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UPPER AND LOWER m-I-CONTINUOUS MULTIFUNCTIONS

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Abstract. In this paper we introduce upper/lower m-I-continuous multifunctions as a multifunction defined on an ideal minimal space (X, m, I). We obtain some characterizations and several properties of such multifunctions. As special cases of these properties, we obtain the properties of upper/lower α-I-continuous [4], upper/lower semi-I-continuous [3] and upper/lower pre-I-continuous [3] multifunctions.

1. Introduction

Semi-open sets, preopen sets, α-sets, β-open sets and b-open sets play an important part in the researches of generalizations of continuity for functions and multifunctions in topological spaces. By using these sets, various types of continuous multifunctions are introduced and studied in [22], [29], [30], [11], [27], [28] and other papers. The notions of minimal structures, m-continuity, and M-continuity are introduced in [32] and [33]. By using these notions, the present authors unified the theory of continuity for multifunctions in [23], [34] and other papers.

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The notion of ideal topological spaces were introduced in [19] and [36]. As generalizations of open sets, the notion of I-open sets, semi-I-open sets, pre-I-open sets, α-I-open sets, β-I-open sets and b-I-open sets are introduced. The notion of I-continuous multifunctions is introduced in [2]. Quite recently, other generalizations of continuous multifunctions are introduced and studied in [3], [4], [5], [6], [7], [35].

Throughout the present paper, (X, τ) and (Y, σ) always denote topological spaces and $F : (X, \tau) \to (Y, \sigma)$ presents a mutifunction. For a multifunction $F : (X, \tau) \to (Y, \sigma)$, we shall denote the upper and lower inverse of a subset B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and}$$
$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

The notion of upper and lower semi-continuous [11], [27] (resp. precontinuous [28], α-continuous [11], [29], β-continuous [30]) multifunctions are introduced and studied. The notions of upper and lower I-continuous [2] (resp. pre-I-continuous and semi-I-continuous [3], α-I-continuous [4]) multifunctions are introduced and studied. In this paper, we introduce the notion of upper and lower m-I-continuous multifunctions and generalize upper and lower pre-I-continuous, upper and lower semi-I-continuous, upper and lower α-I-continuous multifunctions. And we obtain properties of new kind of multifunctions.

2. Preliminaries

Definition 2.1. A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) on X [32], [33] if $\emptyset \in m$ and $X \in m$. By (X, m), we denote a nonempty set X with a minimal structure m on X and call it an m-space. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*.

Definition 2.2. Let (X, m) be an m-space. For a subset A of X, the *m-closure* of A and the *m-interior* of A are defined in [20] as follows:
(1) $\mathrm{mCl}(A) = \bigcap\{F : A \subset F, X - F \in m\}$,
(2) $\mathrm{mInt}(A) = \bigcup\{U : U \subset A, U \in m\}$.

Definition 2.3. A minimal structure m on a nonempty set X is said to have *property $\mathcal{B}$* [20] if the union of any family of subsets belonging to m belongs to m.

Lemma 2.1. ([20]) *Let (X, m) be an m-space. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$*,*
(2) If $(X \setminus A) \in m$*, then* $\mathrm{mCl}(A) = A$ *and if* $A \in m$*, then* $\mathrm{mInt}(A) = A$*,*
(3) $\mathrm{mCl}(\emptyset) = \emptyset$ *and* $\mathrm{mCl}(X) = X$*,* $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$*,*
(4) If $A \subset B$*, then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$*,*
(5) $\mathrm{mInt}(A) \subset A \subset \mathrm{mCl}(A)$*,*
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$*.*

Lemma 2.2. ([23]) *Let* (X, m) *be an* m*-space and* m *have property* $\mathcal{B}$*. For a subset* A *of* X*, the following properties hold:*
(1) $A \in m$ *if and only if* $\mathrm{mInt}(A) = A$*,*
(2) A *is* m*-closed if and only if* $\mathrm{mCl}(A) = A$*,*
(3) $\mathrm{mInt}(A) \in m$ *and* $\mathrm{mCl}(A)$ *is* m*-closed.*

Definition 2.4. ([23]) Let (X, m) be an m-space and (Y, σ) a topological space. A multifunction $F : (X, m) \to (Y, \sigma)$ is said to be

(1) *upper* m*-continuous* at $x \in X$ if for each open set V of Y containing $F(x)$, there exists an m-open set U of X containing x such that $F(U) \subset V$,

(2) *lower* m*-continuous* at $x \in X$ if for each open set V of Y such that $V \cap F(x) \neq \emptyset$, there exists $U \in m$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

(3) F is *upper/lower* m*-continuous* if it has the property at each point $x \in X$.

If m has property $\mathcal{B}$, then by Lemma 2.2 we obtain the following theorems.

Theorem 2.1. ([23]) *For a multifunction* $F : (X, m) \to (Y, \sigma)$*, where* (X, m) *satisfies property* $\mathcal{B}$*, the following properties are equivalent:*
(1) F *is upper* m*-continuous;*
(2) $F^+(V)$ *is* m*-open for every* $V \in \sigma$*;*
(3) $F^-(K)$ *is* m*-closed for every closed set* K *of* Y*;*
(4) $\mathrm{mCl}(F^-(B)) \subset F^-\mathrm{Cl}(B))$ *for every subset* B *of* Y*;*
(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(B))$ *for every subset* B *of* Y*.*

Theorem 2.2. ([23]) *For a multifunction* $F : (X, m) \to (Y, \sigma)$*, where* (X, m) *satisfies property* $\mathcal{B}$*, the following properties are equivalent:*
(1) F *is lower* m*-continuous;*
(2) $F^-(V)$ *is* m*-open for every open set* V *of* Y*;*
(3) $F^+(K)$ *is* m*-closed for every closed set* K *of* Y*;*
(4) $\mathrm{mCl}(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ *for every subset* B *of* Y*;*
(5) $F(\mathrm{mCl}(A)) \subset \mathrm{Cl}(F(A))$ *for every subset* A *of* X*;*
(6) $F^-(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^-(B))$ *for every subset* B *of* Y*.*

Theorem 2.3. ([24]) *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper m-continuous at $x \in X$;

(2) $x \in \mathrm{mInt}(F^+(V))$ for every open set V of Y containing $F(x)$;

(3) $x \in F^-(\mathrm{Cl}(B))$ for every subset B of Y such that $x \in \mathrm{mCl}(F^-(B))$;

(4) $x \in \mathrm{mInt}(F^+(B))$ for every subset B of Y such that $x \in F^+(\mathrm{Int}(B))$

Theorem 2.4. ([24]) *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower m-continuous at $x \in X$;

(2) $x \in \mathrm{mInt}(F^-(V))$ for every open set V of Y such that $F(x) \cap V \neq \emptyset$;

(3) $x \in F^+(\mathrm{Cl}(B))$ for every subset B of Y such that $x \in \mathrm{mCl}(F^+(B))$;

(4) $x \in \mathrm{mInt}(F^-(B))$ for every subset B of Y such that $x \in F^-(\mathrm{Int}(B))$.

(5) $x \in F^+(\mathrm{Cl}(F(A)))$ for every subset A of X such that $x \in \mathrm{mCl}(A)$).

For a multifunction $F : (X, m) \to (Y, \sigma)$, $D^+_m(F)$ and $D^-_m(F)$ are defined in [24] as follows:

$$D^+_m(F) = \{x \in X : F \text{ is not upper } m\text{-continuous at } x\},$$
$$D^-_m(F) = \{x \in X : F \text{ is not lower } m\text{-continuous at } x\}.$$

Theorem 2.5. ([24]) *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties hold:*

$$\begin{aligned} D^+_m(F) &= \textstyle\bigcup_{G\in\sigma}\{F^+(G) - \mathrm{mInt}(F^+(G))]\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\{F^+(\mathrm{Int}(B)) - \mathrm{mInt}(F^+(B))]\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\{\mathrm{mCl}(F^-(B)) - F^-(\mathrm{Cl}(B))\} \\ &= \textstyle\bigcup_{H\in\mathcal{F}}\{\mathrm{mCl}(F^-(H)) - F^-(H)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

Theorem 2.6. ([24]) *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties hold:*

$$\begin{aligned} D^-_m(F) &= \textstyle\bigcup_{G\in\sigma}\{F^-(G) - \mathrm{mInt}(F^-(G))]\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\{F^-(\mathrm{Int}(B)) - \mathrm{mInt}(F^-(B))\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\{\mathrm{mCl}(F^+(B)) - F^+(\mathrm{Cl}(B))\} \\ &= \textstyle\bigcup_{A\in\mathcal{P}(X)}\{\mathrm{mCl}(A) - F^+(\mathrm{Cl}(F(A)))\} \\ &= \textstyle\bigcup_{H\in\mathcal{F}}\{\mathrm{mCl}(F^+(H)) - F^+(H)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

3. Ideal topological spaces

Let (X, τ) be a topological space. The notion of ideals has been introduced in [19] and [36] and further investigated in [17]

Definition 3.1. A nonempty collection I of subsets of a set X is called an *ideal* on X [19], [36] if it satisfies the following two conditions:
(1) $A \in I$ and $B \subset A$ implies $B \in I$,
(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X, τ) with an ideal I on X is called an *ideal topological space* and is denoted by (X, τ, I). Let (X, τ, I) be an ideal topological space. For any subset A of X, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the *local function* of A with respect to τ and I [16], [17]. Hereafter $A^\star(I, \tau)$ is simply denoted by $A^\star$. It is well known that $\mathrm{Cl}^\star(A) = A \cup A^\star$ defines a Kuratowski closure operator on X and the topology generated by $\mathrm{Cl}^\star$ is denoted by $\tau^\star$.

Lemma 3.1. *Let (X, τ, I) be an ideal topological space and A, B be subsets of X. Then the following properties hold:*
(1) $A \subset B$ implies $\mathrm{Cl}^\star(A) \subset \mathrm{Cl}^\star(B)$,
(2) $\mathrm{Cl}^\star(X) = X$ and $\mathrm{Cl}^\star(\emptyset) = \emptyset$,
(3) $\mathrm{Cl}^\star(A) \cup \mathrm{Cl}^\star(B) \subset \mathrm{Cl}^\star(A \cup B)$.

Definition 3.2. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be
(1) *α-I-open* [14] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(A)))$,
(2) *semi-I-open* [14] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(3) *pre-I-open* [10] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A))$,
(4) *b-I-open* [9] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A)) \cup \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(5) *β-I-open* [15] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A)))$,
(6) *weakly semi-I-open* [12] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(7) *weakly b-I-open* [21] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A))) \cup \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(8) *strongly β-I-open* [13] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A)))$.

Among the sets in Definition 3.2, we have the following relation:

DIAGRAM 1

$$\begin{array}{ccccccc}
\text{open} \Rightarrow & \alpha\text{-}I\text{-open} & \Rightarrow & \text{semi-}I\text{-open} & \Rightarrow & \text{weakly semi-}I\text{-open} & \\
& \Downarrow & & \Downarrow & & \Downarrow & \\
& \text{pre-}I\text{-open} & \Rightarrow & b\text{-}I\text{-open} & \Rightarrow & \text{weakly } b\text{-}I\text{-open} & \\
& & & \Downarrow & & \Uparrow & \\
& & & \text{strongly } \beta\text{-}I\text{-open} & \Rightarrow & \beta\text{-}I\text{-open} &
\end{array}$$

The family of all α-I-open (resp. semi-I-open, pre-I-open, b-I-open, β-I-open, weakly semi-I-open, weakly b-I-open, strongly β-I-open) sets in an ideal topological space (X, τ, I) is denoted by $\alpha\mathrm{IO}(X)$ (resp. $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$).

Definition 3.3. By $\mathrm{mIO}(X)$, we denote each one of the families $\alpha\mathrm{IO}(X)$, $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$).

Lemma 3.2. *Let (X, τ, I) be an ideal topological space. Then* $\mathrm{mIO}(X)$ *is a minimal structure and has property $\mathcal{B}$.*

Proof. The proof follows from Lemma 3.1(1)(2). As an example, we shall show that $\alpha\mathrm{IO}(X)$ has property $\mathcal{B}$. Let A_α be an α-I-open set for each $\alpha \in \Lambda$. Then $A_\alpha \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(A_\alpha))) \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(\cup_{\alpha\in\Lambda} A_\alpha)))$ for each $\alpha \in \Lambda$ and hence $\cup_{\alpha\in\Lambda} A_\alpha \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(\cup_{\alpha\in\Lambda} A_\alpha)))$. Therefore, $\cup_{\alpha\in\Lambda} A_\alpha$ is α-I-open.

Remark 3.1. It is shown in Theorem 3.4 of [14] (resp. Theorem 2.10 of [10], Theorem 2.1 of [12], Theorem 2.7 of [21], Proposition 3 of [13]) that $\mathrm{SIO}(X)$ (resp. $\mathrm{PIO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$) has property $\mathcal{B}$.

Definition 3.4. Let (X, τ, I) be an ideal topological space. For a subset A of X, $\mathrm{mCl_I}(A)$ and $\mathrm{mInt_I}(A)$ as follows:
(1) $\mathrm{mCl_I}(A) = \cap\{F : A \subset F, X \setminus F \in \mathrm{mIO(X)}\}$,
(2) $\mathrm{mInt_I}(A) = \cup\{U : U \subset A, U \in \mathrm{mIO(X)}\}$.

Let (X, τ, I) be an ideal topological space and $\mathrm{mIO}(X)$ the m-structure on X. If $\mathrm{mIO}(X) = \alpha\mathrm{IO}(X)$ (resp. $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$), then we have
(1) $\mathrm{mCl_I}(A) = \alpha\mathrm{Cl_I}(A)$ (resp. $\mathrm{sCl_I}(A)$, $\mathrm{pCl_I}(A)$, $\mathrm{bCl_I}(A)$, $\beta\mathrm{Cl_I}(A)$, $\mathrm{wsCl_I}(A)$, $\mathrm{wbCl_I}(A)$, $\mathrm{s}\beta\mathrm{Cl_I}(A)$),
(2) $\mathrm{mInt_I}(A) = \alpha\mathrm{Int_I}(A)$ (resp. $\mathrm{sInt_I}(A)$, $\mathrm{pInt_I}(A)$, $\mathrm{bInt_I}(A)$, $\beta\mathrm{Int_I}(A)$, $\mathrm{wsInt_I}(A)$, $\mathrm{wbInt_I}(A)$, $\mathrm{s}\beta\mathrm{Int_I}(A)$).

4. m-I-continuous multifunctions

Definition 4.1. A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is said to be
(1) *upper m-I-continuous* at $x \in X$ if for each open set V of Y such that $F(x) \subset V$ there exists $U \in \mathrm{mIO}(X)$ containing x such that $F(U) \subset V$,
(2) *lower m-I-continuous* at $x \in X$ if for each open set V of Y such

that $V \cap F(x) \neq \emptyset$, there exists $U \in \mathrm{mIO}(X)$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

(3) F is *upper/lower m-I-continuous* if it has the property at each point $x \in X$.

Lemma 4.1. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties hold: $F : (X, \tau, I) \to (Y, \sigma)$ is upper/lower m-I-continuous at $x \in X$ if and only if $F : (X, \mathrm{mIO}(X)) \to (Y, \sigma)$ is upper/lower m-continuous at $x \in X$.*

Proof. The proof is obvious from the definition.

Definition 4.2. A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is said to be *upper α-I-continuous* (resp. *lower α-I-continuous*) [4] if for each point $x \in X$ and each open set V of Y such that $F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$), there exists an α-I-open set U containing x such that $F(U) \subset V$ (resp. $F(u) \cap V \neq \emptyset$ for each $u \in U$).

Upper/lower semi-I-continuous multifunctions and upper/lower pre-I-continuous multifunctions are defind in [3].

Remark 4.1. If $F : (X, \mathrm{mIO}(X)) \to (Y, \sigma)$ is upper/lower m-continuous and $\mathrm{mIO}(X) = \alpha\mathrm{IO}(X)$ (resp. $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$), then $F : (X, \tau, I) \to (Y, \sigma)$ is upper/lower α-I-continuous (resp. upper/lower semi-I-continuous, upper/lower pre-I-continuous, upper/lower b-I-continuous, upper/lower β-I-continuous, upper/lower strongly β-I-continuous).

By Lemma 3.2, Theorems 2.1 and 2.2, we obtain the following theorems:

Theorem 4.1. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper m-I-continuous;

(2) $F^+(V)$ is mI-open for every $V \in \sigma$;

(3) $F^-(K)$ is mI-closed for every closed set K of Y;

(4) $\mathrm{mCl_I}(F^-(B)) \subset F^-\mathrm{Cl}(B))$ *for every subset B of Y;*

(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt_I}(F^+(B))$ *for every subset B of Y.*

Theorem 4.2. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower m-I-continuous;

(2) $F^-(V)$ is mI-open for every open set V of Y;

(3) $F^+(K)$ is mI-closed for every closed set K of Y;

(4) $\mathrm{mCl_I}(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ *for every subset B of Y;*

(5) $F(\mathrm{mCl_I}(A)) \subset \mathrm{Cl}(F(A))$ *for every subset* A *of* X*;*
(6) $F^-(\mathrm{Int}(B)) \subset \mathrm{mInt_I}(F^-(B))$ *for every subset* B *of* Y*.*

Remark 4.2. If $\mathrm{mIO}(X) = \alpha\mathrm{IO}(X)$, then by Theorems 4.1 and 4.2 we obtain Theorem 1 of [4].

For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, we define $D^+_{mIO}(F)$ and $D^-_{mIO}(F)$ as follows:

$D^+_{mIO}(F) = \{x \in X : F \text{ is not upper } m\text{-}I\text{-continuous at } x\}$,
$D^-_{mIO}(F) = \{x \in X : F \text{ is not lower } m\text{-}I\text{-continuous at } x\}$.

By Lemma 3.2, Theorems 2.5 and 2.6, we obtain the following theorem:

Theorem 4.3. *For a multifunction* $F : (X, \tau, I) \to (Y, \sigma)$*, the following properties hold:*

$$\begin{aligned} D^+_{mIO}(F) &= \bigcup_{G\in\sigma}\{F^+(G) - \mathrm{mInt_I}(F^+(G))]\} \\ &= \bigcup_{B\in\mathcal{P}(Y)}\{F^+(\mathrm{Int}(B)) - \mathrm{mInt_I}(F^+(B))\} \\ &= \bigcup_{B\in\mathcal{P}(Y)}\{\mathrm{mCl_I}(F^-(B)) - F^-(\mathrm{Cl}(B))\} \\ &= \bigcup_{H\in\mathcal{F}}\{\mathrm{mCl_I}(F^-(H)) - F^-(H)\}, \end{aligned}$$

where $\mathcal{F}$ *is the family of closed sets of* (Y, σ)*.*

Theorem 4.4. *For a multifunction* $F : (X, \tau, I) \to (Y, \sigma)$*, the following properties hold:*

$$\begin{aligned} D^-_{mIO}(F) &= \bigcup_{G\in\sigma}\{F^-(G) - \mathrm{mInt_I}(F^-(G))]\} \\ &= \bigcup_{B\in\mathcal{P}(Y)}\{F^-(\mathrm{Int}(B)) - \mathrm{mInt_I}(F^-(B))\} \\ &= \bigcup_{B\in\mathcal{P}(Y)}\{\mathrm{mCl_I}(F^+(B)) - F^+(\mathrm{Cl}(B))\} \\ &= \bigcup_{A\in\mathcal{P}(X)}\{\mathrm{mCl_I}(A) - F^+(\mathrm{Cl}(F(A)))\} \\ &= \bigcup_{H\in\mathcal{F}}\{\mathrm{mCl_I}(F^+(H)) - F^+(H)\}, \end{aligned}$$

where $\mathcal{F}$ *is the family of closed sets of* (Y, σ)*.*

5. Some properties of m-I-continuous multifunctions

Let (X, τ) be a topological space and A a subset of X. A point $x \in X$ is called a θ-closure of A [37] if $\mathrm{Cl}(U) \cap A \neq \emptyset$ for every open open set U containing x. The set of all θ-closure points of A is called the θ-closure of A and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be θ-closed. The complement of a θ-closed set is said to be θ-open. It follows from [37] that the collection of all θ-open sets is a topology for X.

Lemma 5.1. ([25]) *Let* (Y, σ) *be a regular space and* m *a minimal structure having property* $\mathcal{B}$*. For a multifunction* $F : (X, m) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F is upper m-continuous;
(2) $F^-(\mathrm{Cl}_\theta(B))$ is m-closed for every subset B of Y;
(3) $F^-(K)$ is m-closed for every θ-closed set K of Y;
(4) $F^+(V)$ is m-open for every θ-open set V of Y.

Lemma 5.2. ([25]) *Let (Y, σ) be a regular space and m a minimal structure having property $\mathcal{B}$. For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is lower m-continuous;
(2) $F^+(\mathrm{Cl}_\theta(B))$ is m-closed for every subset B of Y;
(3) $F^+(K)$ is m-closed for every θ-closed set K of Y;
(4) $F^-(V)$ is m-open for every θ-open set V of Y.

By Lemmas 3.2, 5.1 and 5.2, we have the following theorems.

Theorem 5.1. *Let (Y, σ) be a regular space. For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is upper m-I-continuous;
(2) $F^-(\mathrm{Cl}_\theta(B))$ is mI-closed for every subset B of Y;
(3) $F^-(K)$ is mI-closed for every θ-closed set K of Y;
(4) $F^+(V)$ is mI-open for every θ-open set V of Y.

Theorem 5.2. *Let (Y, σ) be a regular space. For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is lower m-I-continuous;
(2) $F^+(\mathrm{Cl}_\theta(B))$ is mI-closed for every subset B of Y;
(3) $F^+(K)$ is mI-closed for every θ-closed set K of Y;
(4) $F^-(V)$ is mI-open for every θ-open set V of Y.

Definition 5.1. A subset A of a topological space (Y, σ) is said to be
(1) *α-regular* [18] if for each $a \in A$ and each open set U containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$,
(2) *α-paracompact* [38] if every Y-open cover of A has a Y-open refinement which covers A and is locally finite for each point of X.

For a multifunction $F : X \to (Y, \sigma)$, by $\mathrm{Cl}(F) : X \to (Y, \sigma)$ [8] we denote a multifunction defined as follows: $\mathrm{Cl}(F)(x) = \mathrm{Cl}(F(x))$ for each $x \in X$. Similarly, we denote $\mathrm{sCl}(F)$, $\mathrm{pCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, $b\mathrm{Cl}(F)$.

Lemma 5.3. ([26]) *Let $F : (X, m) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then F is upper m-continuous if and only if G is upper m-continuous, where $G = \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, $bCl(F)$.*

Lemma 5.4. ([26]) *Let $F : (X, m) \to (Y, \sigma)$ be a multifunction. Then F is lower m-continuous if and only if G is lower m-continuous, where $G = \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, bCl(F).*

Theorem 5.3. *Let $F : (X, \tau, I) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then F is upper m-I-continuous if and only if G is upper m-I-continuous, where $G = \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, bCl(F).*

Theorem 5.4. *Let $F : (X, \tau, I) \to (Y, \sigma)$ be a multifunction. Then F is lower m-I-continuous if and only if G is lower m-I-continuous, where $G = \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, bCl(F).*

Definition 5.2. Let (X, τ, I) be an ideal topological space and A a subset of X. The *mI-frontier* of A, $\mathrm{mIFr}(A)$, is defined as follows: $\mathrm{mIFr}(A) = \mathrm{mCl_I}(A) \cap \mathrm{mCl_I}(X \setminus A)$.

Lemma 5.5. ([26]) *The set of all points $x \in X$ at which a multifunction $F : (X, m) \to (Y, \sigma)$ is not upper (resp. lower) m-continuous is identical with the union of the m-frontiers of the upper (resp. lower) inverse images of open sets containing (resp. meeting) F(x).*

Theorem 5.5. *The set of all points $x \in X$ at which a multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is not upper (resp. lower) m-I-continuous is identical with the union of the mI-frontiers of the upper (resp. lower) inverse images of open sets containing (resp. meeting) F(x).*

Definition 5.3. An m-space (X, m) is said to be *m-connected* [32] if X can not be written as the union of two nonempty disjoint m-open sets.

Definition 5.4. An ideal topological space (X, τ, I) is said to be *mI-connected* if X can not be written as the union of two nonempty disjoint mI-open sets.

Lemma 5.6. ([23]) *Let (X, m) be an m-space, m have property $\mathcal{B}$ and (Y, σ) a topological space. If $F : (X, m_X) \to (Y, \sigma)$ is an upper or lower m-continuous surjective multifunction such that $F(x)$ is connected for each $x \in X$ and (X, m) is m-connected, then (Y, σ) is connected.*

Theorem 5.6. *If $F : (X, \tau, I) \to (Y, \sigma)$ is an upper or lower m-I-continuous surjective multifunction such that $F(x)$ is connected for each $x \in X$ and $(X, \mathrm{mIO}(X))$ is mI-connected, then (Y, σ) is connected.*

Remark 5.1. If $\mathrm{mIO}(X) = \alpha\mathrm{IO}(X)$, then we obtain Theorem 12 of [4].

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6. mI-T_2 and m-I-continuous multifunctions

Definition 6.1. An m-space (X, m) is said to be m-T_2 [32] if for each distinct points $x, y \in X$ there exist $U, V \in m$ containing x, y, respectively, such that $U \cap V = \emptyset$.

Definition 6.2. An ideal topological space (X, τ, I) is said to be mI-T_2 if the m-space $(X, \mathrm{mIO}(X))$ is m-T_2. Hence (X, τ, I) is mI-T_2 if for each distinct points $x, y \in X$ there exist $U, V \in mIO(X)$ containing x, y, respectively, such that $U \cap V = \emptyset$.

Definition 6.3. A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is said to be *injective* if $x \neq y$ implies that $F(x) \cap F(y) = \emptyset$.

Lemma 6.1. ([26]) *If $F : (X, m) \to (Y, \sigma)$ is an upper m-continuous injective multifunction into a Hausdorff space (Y, σ) and F(x) is compact for each $x \in X$, then X is m-T_2.*

Theorem 6.1. *If $F : (X, \tau, I) \to (Y, \sigma)$ is an upper m-I-continuous injective multifunction into a Hausdorff space (Y, σ) and F(x) is compact for each $x \in X$, then X is mI-T_2.*

Theorem 6.2. *If $F : (X, \tau, I) \to (Y, \sigma)$ is an upper m-I-continuous multifunction satisfying the following conditions:*
(1) F(x) is closed in Y for each $x \in X$,
(2) $F(x) \cap F(y) = \emptyset$ for each distinct points x, y of X,
(3) (Y, σ) is normal,
then (X, τ, I) is mI-T_2.

Proof. Let x and y be distinct points of X. Then $F(x) \cap F(y) = \emptyset$. Since $F(x)$ and $F(y)$ are closed and (Y, σ) is normal, there exist disjoint open sets V_x and V_y in Y such that $F(x) \subset V_x$ and $F(y) \subset V_y$. Since F is an upper m-I-continuous multifunction, there exist $U_x, U_y \in mIO(X)$ containing x, y, respectively, such that $F(U_x) \subset V_x$ and $F(U_y) \subset V_y$. Therefore, U_x and U_y are disjoint and hence (X, τ, I) is mI-T_2.

Theorem 6.3. *Let $F : (X, \tau, I) \to (Y, \sigma)$ be a multifunction and (Y, σ) a normal space. If for each pair of distinct points $x_1, x_2 \in X$, the following conditions are satisfied:*
(1) $F(x_1)$ and $F(x_2)$ are closed in Y,
(2) $F(x_1) \cap F(x_2) = \emptyset$,
(3) F is upper m-I-continuous at x_1 and x_2.
then (X, τ, I) is mI-T_2.

Proof. The proof is similar with that of Theorem 6.2.

Final Remarks. All properties obtained in Sections 4, 5 and 6 are applied to all families αIO(X), SIO(X), PIO(X), BIO(X), βIO(X), WSIO(X), WBIO(X), SβIO(X).

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