's and let $g : X \to X \times Y$ be the graph of a function $f : X \to Y$. Then if A, B are fuzzy sets in X and Y respectively, $g^{-1}(A \times B) = A \bigcap f^{-1}(B)$.

Theorem 4.23. Let $f : X \to Y$ be a function from an fts X to an fts Y and $g : X \to X \times Y$ be the graph function of f. If g is fuzzy pre-γ-continuous function, then f is so.

Proof. Let g be fuzzy pre-γ-continuous function and B be a fuzzy set in Y. Then by Lemma 4.22, $f^{-1}(B) = 1_X \bigcap f^{-1}(B) = g^{-1}(1_Y \times B)$. Now if B is fuzzy open in Y, then $1_Y \times B$ is fuzzy open in $X \times Y$. Again, $g^{-1}(1_Y \times B) = f^{-1}(B) \in FP\gamma O(X)$ (by hypothesis) $\Rightarrow f$ is fuzzy pre-γ-continuous function.

5. **Fuzzy Almost Pre-γ-Continuous Function: Some Characterizations**

In this section we introduce a new type of fuzzy continuous-like function, viz., fuzzy almost pre-γ-continuous function which is fuzzy pre-γ-continuous function and the reverse implication is true under certain condition.

Definition 5.1. A function $f : X \to Y$ is called fuzzy almost pre-γ-continuous if the inverse image of every fuzzy pre-γ-open set in Y is fuzzy pre-γ-open set in X.

Theorem 5.2. For a function $f : X \to Y$, the following statements are equivalent :
(a) f is fuzzy almost pre-γ-continuous function,
(b) for each fuzzy point x_α in X and each fuzzy pre-γ-open nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy pre-γ-open nbd U of x_α in X such that $f(U) \leq V$,
(c) $f^{-1}(F) \in FP\gamma C(X)$, for all $F \in FP\gamma C(Y)$,
(d) for each fuzzy point x_α in X, the inverse image under f of every fuzzy pre-γ-open nbd of $f(x_\alpha)$ in Y is a fuzzy pre-γ-open nbd of x_α in X,
(e) $f(p\gamma clA) \leq p\gamma cl(f(A))$, for all $A \in I^X$,
(f) $p\gamma cl(f^{-1}(B)) \leq f^{-1}(p\gamma clB)$, for all $B \in I^Y$,
(g) $f^{-1}(p\gamma intB) \leq p\gamma int(f^{-1}(B))$, for all $B \in I^Y$.

Proof. The proof is similar to that of Theorem 4.4 and hence is omitted.

Theorem 5.3. A function $f : X \to Y$ is fuzzy almost pre-γ-continuous iff for each fuzzy point x_α in X and corresponding to any fuzzy pre-γ-open q-nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy pre-γ-open q-nbd W of x_α in X such that $f(W) \leq V$.

Proof. The proof is similar to that of Theorem 4.5 and hence is omitted.

Note 5.4. It is clear from definition that composition of two fuzzy almost pre-γ-continuous functions is fuzzy almost pre-γ-continuous function.

Theorem 5.5. If $f : X \to Y$ is fuzzy almost pre-γ-continuous function and $g : Y \to Z$ is fuzzy pre-γ-continuous function, then $g \circ f : X \to Z$ is fuzzy pre-γ-continuous function.

Proof. Obvious.

Remark 5.6. It is obvious that fuzzy almost pre-γ-continuous function is fuzzy pre-γ-continuous function, but the converse is not true, in general, follows from the following example.

Example 5.7. Fuzzy pre-γ-continuous function $\not\Rightarrow$ fuzzy almost pre-γ-continuous function

Consider Example 4.9. Here i is fuzzy pre-γ-continuous function. Now consider the fuzzy set C defined by $C(a) = 0.6$, $C(b) = 0.2$.

Then $C \in FP\gamma O(X, \tau_2)$. Now $i^{-1}(C) = C \notin FP\gamma O(X, \tau_1) \Rightarrow i$ is not fuzzy almost pre-γ-continuous function.

Remark 5.8. Fuzzy continuity and fuzzy almost pre-γ-continuity are independent concepts follow from the following examples.

Example 5.9. Fuzzy almost pre-γ-continuous function $\not\Rightarrow$ fuzzy continuous function
Consider Example 4.13. Here every fuzzy set in (X, τ_1) is fuzzy pre-γ-open set and so clearly i is fuzzy almost pre-γ-continuous function. But i is not fuzzy continuous function.

Example 5.10. Fuzzy continuous function $\not\Rightarrow$ fuzzy almost pre-γ-continuous function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Clearly i is fuzzy continuous function. Now every fuzzy set in (X, τ_2) is fuzzy pre-γ-open set in (X, τ_2). Consider the fuzzy set B defined by $B(a) = 0.5, B(b) = 0.3$. Then $B \in FP\gamma O(X, \tau_2)$. Now $i^{-1}(B) = B \not\leq int_{\tau_1}\gamma cl_{\tau_1} B = int_{\tau_1} B = 0_X \Rightarrow B \notin FP\gamma O(X, \tau_1) \Rightarrow i$ is not fuzzy almost pre-γ-continuous function.

Note 5.11. From Remark 5.6 and Remark 4.12, we say that fuzzy almost pre-γ-continuous function is fuzzy almost continuous. But the converse is not necessary true follows from Example 4.14. Here i is fuzzy almost continuous function. Since here i is not fuzzy pre-γ-continuous, it is not fuzzy almost pre-γ-continuous function.

To achieve the converse of Remark 5.6, we have to define some sort of fuzzy open-like function, as follows.

Definition 5.12. A function $f : X \to Y$ is said to be fuzzy γ-open if $f(U)$ is fuzzy γ-open in Y for every fuzzy γ-open set U in X.

Lemma 5.13. If $f : X \to Y$ is fuzzy γ-open function, then $f^{-1}(\gamma cl U) \leq \gamma cl(f^{-1}(U))$, for any fuzzy set U in Y.

Proof. let $x_\alpha \not\leq \gamma cl(f^{-1}(U))$ for some fuzzy set U in Y. Then there exists $W \in F\gamma O(X)$ such that $x_\alpha q W$, $W \not{q} f^{-1}(U) \Rightarrow f(W) \not{q} U$ ... (1). As f is fuzzy γ-open function, $f(W) \in F\gamma O(Y)$. Now

$x_\alpha qW \Rightarrow f(x_\alpha)qf(W) \Rightarrow f(W)$ is a fuzzy γ-open q-nbd of $f(x_\alpha)$ in Y. By (1), $f(x_\alpha) \not\leq \gamma clU \Rightarrow x_\alpha \not\leq f^{-1}(\gamma clU)$.

Theorem 5.14. If $f : X \to Y$ is fuzzy pre-γ-continuous and fuzzy γ-open function, then f is fuzzy almost pre-γ-continuous function.

Proof. Let $V \in FP\gamma O(Y)$. Then $V \leq int(\gamma clV)$. Since f is fuzzy pre-γ-continuous function, $f^{-1}(V) \leq f^{-1}(int(\gamma clV)) \leq int(\gamma cl(f^{-1}(int(\gamma clV))))$ (by Theorem 4.4 (a)$\Leftrightarrow$ (b)) $\leq int(\gamma cl(f^{-1}(\gamma clV))) \leq int(\gamma cl(\gamma cl(f^{-1}(V))))$ as f is fuzzy γ-open function (by Lemma 5.13) $= int(\gamma cl(f^{-1}(V))) \Rightarrow f^{-1}(V) \in FP\gamma O(X) \Rightarrow f$ is fuzzy almost pre-γ-continuous.

Let us now recall the following definition from [11] for ready references.

Definition 5.15 [11]. A function $f : X \to Y$ is called a fuzzy open function if $f(U)$ is a fuzzy open set in Y for every fuzzy open set U in X.

Remark 5.16. Fuzzy open function and fuzzy γ-open function are independent concepts follow from the following examples.

Example 5.17. Fuzzy open function $\not\Rightarrow$ fuzzy γ-open function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Clearly i is fuzzy open function. Now consider the fuzzy set B defined by $B(a) = 0.5, B(b) = 0.3$. Since every fuzzy set in (X, τ_1) is fuzzy γ-open set in (X, τ_1), $B \in F\gamma O(X, \tau_1)$. Now $i(B) = B \not\leq (int_{\tau_2} cl_{\tau_2} B) \bigcup (cl_{\tau_2} int_{\tau_2} B) = 0_X \Rightarrow B \notin F\gamma O(X, \tau_2) \Rightarrow i$ is not fuzzy γ-open function.

Example 5.18. Fuzzy γ-open function $\not\Rightarrow$ fuzzy open function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$ where $A(a) = A(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Since every fuzzy set in (X, τ_2) is fuzzy γ-open set in (X, τ_2), clearly i is fuzzy γ-open function. Also i is not fuzzy open function as $A \in \tau_1$, but $i(A) = A \notin \tau_2$.

6. **Fuzzy Pre-γ-Regular Space**

In this section a new type of fuzzy regularity is introduced and shown that in this space fuzzy closed set and fuzzy pre-γ-closed set coincide.

Definition 6.1. An fts (X,τ) is said to be fuzzy pre-γ-regular space if for each fuzzy pre-γ-closed set F in X and each fuzzy point x_α in X with $x_\alpha q(1_X \setminus F)$, there exist a fuzzy open set U in X and a fuzzy pre-γ-open set V in X such that $x_\alpha qU$, $F \leq V$ and $U \not{q} V$.

Theorem 6.2. For an fts (X,τ), the following statements are equivalent:
(a) X is fuzzy pre-γ-regular space,
(b) for each fuzzy point x_α in X and each fuzzy pre-γ-open set U in X with $x_\alpha qU$, there exists a fuzzy open set V in X such that $x_\alpha qV \leq p\gamma clV \leq U$,
(c) for each fuzzy pre-γ-closed set F in X, $\bigcap\{clV : F \leq V, V \in FP\gamma O(X)\} = F$,
(d) for each fuzzy set G in X and each fuzzy pre-γ-open set U in X such that GqU, there exists a fuzzy open set V in X such that $GqV \leq p\gamma clV \leq U$.

Proof (a)$\Rightarrow$(b). Let x_α be a fuzzy point in X and U, a fuzzy pre-γ-open set in X with $x_\alpha qU$. By (a), there exist a fuzzy open set V and a fuzzy pre-γ-open set W in X such that $x_\alpha qV$, $1_X \setminus U \leq W$, $V \not{q} W$. Then $x_\alpha qV \leq 1_X \setminus W \leq U \Rightarrow x_\alpha qV$ and $p\gamma clV \leq p\gamma cl(1_X \setminus W) = 1_X \setminus W \leq U \Rightarrow x_\alpha qV \leq p\gamma clV \leq U$.
(b)$\Rightarrow$(a). Let F be a fuzzy pre-γ-closed set in X and x_α be a fuzzy point in X with $x_\alpha q(1_X \setminus F)$. Then $1_X \setminus F \in FP\gamma O(X)$. By (b), there exists a fuzzy open set V in X such that $x_\alpha qV \leq p\gamma clV \leq 1_X \setminus F$. Put $U = 1_X \setminus p\gamma clV$. Then $U \in FP\gamma O(X)$ and $x_\alpha qV$, $F \leq U$ and $U \not{q} V$.
(b)$\Rightarrow$(c). Let F be fuzzy pre-γ-closed set in X. It is clear that $F \leq \bigcap\{clV : F \leq V, V \in FP\gamma O(X)\}$.
Conversely, let $x_\alpha \not\leq F$. Then $F(x) < \alpha \Rightarrow x_\alpha q(1_X \setminus F)$ where $1_X \setminus F \in FP\gamma O(X)$. By (b), there exists a fuzzy open set U in X such that $x_\alpha qU \leq p\gamma clU \leq 1_X \setminus F$. Put $V = 1_X \setminus p\gamma clU$. Then $F \leq V$ and $U \not{q} V \Rightarrow x_\alpha \not\leq clV \Rightarrow \bigcap\{clV : F \leq V, V \in FP\gamma O(X)\} \leq F$.
(c)$\Rightarrow$(b). Let V be any fuzzy pre-γ-open set in X and x_α be any fuzzy point in X with $x_\alpha qV$. Then $V(x) + \alpha > 1 \Rightarrow x_\alpha \not\leq (1_X \setminus V)$ where $1_X \setminus V \in FP\gamma C(X)$. By (c), there exists $G \in FP\gamma O(X)$

such that $1_X \setminus V \leq G$ and $x_\alpha \not\leq clG \Rightarrow$ there exists a fuzzy open set U in X with $x_\alpha qU$, $U \ /qG \Rightarrow U \leq 1_X \setminus G \leq V$ $\Rightarrow x_\alpha qU \leq p\gamma clU \leq p\gamma cl(1_X \setminus G) = 1_X \setminus G \leq V$.
(c)$\Rightarrow$(d). Let G be any fuzzy set in X and U be any fuzzy pre-γ-open set in X with GqU. Then there exists $x \in X$ such that $G(x) + U(x) > 1$. Let $G(x) = \alpha$. Then $x_\alpha qU \Rightarrow x_\alpha \not\leq 1_X \setminus U$ where $1_X \setminus U \in FP\gamma C(X)$. By (c), there exists $W \in FP\gamma O(X)$ such that $1_X \setminus U \leq W$ and $x_\alpha \not\leq clW \Rightarrow (clW)(x) < \alpha \Rightarrow x_\alpha q(1_X \setminus clW)$. Let $V = 1_X \setminus clW$. Then V is a fuzzy open set in X and $V(x) + \alpha > 1 \Rightarrow V(x) + G(x) > 1 \Rightarrow VqG$ and $p\gamma clV = p\gamma cl(1_X \setminus clW) \leq p\gamma cl(1_X \setminus W) = 1_X \setminus W \leq U$.
(d)$\Rightarrow$(b). Obvious.

Note 6.3. It is clear from Theorem 6.2 that in a fuzzy pre-γ-regular space, every fuzzy pre-γ-closed set is fuzzy closed and hence every fuzzy pre-γ-open set is fuzzy open. As a result, in a fuzzy pre-γ-regular space, the collection of all fuzzy closed (resp., fuzzy open) sets and fuzzy pre-γ-closed (resp., fuzzy pre-γ-open) sets coincide.

Theorem 6.4. If $f : X \to Y$ is fuzzy pre-γ-continuous function and Y is fuzzy pre-γ-regular space, then f is fuzzy almost pre-γ-continuous function.

Proof. Let x_α be a fuzzy point in X and V be any fuzzy pre-γ-open q-nbd of $f(x_\alpha)$ in Y where Y is fuzzy pre-γ-regular space. By Theorem 6.2 (a)$\Rightarrow$(b), there exists a fuzzy open set W in Y such that $f(x_\alpha)qW \leq p\gamma clW \leq V$. Since f is fuzzy pre-γ-continuous function, by Theorem 4.5, there exists $U \in FP\gamma O(X)$ with $x_\alpha qU$ and $f(U) \leq W \leq V$. By Theorem 5.3, f is fuzzy almost pre-γ-continuous function.

Let us now recall following definitions from [6, 7] for ready references.

Definition 6.5 [6]. A collection $\mathcal{U}$ of fuzzy sets in an fts X is said to be a fuzzy cover of X if $\bigcup \mathcal{U} = 1_X$. If, in addition, every member of $\mathcal{U}$ is fuzzy open, then $\mathcal{U}$ is called a fuzzy open cover of X.

Definition 6.6 [6]. A fuzzy cover $\mathcal{U}$ of an fts X is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that

$\bigcup \mathcal{U}_0 = 1_X$.

Definition 6.7 [7]. An fts (X, τ) is said to be fuzzy almost compact if every fuzzy open cover $\mathcal{U}$ of X has a finite proximate subcover, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{clU : U \in \mathcal{U}_0\}$ is again a fuzzy cover of X.

Theorem 6.8. If $f : X \to Y$ is a fuzzy pre-γ-continuous surjective function and X is fuzzy pre-γ-regular as well as fuzzy almost compact space, then Y is fuzzy almost compact space.

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy open cover of Y. Then as f is fuzzy pre-γ-continuous function, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy pre-γ-open cover and hence fuzzy open cover of X as X is fuzzy pre-γ-regular space. Since X is fuzzy almost compact, there are finitely many members $U_1, U_2, ..., U_n$ of $\mathcal{U}$ such that $\bigcup\limits_{i=1}^{n} cl(f^{-1}(U_i)) = 1_X$. Since X is fuzzy pre-γ-regular space, by Note 6.3, $cl(f^{-1}(U_i)) = p\gamma cl(f^{-1}(U_i))$, for $i = 1, 2, ..., n$ and so $1_X = \bigcup\limits_{i=1}^{n} p\gamma cl(f^{-1}(U_i)) \Rightarrow 1_Y = f(\bigcup\limits_{i=1}^{n} p\gamma cl(f^{-1}(U_i))) = \bigcup\limits_{i=1}^{n} f(p\gamma cl(f^{-1}(U_i))) \leq \bigcup\limits_{i=1}^{n} cl(f(f^{-1}(U_i)))$ (by Theorem 4.4 (a)$\Rightarrow$(f)) $\leq \bigcup\limits_{i=1}^{n} cl(U_i) \Rightarrow \bigcup\limits_{i=1}^{n} cl(U_i) = 1_Y \Rightarrow Y$ is fuzzy almost compact space.

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