"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 29 (2019), No. 2, 103-120

A CONTRIBUTION ON CONVEX AND STRICTLY PLURISUBHARMONIC FUNCTIONS DEFINED BY HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES AND APPLICATIONS

ABIDI JAMEL

Abstract. Let $A_1, A_2 \in \mathbb{C}\backslash\{0\}$ and $n, m \in \mathbb{N}\backslash\{0\}$. Using algebraic methods, we prove that there exist three analytic functions $\varphi : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that v is convex and strictly plurisubharmonic on $\mathbb{C}^n \times \mathbb{C}^m$ if and only if $m = 1$, $n \in \{1, 2\}$, there exists $c \in \mathbb{C}$ such that $\mid \varphi + c \mid^2$ is convex and strictly subharmonic on $\mathbb{C}$ and the functions g_1 and g_2 have fundamental representations over $\mathbb{C}^n$. $v(z, w) =\mid A_1\varphi(w) - \overline{g_1}(z) \mid^2 + \mid A_2\varphi(w) - \overline{g_2}(z) \mid^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$. At the end, we prove an additional theorem by analytic and algebraic methods.

1. Introduction

Let D be a convex domain of $\mathbb{C}^n$ and $\psi : \mathbb{C} \to \mathbb{C}$ be a holomorphic not constant function, $n \geq 1$. Assume that $g : D \to \mathbb{C}$ and $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ be two holomorphic functions. Put $u_1(z, w) =\mid \psi(w) - g(z) \mid^2$ and $u_2(\xi, w) =\mid w - f_1(\xi) \mid^2 + \mid w - f_2(\xi) \mid^2$, $(z, w, \xi) \in D \times \mathbb{C} \times \mathbb{C}$. Assume that u_1 is convex on $D \times \mathbb{C}$. By [4], we have the following two cases.

Keywords and phrases: holomorphic, convex, plurisubharmonic functions, inequalities, holomorphic partial differential equation, strictly, maximum principle.
(2010) Mathematics Subject Classification: 32A10, 32A60, 32F17, 32U05

Case 1. ψ is affine on $\mathbb{C}$. Therefore g is affine on D.
Case 2. ψ is not affine on $\mathbb{C}$. Then g is constant on D. (Observe that if we consider the sum of two absolute values like u_2, we have another situation).
Moreover, u_1 is convex and strictly psh on $D \times \mathbb{C}$ if and only if $n = 1$, ψ is affine on $\mathbb{C}$, g is affine on D with the modulus $\mid \frac{\partial \psi}{\partial w} \mid > 0$ and $\mid \frac{\partial g}{\partial \overline{z}} \mid > 0$.
Now let $f : \mathbb{C}^n \to \mathbb{C}$ be holomorphic and $v_1(z,w) = \mid w - f(z) \mid^2$, $v_2(z,w) = \displaystyle\int_{B(0,1)} v_1(z+\xi, w) dm_{2n}(\xi)$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.
We have v_1 is not strictly psh at every point of $\mathbb{C}^n \times \mathbb{C}$. While v_2 is strictly psh at all $\mathbb{C}^n \times \mathbb{C}$, if f shall satisfy a suitable condition. But for example, there exists several cases where u_2 is convex and strictly psh on $\mathbb{C} \times \mathbb{C}$. This proves that we have a great differences between the family of functions defined like v_1 and the class of functions like v_2.
The original problem is to find all the analytic functions $\varphi_1, \varphi_2, \varphi_3, \varphi_4 : \mathbb{C}^m \to \mathbb{C}$ and $f_1, f_2, f_3, f_4 : \mathbb{C}^n \to \mathbb{C}$ such that u_3 and u_4 are convex functions on $\mathbb{C}^n \times \mathbb{C}^m$ and $u_5 = (u_3 + u_4)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$. Where $u_3(z,w) = \mid \varphi_1(w) - f_1(z) \mid^2 + \mid \varphi_2(w) - f_2(z) \mid^2$, $u_4(z,w) = \mid \varphi_3(w) - f_3(z) \mid^2 + \mid \varphi_4(w) - f_4(z) \mid^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$.

In this paper, we consider application of the following complex analysis property. Let $g, f : G \to \mathbb{C}^t$ be two analytic functions, $s, t \in \mathbb{N}\backslash\{0\}$ and G a domain of $\mathbb{C}^s$. Then $\parallel f + \overline{g} \parallel^2$ and $(\parallel f \parallel^2 + \parallel g \parallel^2)$ have the same hermitian Levi form over G. This criterion plays a particular role in several questions of complex analysis.
We are first interested, in section 2, to answer of the following question and related topics.
Let $A_1, A_2 \in \mathbb{C}\backslash\{0\}$ and $n, m \in \mathbb{N}\backslash\{0\}$. Find exactly all the three analytic functions $\varphi : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that v is convex and strictly plurisubharmonic on $\mathbb{C}^n \times \mathbb{C}^m$, where $v(z,w) =$
$\mid A_1\varphi(w) - \overline{g_1}(z) \mid^2 + \mid A_2\varphi(w) - \overline{g_2}(z) \mid^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$.
In this case find exactly φ, g_1 and g_2 by their expressions.
Similarly, using the methods based on the idea of this paper, we can discuss the several cases, v is convex and strictly psh but not strictly convex on $\mathbb{C}^n \times \mathbb{C}^m$, v is convex strictly psh but not strictly convex on any not empty open ball of $\mathbb{C}^n \times \mathbb{C}^m$, v is strictly convex on $\mathbb{C}^n \times \mathbb{C}^m$, ...

We obtain several representations which exhibit new classes of functions in the above type. We study now some good family of plurisubharmonic (psh) functions, that is the class of convex and strictly psh functions over $\mathbb{C}^N$, $N \geq 1$.
The following classes ((convex and strictly psh functions), (convex strictly psh and not strictly convex functions), (convex strictly psh and not strictly convex in any not empty Euclidean open ball of $\mathbb{C}^n \times \mathbb{C}^m$), ...) play a classical role on many problems of complex analysis, convex analysis and harmonic analysis (representation theory).
Several papers appeared recently related to this topic, let us mention [3], [5], [4], the monograph [6] and others.
In section 3, some auxiliary results are proved, while we will need a key lemma and several algebraic methods.
Let U be a domain of $\mathbb{R}^d, (d \geq 2)$. sh(U) is the class of subharmonic functions on U and m_d is the Lebesgue measure on $\mathbb{R}^d$. For $N \geq 1$ and $h = (h_1, ..., h_N)$, where $h_1, ..., h_N : U \to \mathbb{C}$, $\| h \| = (| h_1 |^2 + ... + | h_N |^2)^{\frac{1}{2}}$.
Let $g : D \to \mathbb{C}$ be a analytic function, where D is a domain of $\mathbb{C}$. We denote by $g^{(m)} = \frac{\partial^m g}{\partial z^m}$ the holomorphic derivative of g of order m, for all $m \in \mathbb{N}$.
If $z = (z_1, ..., z_n)$, $\xi = (\xi_1, ..., \xi_n) \in \mathbb{C}^n$, we denote $< z/\xi > = z_1\overline{\xi_1} + ... + z_n\overline{\xi_n}$ and $B(\xi, r) = \{\zeta \in \mathbb{C}^n / \| \zeta - \xi \| < r\}$ for $r > 0$, where $\sqrt{< \xi/\xi >} = \| \xi \|$ is the Euclidean norm of ξ. Denote $C^\infty(U) = \{\varphi : U \to \mathbb{C} / \varphi \text{ is of class } C^\infty \text{ on } U\}$.
Let D be a domain of $\mathbb{C}^n, (n \geq 1)$. psh(D) and prh(D) are respectively the class of plurisubharmonic and pluriharmonic functions on D.
For the study of properties of analytic and plurisubharmonic functions we cite the references [1], [7], [8], [9], [11], [12], [13] and [14]. For the study of convex functions in complex convex domains, we cite [10], [6] and [12].

2. The representation of analytic functions in real and complex convexity

Throughout this section, $A_1, A_2 \in \mathbb{C}\backslash\{0\}, n, m \in \mathbb{N}\backslash\{0\}$, $\varphi : \mathbb{C}^m \to \mathbb{C}$ be analytic and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions. Also we define $u(z, w) = | A_1\varphi(w) - g_1(z) |^2 + | A_2\varphi(w) - g_2(z) |^2$, $v(z, w) =$
$| A_1\varphi(w) - \overline{g_1}(z) |^2 + | A_2\varphi(w) - \overline{g_2}(z) |^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$.
The following theorem is an important technical result, which we will

need for our purposes.

Theorem 1. Assume that u is convex on $\mathbb{C}^n \times \mathbb{C}^m$. Then there exist $c \in \mathbb{C}$ such that $\mid \varphi + c \mid^2$ is convex on $\mathbb{C}^m$.

Proof. u is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}^m$. Let $w = (w_1, ..., w_m) \in \mathbb{C}^m$, $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{C}^m$ and $z \in \mathbb{C}^n$. The function $u(z, .)$ is convex on $\mathbb{C}^m$. Assume that φ is not affine on $\mathbb{C}^m$.

Therefore $\mid \sum_{j,k=1}^{m} \frac{\partial^2 u}{\partial w_j \partial w_k}(z, w)\alpha_j\alpha_k \mid \leq \sum_{j,k=1}^{m} \frac{\partial^2 u}{\partial w_j \partial \overline{w_k}}(z, w)\alpha_j\overline{\alpha}_k.$

Then $\mid \sum_{j,k=1}^{m} \frac{\partial^2 \varphi}{\partial w_j \partial w_k}(w)(\mid A_1 \mid^2 \overline{\varphi}(w) - A_1\overline{g_1}(z))\alpha_j\alpha_k +$
$\sum_{j,k=1}^{m} \frac{\partial^2 \varphi}{\partial w_j \partial w_k}(w)(\mid A_2 \mid^2 \overline{\varphi}(w) - A_2\overline{g_2}(z))\alpha_j\alpha_k \mid \leq \mid A_1 \sum_{j=1}^{m} \frac{\partial \varphi}{\partial w_j}(w)\alpha_j \mid^2 +$
$\mid A_2 \sum_{j=1}^{m} \frac{\partial \varphi}{\partial w_j}(w)\alpha_j \mid^2 .$

It follows that

$$\mid \sum_{j,k=1}^{m} \frac{\partial^2 \varphi}{\partial w_j \partial w_k}(w)[(\mid A_1 \mid^2 + \mid A_2 \mid^2)\overline{\varphi}(w) - A_1\overline{g_1}(z) - A_2\overline{g_2}(z)]\alpha_j\alpha_k \mid \leq$$

$$(\mid A_1 \mid^2 + \mid A_2 \mid^2) \mid \sum_{j=1}^{m} \frac{\partial \varphi}{\partial w_j}(w)\alpha_j \mid^2$$

for all $z \in \mathbb{C}^n$.

It follows that $(\overline{A_1}g_1 + \overline{A_2}g_2)$ is analytic and bounded on $\mathbb{C}^n$.

Therefore $(\overline{A_1}g_1 + \overline{A_2}g_2)$ is constant on $\mathbb{C}^n$, by Liouville theorem.

Thus $A_1\overline{g_1}(z) + A_2\overline{g_2}(z) = -\overline{c}(\mid A_1 \mid^2 + \mid A_2 \mid^2)$, for all $z \in \mathbb{C}^n$, where $c \in \mathbb{C}$.

Now we have
$\mid \sum_{j,k=1}^{m} \frac{\partial^2 \varphi}{\partial w_j \partial w_k}(w)(\mid A_1 \mid^2 + \mid A_2 \mid^2)(\overline{\varphi}(w) + \overline{c})\alpha_j\alpha_k \mid \leq$
$(\mid A_1 \mid^2 + \mid A_2 \mid^2) \mid \sum_{j=1}^{m} \frac{\partial \varphi}{\partial w_j}(w)\alpha_j \mid^2 .$

Thus
$\mid \sum_{j,k=1}^{m} \frac{\partial^2 \varphi}{\partial w_j \partial w_k}(w)(\overline{\varphi}(w) + \overline{c})\alpha_j\alpha_k \mid \leq \mid \sum_{j=1}^{m} \frac{\partial \varphi}{\partial w_j}(w)\alpha_j \mid^2$, $\forall w = (w_1, ..., w_m) \in \mathbb{C}^m$.

Consequently, $\mid \varphi + c \mid^2$ is convex on $\mathbb{C}^m$. This follows from [4].

Theorem 2. Assume that u is convex on $\mathbb{C}^n \times \mathbb{C}^m$ and φ is not constant in $\mathbb{C}^m$. Then there exists a constant $c \in \mathbb{C}$ such that $\mid \varphi + c \mid^2$ is convex on $\mathbb{C}^m$. c is independent of n, g_1, g_2 and we have the following three cases.

(A) $\varphi(w) =< w/a > +b$, for all $w \in \mathbb{C}^m$, where $a \in \mathbb{C}^m \backslash \{0\}$ and $b \in \mathbb{C}$. Then we have the representation

$$\begin{cases} g_1(z) = A_1(< z/\lambda > +\mu) + \overline{A_2}(< z/\lambda_1 > +\mu_1)^s \\ g_2(z) = A_2(< z/\lambda > +\mu) - \overline{A_1}(< z/\lambda_1 > +\mu_1)^s \end{cases}$$

(for all $z \in \mathbb{C}^n$, where $\lambda, \lambda_1 \in \mathbb{C}^n$, $\mu, \mu_1 \in \mathbb{C}$, $s \in \mathbb{N}$), or

$$\begin{cases} g_1(z) = A_1(< z/\lambda_2 > +\mu_2) + \overline{A_2}e^{(<z/\lambda_3>+\mu_3)} \\ g_2(z) = A_2(< z/\lambda_2 > +\mu_2) - \overline{A_1}e^{(<z/\lambda_3>+\mu_3)} \end{cases}$$

(for all $z \in \mathbb{C}^n$, where $\lambda_2, \lambda_3 \in \mathbb{C}^n$, $\mu_2, \mu_3 \in \mathbb{C}$).

(B) $\varphi(w) = (< w/a > +b)^k - c$, for all $w \in \mathbb{C}^m$, with $a \in \mathbb{C}^m \backslash \{0\}$, $b \in \mathbb{C}$, $k \in \mathbb{N}$, $k \geq 2$. We have then the representation

$$\begin{cases} g_1(z) = -A_1 c + \overline{A_2}(< z/\lambda > +\mu)^s \\ g_2(z) = -A_2 c - \overline{A_1}(< z/\lambda > +\mu)^s \end{cases}$$

(for all $z \in \mathbb{C}^n$, where $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}, s \in \mathbb{N}$), or

$$\begin{cases} g_1(z) = -A_1 c + \overline{A_2}e^{(<z/\lambda_1>+\mu_1)} \\ g_2(z) = -A_2 c - \overline{A_1}e^{(<z/\lambda_1>+\mu_1)} \end{cases}$$

(for all $z \in \mathbb{C}^n$, where $\lambda_1 \in \mathbb{C}^n$, $\mu_1 \in \mathbb{C}$).

(C) $\varphi(w) = e^{(<w/a>+b)} - c$, for all $w \in \mathbb{C}^m$, with $a \in \mathbb{C}^m \backslash \{0\}$ and $b \in \mathbb{C}$. Then we have the representation

$$\begin{cases} g_1(z) = -A_1 c + \overline{A_2}(< z/\lambda > +\mu)^s \\ g_2(z) = -A_2 c - \overline{A_1}(< z/\lambda > +\mu)^s \end{cases}$$

(for all $z \in \mathbb{C}^n$, where $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}, s \in \mathbb{N}$), or

$$\begin{cases} g_1(z) = -A_1 c + \overline{A_2}e^{(<z/\lambda_1>+\mu_1)} \\ g_2(z) = -A_2 c - \overline{A_1}e^{(<z/\lambda_1>+\mu_1)} \end{cases}$$

(for all $z \in \mathbb{C}^n$, where $\lambda_1 \in \mathbb{C}^n$, $\mu_1 \in \mathbb{C}$).

The proof follows from the case $m = 1$, (see also [3]).

We can also consider the study of prh functions and establish several representations.

The following lemma is fundamental, we will use it as an important tool in pluripotential theory and in this paper.

Lemma 1. Let $g = (g_1, ..., g_N)$ and $f = (f_1, ..., f_N)$ be 2 holomorphic functions on D, where $N \geq 1$, D is a domain of $\mathbb{C}^n$, $n \geq 1$, $(f_j, g_j : D \to \mathbb{C})$, for all $j \in \{1, ..., N\}$. Then $\| g + \overline{f} \|^2$ and $(\| g \|^2 + \| f \|^2)$ have the same hermitian Levi form on D.
On the other hand, let $u : D \to \mathbb{R}$ be a function of class C^2. Define $u_1 = (u + \| g + \overline{f} \|^2)$, $u_2 = (u + \| f \|^2 + \| g \|^2)$.
Then u_1 and u_2 are functions of class C^2 on D and we have the assertion.
u_1 is strictly psh on D if and only if u_2 is strictly psh on D.
(Observe that if $N < n$, then $\| g \|^2$ is not strictly psh at each point of D).
Proof. We have $\| g + \overline{f} \|^2 = | g_1 + \overline{f_1} |^2 + ... + | g_N + \overline{f_N} |^2 = | g_1 |^2 + | f_1 |^2 + ... + | g_N |^2 + | f_N |^2 + \sum_{j=1}^{N} (g_j f_j + \overline{g_j} \overline{f_j}) = \| g \|^2 + \| f \|^2 + \sum_{j=1}^{N} (g_j f_j + \overline{g_j} \overline{f_j})$.

Since $(g_j f_j + \overline{g_j} \overline{f_j})$ is prh on D, then $\sum_{j=1}^{N} (g_j f_j + \overline{g_j} \overline{f_j})$ is prh on D.
Consequently, $\| g + \overline{f} \|^2$ and $(\| g \|^2 + \| f \|^2)$ have the same hermitian Levi form on D.
Several fundamental properties can be deduced from the above lemma 1. As an example, we cite theorem 4, theorem 5 and theorem 6.
Theorem 3. The following assertions are equivalent
(A) u is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$;
(B) $n = m = 1$ and we have the following two cases.
(I) $\varphi(w) = aw + b$, with $a \in \mathbb{C} \backslash \{0\}$, $b \in \mathbb{C}$ and we have the representation

$$\begin{cases} g_1(z) = A_1(\lambda z + \mu) + \overline{A}_2 \varphi_1(z) \\ g_2(z) = A_2(\lambda z + \mu) - \overline{A}_1 \varphi_1(z) \end{cases}$$

(for all $z \in \mathbb{C}$, with $\lambda, \mu \in \mathbb{C}$ and $\varphi_1 : \mathbb{C} \to \mathbb{C}$ be analytic affine bijective on $\mathbb{C}$), or

$$\begin{cases} g_1(z) = A_1(\lambda_2 z + \mu_2) + \overline{A}_2 e^{\varphi_2(z)} \\ g_2(z) = A_2(\lambda_2 z + \mu_2) - \overline{A}_1 e^{\varphi_2(z)} \end{cases}$$

(for all $z \in \mathbb{C}$, with $\lambda_2, \mu_2 \in \mathbb{C}, \varphi_2 : \mathbb{C} \to \mathbb{C}$ be analytic affine bijective on $\mathbb{C}$).
(II) $\varphi(w) = e^{(aw+b)} - c$, for all $w \in \mathbb{C}$, with $a \in \mathbb{C} \backslash \{0\}$, $b \in \mathbb{C}$ and

$c \in \mathbb{C}$.
Then we have the representation

$$\begin{cases} g_1(z) = \overline{A_2}(\gamma_1 z + \delta_1) - A_1 c \\ g_2(z) = -\overline{A_1}(\gamma_1 z + \delta_1) - A_2 c \end{cases}$$

(for all $z \in \mathbb{C}$, where $\gamma_1 \in \mathbb{C}\backslash\{0\}$, $\delta_1 \in \mathbb{C}$), or

$$\begin{cases} g_1(z) = \overline{A_2} e^{(\gamma_2 z + \delta_2)} - A_1 c \\ g_2(z) = -\overline{A_1} e^{(\gamma_2 z + \delta_2)} - A_2 c \end{cases}$$

(for all $z \in \mathbb{C}$, where $\gamma_2 \in \mathbb{C}\backslash\{0\}$, $\delta_2 \in \mathbb{C}$).

Proof. (A) implies (B). u is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}^m$. $u(0,.)$ is convex on $\mathbb{C}^m$. Then
$\mid \sum_{j,k=1}^{m} \frac{\partial^2 u}{\partial w_j \partial w_k}(0,w)\alpha_j \alpha_k \mid \leq \sum_{j,k=1}^{m} \frac{\partial^2 u}{\partial w_j \partial \overline{w_k}}(0,w)\alpha_j \overline{\alpha_k}$, $\forall w = (w_1, ..., w_m) \in \mathbb{C}^m$, $\forall \alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{C}^m$.
Thus
$\mid \sum_{j,k=1}^{m} \frac{\partial^2 \varphi}{\partial w_j \partial w_k}(w)[(\mid A_1 \mid^2 + \mid A_2 \mid^2)\overline{\varphi}(w) - A_1\overline{g_1}(0) - A_2\overline{g_2}(0)]\alpha_j \alpha_k \mid \leq$
$(\mid A_1 \mid^2 + \mid A_2 \mid^2) \mid \sum_{j=1}^{m} \frac{\partial \varphi}{\partial w_j}(w)\alpha_j \mid^2$.
Then
$\mid \sum_{j,k=1}^{m} \frac{\partial^2 \varphi}{\partial w_j \partial w_k}(w)[\overline{\varphi}(w) + c]\alpha_j \alpha_k \mid \leq \mid \sum_{j=1}^{m} \frac{\partial \varphi}{\partial w_j}(w)\alpha_j \mid^2$, $\forall w = (w_1, ..., w_m) \in \mathbb{C}^m$, $\forall \alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{C}^m$, where $c \in \mathbb{C}$. Therefore $\mid \varphi + c \mid^2$ is convex on $\mathbb{C}^m$.

Since $u(0,.)$ is strictly psh on $\mathbb{C}^m$, then $\mid \sum_{j=1}^{m} \frac{\partial \varphi}{\partial w_j}(w)\alpha_j \mid^2 > 0$, $\forall w = (w_1, ..., w_m) \in \mathbb{C}^m$, $\forall \alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{C}^m\backslash\{0\}$. Therefore $m = 1$.
Consequently, $\mid \varphi + c \mid^2$ is convex and strictly sh on $\mathbb{C}$. By Abidi [2], it follows that
$\varphi(w) = aw + b$, for all $w \in \mathbb{C}$, where $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$, or
$\varphi(w) = e^{(a_1 w + b_1)} - c$, for all $w \in \mathbb{C}$, where $a_1 \in \mathbb{C}\backslash\{0\}$ and $b_1 \in \mathbb{C}$.
(I) $\varphi(w) = aw + b$, for all $w \in \mathbb{C}$. Then we have the representation

$$\begin{cases} g_1(z) = A_1(< z/\lambda > + \mu) + \overline{A_2}\varphi_1(z) \\ g_2(z) = A_2(< z/\lambda > + \mu) - \overline{A_1}\varphi_1(z) \end{cases}$$

(for all $z \in \mathbb{C}^n$, with $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$, $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ be analytic, $\mid \varphi_1 \mid^2$ is convex on $\mathbb{C}^n$).
Then $u(z,w) = (\mid A_1 \mid^2 + \mid A_2 \mid^2)(\mid aw + b- < z/\lambda > -\mu \mid^2 + \mid \varphi_1(z) \mid^2)$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Put $T(z,w) = (z, w + \frac{1}{a} < z/\lambda >)$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. T is a $\mathbb{C}$ linear bijective transformation on $\mathbb{C}^n \times \mathbb{C}$. Let $u_1 = \frac{1}{(|A_1|^2+|A_2|^2)} uoT$.
u_1 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$ and we have
u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
But $u_1(z,w) = \mid aw + b - \mu \mid^2 + \mid \varphi_1(z) \mid^2$.
Observe now that u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $\mid \varphi_1 \mid^2$ is strictly psh on $\mathbb{C}^n$. Therefore $n = 1$ and $\mid \varphi_1 \mid^2$ is convex and strictly sh on $\mathbb{C}$.
Thus $\varphi_1(z) = \lambda_3 z + \mu_3$, for all $z \in \mathbb{C}$, where $\lambda_3 \in \mathbb{C}\backslash\{0\}$ and $\mu_3 \in \mathbb{C}$, or
$\varphi_1(z) = e^{(\lambda_4 z + \mu_4)}$, for all $z \in \mathbb{C}$, with $\lambda_4 \in \mathbb{C}\backslash\{0\}$ and $\mu_4 \in \mathbb{C}$.
(II) $\varphi(w) = e^{(aw+b)} - c$, for all $w \in \mathbb{C}$ $(a_1 = a, b_1 = b)$.
By theorem 2, we have

$$\begin{cases} g_1(z) = -A_1 c + \overline{A_2}\varphi_1(z) \\ g_2(z) = -A_2 c - \overline{A_1}\varphi_1(z) \end{cases}$$

for all $z \in \mathbb{C}^n$, with $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ be analytic, $\mid \varphi_1 \mid^2$ is convex on $\mathbb{C}^n$.
$u(z,w) = \mid A_1 e^{(aw+b)} - \overline{A_2}\varphi_1(z) \mid^2 + \mid A_2 e^{(aw+b)} + \overline{A_1}\varphi_1(z) \mid^2 =$
$(\mid A_1 \mid^2 + \mid A_2 \mid^2)(\mid e^{(aw+b)} \mid^2 + \mid \varphi_1(z) \mid^2)$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.
Observe that u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $\mid \varphi_1 \mid^2$ is strictly psh on $\mathbb{C}^n$. Therefore $n = 1$. Consequently, $\mid \varphi_1 \mid^2$ is convex and strictly sh on $\mathbb{C}$. Thus
$\varphi_1(z) = \gamma_1 z + \delta_1$, for all $z \in \mathbb{C}$, where $\gamma_1 \in \mathbb{C}\backslash\{0\}$ and $\delta_1 \in \mathbb{C}$, or
$\varphi_1(z) = e^{(\gamma_2 z + \delta_2)}$, for all $z \in \mathbb{C}$, with $\gamma_2 \in \mathbb{C}\backslash\{0\}$ and $\delta_2 \in \mathbb{C}$.
Theorem 4. The following conditions are equivalent
(A) v is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$;
(B) $m = 1$, $n \in \{1,2\}$ and we have the following two cases.
(I) $\varphi(w) = aw + b$, for all $w \in \mathbb{C}$, where $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$.

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda > +\mu) + A_2\varphi_1(z) \\ g_2(z) = \overline{A_2}(< z/\lambda > +\mu) - A_1\varphi_1(z) \end{cases}$$

$(\forall z \in \mathbb{C}^n$, with $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C})$, $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ be analytic, $\mid \varphi_1 \mid^2$ is convex on $\mathbb{C}^n$ such that
Case 1. $\varphi_1(z) = (< z/a_1 > +b_1)^{s_1}$, for all $z \in \mathbb{C}^n$, where $a_1 \in \mathbb{C}^n$, $b_1 \in \mathbb{C}$, $s_1 \in \mathbb{N}$. Then $(n = 1, \lambda \neq 0)$, or $(n = 1, s_1 = 1$ and $a_1 \neq 0)$, or $(n = 2, s_1 = 1$ and (λ, a_1) is a basis of the complex vector space $\mathbb{C}^2)$.

Case 2. $\varphi_1(z) = e^{(<z/a_2>+b_2)}$, for all $z \in \mathbb{C}^n$, where $a_2 \in \mathbb{C}^n$ and $b_2 \in \mathbb{C}$. Then
($n = 1$ and $\lambda \neq 0$), or ($n = 1$ and $a_2 \neq 0$), or ($n = 2$ and (λ, a_2) is a basis of the complex vector space $\mathbb{C}^2$).
(II) $\varphi(w) = e^{(aw+b)} - c$, for all $w \in \mathbb{C}$, where $a \in \mathbb{C}\backslash\{0\}$, $b \in \mathbb{C}$ and $c \in \mathbb{C}$.
Then $n = 1$ and

$$\begin{cases} g_1(z) = A_2(\gamma_1 z + \delta_1) - \overline{A_1}\overline{c} \\ g_2(z) = -A_1(\gamma_1 z + \delta_1) - \overline{A_2}\overline{c} \end{cases}$$

($\forall z \in \mathbb{C}$, with $\gamma_1 \in \mathbb{C}\backslash\{0\}, \delta_1 \in \mathbb{C}$), or

$$\begin{cases} g_1(z) = A_2 e^{(\gamma_2 z+\delta_2)} - \overline{A_1}\overline{c} \\ g_2(z) = -A_1 e^{(\gamma_2 z+\delta_2)} - \overline{A_2}\overline{c} \end{cases}$$

($\forall z \in \mathbb{C}$, with $\gamma_2 \in \mathbb{C}\backslash\{0\}$ and $\delta_2 \in \mathbb{C}$).
Proof. (A) implies (B). Note that v is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}^m$. The function $v(0, .)$ is then strictly psh on $\mathbb{C}^m$.
Therefore $0 < \sum_{j,k=1}^{m} \frac{\partial^2 v}{\partial w_j \partial \overline{w_k}}(0, w)\alpha_j \overline{\alpha_k}$, $\forall w = (w_1, ..., w_m) \in \mathbb{C}^m$, $\forall \alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{C}^m\backslash\{0\}$. Thus $\mid \sum_{j=1}^{m} \frac{\partial \varphi}{\partial w_j}(0)\alpha_j \mid > 0$, $\forall \alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{C}^m\backslash\{0\}$.
Then $m = 1$, because if $m \geq 2$, there exists always $\beta = (\beta_1, ..., \beta_m) \in \mathbb{C}^m\backslash\{0\}$ such that $\sum_{j=1}^{m} \frac{\partial \varphi}{\partial w_j}(0)\beta_j = 0$. By theorem 1, there exists $c \in \mathbb{C}$ such that $\mid \varphi + c \mid^2$ is convex on $\mathbb{C}$. Since $\mid \varphi \mid^2$ is strictly sh on $\mathbb{C}$, then $\mid \varphi + c \mid^2$ is convex and strictly sh on $\mathbb{C}$. It follows that
$\varphi(w) = aw + b$, for all $w \in \mathbb{C}$, where $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$, or
$\varphi(w) = e^{(c_1 w+d_1)} - c$, for all $w \in \mathbb{C}$, with $c_1 \in \mathbb{C}\backslash\{0\}$ and $d_1 \in \mathbb{C}$.
(I) $\varphi(w) = aw + b$, for all $w \in \mathbb{C}$.
Let $T(z, w) = (z, \overline{w})$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. T is an $\mathbb{R}$ linear bijective transformation on $\mathbb{C}^n \times \mathbb{C}$. Since v is convex on $\mathbb{C}^n \times \mathbb{C}$, then $u = voT$ is convex and of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. $u(z, w) = \mid \overline{A_1}\overline{a}w - (g_1(z) - \overline{A_1 b}) \mid^2 + \mid \overline{A_2}\overline{a}w - (g_2(z) - \overline{A_2 b}) \mid^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.
By theorem 3, it follows that

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda > +\mu) + A_2\varphi_1(z) \\ g_2(z) = \overline{A_2}(< z/\lambda > +\mu) - A_1\varphi_1(z) \end{cases}$$

$\forall z \in \mathbb{C}^n$, with $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$, $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ be analytic, $\mid \varphi_1 \mid^2$ is convex on $\mathbb{C}^n$.
$v(z,w) = (\mid A_1 \mid^2 + \mid A_2 \mid^2)(\mid \varphi(w) - \overline{< z/\lambda >} - \overline{\mu} \mid^2 + \mid \varphi_1(z) \mid^2)$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.
Let $v_1(z,w) = \mid \varphi(w) - \overline{< z/\lambda >} - \overline{\mu} \mid^2 + \mid \varphi_1(z) \mid^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.
v_1 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. We have
v is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v_1 is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
Define $v_2(z) = \mid < z/\lambda > \mid^2 + \mid \varphi_1(z) \mid^2$, $z \in \mathbb{C}^n$. v_2 is a function of class C^∞ on $\mathbb{C}^n$ and we have the assertion
v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v_2 is strictly psh on $\mathbb{C}^n$.
By the lemma 1, we have $n \leq 2$. Then $n \in \{1,2\}$.
The Levi hermitian form of v_2 is
$L(v_2)(z)(\alpha) = \mid < \alpha/\lambda > \mid^2 + \mid \sum_{j=1}^{n} \frac{\partial \varphi_1}{\partial z_j}(z)\alpha_j \mid^2$, $z, \alpha \in \mathbb{C}^n$.
Case 1. $\varphi_1(z) = (< z/a_1 > + b_1)^{s_1}$, for all $z \in \mathbb{C}^n$, where $a_1 \in \mathbb{C}^n$, $b_1 \in \mathbb{C}$ and $s_1 \in \mathbb{N}$.
$L(v_2)(z)(\alpha) = \mid < \alpha/\lambda > \mid^2 + s_1^2 \mid < \alpha/a_1 > \mid^2 \mid < z/a_1 > + b_1 \mid^{2s_1-2} > 0$, for all $z \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}^n \backslash \{0\}$.
Then ($n = 1$ and $\alpha \neq 0$), or ($n = 1, s_1 = 1$ and $a_1 \neq 0$), or ($n = 2, s_1 = 1$ and (α, a_1) is a basis of the complex vector space $\mathbb{C}^2$).
Case 2. $\varphi_1(z) = e^{(<z/a_2> + b_2)}$, for all $z \in \mathbb{C}^n$, where $a_2 \in \mathbb{C}^n$ and $b_2 \in \mathbb{C}$.
$L(v_2)(z)(\alpha) = \mid < \alpha/\lambda > \mid^2 + \mid < \alpha/a_2 > \mid^2 \mid e^{(<z/a_2> + b_2)} \mid^2 > 0$, $\forall z \in \mathbb{C}^n$, $\forall \alpha \in \mathbb{C}^n \backslash \{0\}$.
Therefore ($n = 1$ and $\lambda \neq 0$), or ($n = 1$ and $a_2 \neq 0$), or ($n = 2$ and (λ, a_2) is a basis of the complex vector space $\mathbb{C}^2$).
(II) $\varphi(w) = e^{(aw+b)} - c$, for all $w \in \mathbb{C}$, $(a = c_1, b = d_1)$.
By theorem 3, we have

$$\begin{cases} g_1(z) = A_2\varphi_1(z) - \overline{A_1}\overline{c} \\ g_2(z) = -A_1\varphi_1(z) - \overline{A_2}\overline{c} \end{cases}$$

$\forall z \in \mathbb{C}^n$, with $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ be analytic, $\mid \varphi_1 \mid^2$ is convex on $\mathbb{C}^n$.
$v(z,w) = (\mid A_1 \mid^2 + \mid A_2 \mid^2)(\mid e^{(aw+b)} \mid^2 + \mid \varphi_1(z) \mid^2)$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.
Observe that v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $\mid \varphi_1 \mid^2$ is strictly psh on $\mathbb{C}^n$.
By lemma 1, we have $n = 1$. Consequently, $\mid \varphi_1 \mid^2$ is strictly sh on $\mathbb{C}$. Therefore
$\varphi_1(z) = (\gamma_1 z + \delta_1)$, for all $z \in \mathbb{C}$, where $\gamma_1 \in \mathbb{C}\backslash\{0\}$ and $\delta_1 \in \mathbb{C}$, or
$\varphi_1(z) = e^{(\gamma_2 z + \delta_2)}$, for all $z \in \mathbb{C}$, with $\gamma_2 \in \mathbb{C}\backslash\{0\}$ and $\delta_2 \in \mathbb{C}$.
(B) implies (A) is evident.

Theorem 5. The following assertions are equivalent
(A) u is a convex function on $\mathbb{C}^n \times \mathbb{C}^m$, v is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$ and u is not strictly psh on any not empty Euclidean open ball of $\mathbb{C}^n \times \mathbb{C}^m$;
(B) $m = 1$, $\varphi(w) = aw + b$, for all $w \in \mathbb{C}$, where $a \in \mathbb{C}\backslash\{0\}$, $b \in \mathbb{C}$ and $n \in \{1, 2\}$ with
($n = 1, \lambda \neq 0$ and φ_1 is constant on $\mathbb{C}$), or ($n = 2$ and $(\lambda, (\overline{\frac{\partial\varphi_1}{\partial z_1}(z)}, \overline{\frac{\partial\varphi_1}{\partial z_2}(z)}))$
is a basis of the complex vector space $\mathbb{C}^2, \forall z = (z_1, z_2) \in \mathbb{C}^2$), where

$$\begin{cases} g_1(z) = A_1(< z/\lambda > +\mu) + \overline{A_2}\varphi_1(z) \\ g_2(z) = A_2(< z/\lambda > +\mu) - \overline{A_1}\varphi_1(z) \end{cases}$$

for all $z \in \mathbb{C}^n$, where $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$, $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ be analytic, $\mid \varphi_1 \mid^2$ is convex on $\mathbb{C}^n$.

Proof. (A) implies (B). u and v are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}^m$. Since v is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$, then $v(0, .)$ is strictly psh on $\mathbb{C}^m$. Therefore $m = 1$ and consequently, $\mid \varphi \mid^2$ is strictly sh on $\mathbb{C}$.
Now since u is convex on $\mathbb{C}^n \times \mathbb{C}$, then $u(0, .)$ is convex on $\mathbb{C}$. It follows that $\mid \varphi + c \mid^2$ is convex on $\mathbb{C}$, where $c \in \mathbb{C}$.
Since φ is analytic on $\mathbb{C}$, then $\mid \varphi + c \mid^2$ is convex and strictly sh on $\mathbb{C}$. Consequently,
$\varphi(w) + c = aw + b$, $\forall w \in \mathbb{C}$, where $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$, or
$\varphi(w) + c = e^{(a_1 w + b_1)}$, $\forall w \in \mathbb{C}$, where $a_1 \in \mathbb{C}\backslash\{0\}$ and $b_1 \in \mathbb{C}$, by Abidi [4].
Case 1. $\varphi(w) = e^{(a_1 w + b_1)} - c$, for $w \in \mathbb{C}$.
Put $k_1 = A_1 c + g_1$, $k_2 = A_2 c + g_2$; k_1 and k_2 are holomorphic functions on $\mathbb{C}^n$.
After an holomorphic affine change of variable, the function ψ is C^∞ and convex on $\mathbb{C}^n \times \mathbb{C}$, $\psi(z, w) = \mid A_1 e^w - k_1(z) \mid^2 + \mid A_2 e^w - k_2(z) \mid^2$.
Fix $z \in \mathbb{C}^n$. Since $\psi_1 = \psi(z, .)$ is convex on $\mathbb{C}$, then

$$\mid \frac{\partial^2\psi_1}{\partial w^2}(w) \mid \leq \frac{\partial^2\psi_1}{\partial w \partial \overline{w}}(w)$$

for any $w \in \mathbb{C}$. Thus
$\mid (\mid A_1 \mid^2 + \mid A_2 \mid^2) e^{(w+\overline{w})} - e^w(A_1\overline{k_1}(z) + A_2\overline{k_2}(z)) \mid \leq$
$(\mid A_1 \mid^2 + \mid A_2 \mid^2) \mid e^w \mid^2$, for every $w \in \mathbb{C}$.
It follows that
$\mid (\mid A_1 \mid^2 + \mid A_2 \mid^2) e^{(\overline{w})} - (A_1\overline{k_1}(z) + A_2\overline{k_2}(z)) \mid \leq$
$(\mid A_1 \mid^2 + \mid A_2 \mid^2) \mid e^w \mid$, for each $w \in \mathbb{C}$.
For $w = x_1 \in \mathbb{R}$, we have $\lim_{x_1 \to (-\infty)} e^{x_1} = 0$. Then $(\overline{A_1}k_1 + \overline{A_2}k_2) = 0$ on

$\mathbb{C}^n$. Since $k_2 = -\frac{\overline{A_1}}{\overline{A_2}} k_1$, we can prove that $k_1 = \overline{A_2}\varphi_1$ and $k_2 = -\overline{A_1}\varphi_1$, where $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ be analytic and $\mid \varphi_1 \mid^2$ is convex on $\mathbb{C}^n$.
We prove that this case is impossible.
For the rest of the proof we use a similar technical idea developed at the proof of theorem 4.

3. A characterization satisfying some specific conditions

Using the same notation of the below theorem 6. We show that we have a great differences in the theory of convex and strictly psh functions (which involve the complex structure), between the class of functions defined like u_1 and the family of functions like v. Precisely, we prove that there exist a family of convex functions belonging to the class of functions like u_1 and u_2 such that v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, but u is not. On the other hand, when the number N of functions like u_1 or (u_2) satisfy $N \geq 2$, we show the desired result. **Theorem 6.** Let $g_1, f_1, g_2, f_2 : \mathbb{C}^n \to \mathbb{C}$ be 4 analytic functions, $n \geq 1$ and $A_1, A_2, B_1, B_2 \in \mathbb{C}\backslash\{0\}$. Put $u_1(z,w) =\mid A_1 w - g_1(z) \mid^2 + \mid B_1 w - f_1(z) \mid^2$, $u_2(z,w) =\mid A_2 w - g_2(z) \mid^2 + \mid B_2 w - f_2(z) \mid^2$, $u = u_1 + u_2$, $v(z,w) = \displaystyle\int_{B(0,1)} u(z+\xi, w) dm_{2n}(\xi)$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.
The following conditions are equivalent
(A) $n = 3$, u_1 and u_2 are convex functions on $\mathbb{C}^3 \times \mathbb{C}$, v is strictly psh on $\mathbb{C}^3 \times \mathbb{C}$ but u is not strictly psh on $\mathbb{C}^3 \times \mathbb{C}$;
(B) We have one of the following 3 cases.
Case 1. We have $B_1 g_1(z) - A_1 f_1(z) = (\mid A_1 \mid^2 + \mid A_2 \mid^2)(< z/\lambda_1 > +\mu_1)^{s_1}$, $B_2 g_2(z) - A_2 f_2(z) = (\mid A_2 \mid^2 + \mid B_2 \mid^2)(< z/\lambda_2 > +\mu_2)^{s_2}$, $\overline{A_1} g_1(z) + \overline{B_1} f_1(z) = (\mid A_1 \mid^2 + \mid B_1 \mid^2)(< z/a_1 > +b_1)$, $\overline{A_2} g_2(z) + \overline{B_2} f_2(z) = (\mid A_2 \mid^2 + \mid B_2 \mid^2)(< z/a_2 > +b_2)$, for all $z \in \mathbb{C}^n$, where $\lambda_1, \lambda_2, a_1, a_2 \in \mathbb{C}^n$, $\mu_1, \mu_2, b_1, b_2 \in \mathbb{C}$ and $s_1, s_2 \in \mathbb{N}$.
$(a_1 - a_2, \lambda_1, \lambda_2)$ is a basis of the complex vector space $\mathbb{C}^3$ and $s_1 \geq 2$ or $s_2 \geq 2$.
Case 2. We have $B_1 g_1(z) - A_1 f_1(z) = (\mid A_1 \mid^2 + \mid B_1 \mid^2)(< z/\lambda_1 > +\mu_1)^{s_1}$, $B_2 g_2(z) - A_2 f_2(z) = (\mid A_2 \mid^2 + \mid B_2 \mid^2) e^{(< z/\gamma_2 > + \delta_2)}$, $\overline{A_1} g_1(z) + \overline{B_1} f_1(z) = (\mid A_1 \mid^2 + \mid B_1 \mid^2)(< z/a_1 > +b_1)$, $\overline{A_2} g_2(z) + \overline{B_2} f_2(z) = (\mid A_2 \mid^2 + \mid B_2 \mid^2)(< z/a_2 > +b_2)$, for all $z \in \mathbb{C}^n$, where $\lambda_1, \gamma_2, a_1, a_2 \in \mathbb{C}^n$, $\mu_1, \delta_2, b_1, b_2 \in \mathbb{C}$ and $s_1 \in \mathbb{N}, s_1 \geq 2$.
$(a_1 - a_2, \lambda_1, \gamma_2)$ is a basis of the complex vector space $\mathbb{C}^3$.
Case 3. $B_1 g_1(z) - A_1 f_1(z) = (\mid A_1 \mid^2 + \mid B_1 \mid^2) e^{(< z/\gamma_1 > + \delta_1)}$,

$B_2g_2(z) - A_2f_2(z) = (| A_2 |^2 + | B_2 |^2)(< z/\lambda_2 > +\mu_2)^{s_2}$, $\overline{A_1}g_1(z) + \overline{B_1}f_1(z) = (| A_1 |^2 + | B_1 |^2)(< z/a_1 > +b_1)$, $\overline{A_2}g_2(z) + \overline{B_2}f_2(z) = (| A_2 |^2 + | B_2 |^2)(< z/a_2 > +b_2)$, for all $z \in \mathbb{C}^n$, where $\gamma_1, \lambda_2, a_1, a_2 \in \mathbb{C}^n$, $\delta_1, \mu_2, b_1, b_2 \in \mathbb{C}$ and $s_2 \in \mathbb{N}$ with $s_2 \geq 2$. $(a_1 - a_2, \gamma_1, \lambda_2)$ is a basis of the complex vector space $\mathbb{C}^3$.

Proof. (A) implies (B). Denote by $(| A_1 |^2 + | B_1 |^2)\varphi_1 = (B_1g_1 - A_1f_1)$ and $(| A_2 |^2 + | B_2 |^2)\varphi_2 = (B_2g_2 - A_2f_2)$. φ_1 and φ_2 are analytic functions on $\mathbb{C}^3$. By theorem 2, $| \varphi_1 |$ and $| \varphi_2 |$ are convex functions on $\mathbb{C}$, $(\overline{A_1}g_1 + \overline{B_1}f_1)$ and $(\overline{A_2}g_2 + \overline{B_2}f_2)$ are affine functions on $\mathbb{C}^3$.

Therefore $(\overline{A_1}g_1(z) + \overline{B_1}f_1(z)) = (| A_1 |^2 + | B_1 |^2)(< z/a_1 > +b_1)$, $(\overline{A_2}g_2(z) + \overline{B_2}f_2(z)) = (| A_2 |^2 + | B_2 |^2)(< z/a_2 > +b_2)$, for all $z \in \mathbb{C}^n$, where $a_1, a_2 \in \mathbb{C}^n$, $b_1, b_2 \in \mathbb{C}$.

We have $u(z, w) = (| A_1 |^2 + | B_1 |^2)(| w- < z/a_1 > -b_1 |^2 + | \varphi_1(z) |^2) + (| A_2 |^2 + | B_2 |^2)(| w- < z/a_2 > -b_2 |^2 + | \varphi_2(z) |^2)$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.

$v(z, w) = (| A_1 |^2 + | B_1 |^2)(\int_{B(0,1)} | w- < z/a_1 > - < \xi/a_1 > -b_1 |^2 dm_6(\xi) + \int_{B(0,1)} | \varphi_1(z + \xi) |^2 dm_6(\xi)) + (| A_2 |^2 + | B_2 |^2)(\int_{B(0,1)} | w+ - < z/a_2 > - < \xi/a_2 > -b_2 |^2 dm_6(\xi) + \int_{B(0,1)} | \varphi_2(z + \xi) |^2 dm_6(\xi))$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.

Define

$v_1(z, w) = | w- < z/a_1 > -b_1 |^2 + | \varphi_1(z) |^2 + | w- < z/a_2 > -b_2 |^2 + | \varphi_2(z) |^2$, $v_2(z, w) = \int_{B(0,1)} | w- < z/a_1 > - < \xi/a_1 > -b_1 |^2 dm_6(\xi) + \int_{B(0,1)} | \varphi_1(z + \xi) |^2 dm_6(\xi) + \int_{B(0,1)} | w- < z/a_2 > - < \xi/a_2 > -b_2 |^2 dm_6(\xi) + \int_{B(0,1)} | \varphi_2(z + \xi) |^2 dm_6(\xi)$, $(z, w) \in \mathbb{C}^3 \times \mathbb{C}$.

u, v, v_1, v_2 are functions of class C^∞ on $\mathbb{C}^3 \times \mathbb{C}$.

Note that u is strictly psh on $\mathbb{C}^3 \times \mathbb{C}$ if and only if v_1 is strictly psh on $\mathbb{C}^3 \times \mathbb{C}$.

v is strictly psh on $\mathbb{C}^3 \times \mathbb{C}$ if and only if v_2 is strictly psh on $\mathbb{C}^3 \times \mathbb{C}$.

Let $T : \mathbb{C}^3 \times \mathbb{C} \to \mathbb{C}^3 \times \mathbb{C}$, $T(z, w) = (z, w+ < z/a_1 >)$, for $(z, w) \in \mathbb{C}^3 \times \mathbb{C}$.

T is a $\mathbb{C}$ linear bijective transformation on $\mathbb{C}^3 \times \mathbb{C}$.

Define $v_3 = v_1 oT$, $v_4 = v_2 oT$. v_3 and v_4 are functions of class C^∞ on

$\mathbb{C}^3 \times \mathbb{C}$.
We have v_1 is strictly psh on $\mathbb{C}^3 \times \mathbb{C}$ if and only if v_3 is strictly psh on $\mathbb{C}^3 \times \mathbb{C}$.
v_2 is strictly psh on $\mathbb{C}^3 \times \mathbb{C}$ if and only if v_4 is strictly psh on $\mathbb{C}^3 \times \mathbb{C}$.
$v_3(z,w) =\mid w - b_1 \mid^2 + \mid \varphi_1(z) \mid^2 + \mid w + < z/a_1 - a_2 > -b_2 \mid^2 + \mid \varphi_2(z) \mid^2$, $(z,w) \in \mathbb{C}^3 \times \mathbb{C}$.
By an examination of the hermitian Levi form of v_3, we observe that v_3 is strictly psh on $\mathbb{C}^3 \times \mathbb{C}$ if and only if v_5 is strictly psh on $\mathbb{C}^3$, where $v_5(z) = \mid \varphi_1(z) \mid^2 + \mid < z/a_1 - a_2 > -b_2 \mid^2 + \mid \varphi_2(z) \mid^2$, $z \in \mathbb{C}^3$, (v_5 is a C^∞ function on $\mathbb{C}^3$).
Put $v_6(z) = \int_{B(0,1)} \mid \varphi_1(z+\xi) \mid^2 dm_6(\xi) + \int_{B(0,1)} \mid < z/a_1 - a_2 > + - < \xi/a_2 > -b_2 \mid^2 dm_6(\xi) + \int_{B(0,1)} \mid \varphi_2(z+\xi) \mid^2 dm_6(\xi)$, $z \in \mathbb{C}^3$.
v_6 is a function of class C^∞ on $\mathbb{C}^3$.
Observe that v_4 is strictly psh on $\mathbb{C}^3 \times \mathbb{C}$ if and only if v_6 is strictly psh on $\mathbb{C}^3$.
The Levi hermitian form of v_5 is

$$L(v_5)(z)(\alpha) =\mid < \alpha/a_1 - a_2 >\mid^2 + \mid \sum_{j=1}^{3} \frac{\partial \varphi_1}{\partial z_j}(z)\alpha_j \mid^2 + \mid \sum_{j=1}^{3} \frac{\partial \varphi_2}{\partial z_j}(z)\alpha_j \mid^2,$$

$z = (z_1, z_2, z_3)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$.
The Levi hermitian form of v_6 is

$$L(v_6)(z)(\alpha) =\mid < \alpha/a_1 - a_2 >\mid^2 \int_{B(0,1)} 1 dm_6(\xi) + \int_{B(0,1)} \mid \sum_{j=1}^{3} \frac{\partial \varphi_1}{\partial z_j}(z+\xi)\alpha_j \mid^2 dm_6(\xi) + \int_{B(0,1)} \mid \sum_{j=1}^{3} \frac{\partial \varphi_2}{\partial z_j}(z+\xi)\alpha_j \mid^2 dm_6(\xi),$$

$z = (z_1, z_2, z_3)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$.
Case 1. $\varphi_1(z) = (< z/\lambda_1 > +\mu_1)^{s_1}$, $\varphi_2(z) = (< z/\lambda_2 > +\mu_2)^{s_2}$, for all $z \in \mathbb{C}^3$, where $\lambda_1, \lambda_2 \in \mathbb{C}^3$, $\mu_1, \mu_2 \in \mathbb{C}$ and $s_1, s_2 \in \mathbb{N}$.
We have

$$L(v_6)(z)(\alpha) =\mid < \alpha/a_1 - a_2 >\mid^2 \int_{B(0,1)} 1 dm_6(\xi) + s_1^2 \mid < \alpha/\lambda_1 >\mid^2 \int_{B(0,1)} \mid < z/\lambda_1 > + < \xi/\lambda_1 > +\mu_1 \mid^{2s_1-2} dm_6(\xi) + s_2^2 \mid < \alpha/\lambda_2 >\mid^2 \int_{B(0,1)} \mid < z/\lambda_2 > + < \xi/\lambda_2 > +\mu_2 \mid^{2s_2-2} dm_6(\xi),$$

$z, \alpha \in \mathbb{C}^3$.
Observe that if ($s_1 = 0$ or $\lambda_1 = 0$), then v_6 is not strictly psh at any point of $\mathbb{C}^3$.
Therefore $s_1 > 0$ and $\lambda_1 \neq 0$. Also we have $s_2 > 0$ and $\lambda_2 \neq 0$.
Fix $z \in \mathbb{C}^3$. Then
$\int_{B(0,1)} |< z/\lambda_1 > + < \xi/\lambda_1 > +\mu_1 |^{2s_1-2} dm_6(\xi) > 0$,
$\int_{B(0,1)} |< z/\lambda_2 > + < \xi/\lambda_2 > +\mu_2 |^{2s_2-2} dm_6(\xi) > 0$
and $\int_{B(0,1)} 1 dm_6(\xi) > 0$.
Therefore v_6 is strictly psh on $\mathbb{C}^3$ if and only if for all $(z, \alpha) \in \mathbb{C}^3 \times \mathbb{C}^3$, the condition $|< \alpha/a_1 - a_2 >|^2 + s_1^2 |< \alpha/\lambda_1 >|^2 + s_2^2 |< \alpha/\lambda_2 >|^2 = 0$ implies that $\alpha = 0 \in \mathbb{C}^3$.
Thus the system $< \alpha/a_1 - a_2 >= 0$, $< \alpha/\lambda_1 >= 0$, $< \alpha/\lambda_2 >= 0$ and $\alpha \in \mathbb{C}^3$ implies that $\alpha = 0$. It follows that $(a_1 - a_2, \lambda_1, \lambda_2)$ is a basis of the complex vector space $\mathbb{C}^3$.
We have $L(v_5)(z)(\alpha) = |< \alpha/a_1 - a_2 >|^2 +$
$s_1^2 |< \alpha/\lambda_1 >|^2 |< z/\lambda_1 > +\mu_1 |^{2s_1-2} + s_2^2 |< \alpha/\lambda_2 >|^2 |< z/\lambda_2 > +\mu_2 |^{2s_2-2}$.
Since u is not strictly psh on $\mathbb{C}^3$, then v_5 is not strictly psh on $\mathbb{C}^3$.
It follows that $s_1 \geq 2$ or $s_2 \geq 2$.
Case 2. $\varphi_1(z) = (< z/\lambda_1 > +\mu_1)^{s_1}$, $\varphi_2(z) = e^{(<z/\gamma_2>+\delta_2)}$, for all $z \in \mathbb{C}^3$.
$L(v_6)(\alpha) = |< \alpha/a_1 - a_2 >|^2 \int_{B(0,1)} 1 dm_6(\xi) +$
$s_1^2 |< \alpha/\lambda_1 >|^2 \int_{B(0,1)} |< z/\lambda_1 > + < \xi/\lambda_1 > +\mu_1 |^{2s_1-2} dm_6(\xi) +$
$|< \alpha/\gamma_2 >|^2 \int_{B(0,1)} | e^{(<z/\gamma_2>+<\xi/\gamma_2>+\delta_2)} |^2 dm_6(\xi)$.
If $s_1 = 0$ or $\lambda_1 = 0$, then v_6 is not strictly psh at any point of $\mathbb{C}^3$.
If $\gamma_2 = 0$, then v_6 is not strictly psh at each point of $\mathbb{C}^3$.
Therefore $s_1 > 0$, $\lambda_1 \neq 0$ and $\gamma_2 \neq 0$.
Fix $z \in \mathbb{C}^3$. Observe now that
$\int_{B(0,1)} |< z/\lambda_1 > + < \xi/\lambda_1 > +\mu_1 |^{2s_1-2} dm_6(\xi) > 0$,
$\int_{B(0,1)} | e^{(<z/\gamma_2>+<\xi/\gamma_2>+\delta_2)} |^2 dm_6(\xi) > 0$ and $\int_{B(0,1)} 1 dm_6(\xi) > 0$.
It follows that $L(v_6)(z)(\alpha) = 0$ if and only if $|< \alpha/a_1 - a_2 >|^2 +$
$|< \alpha/\lambda_1 >|^2 + |< \alpha/\gamma_2 >|^2 = 0$.

Thus for all fixed $z \in \mathbb{C}^3$, the system $<\alpha/a_1 - a_2> = 0$, $<\alpha/\lambda_1> = 0$, $<\alpha/\gamma_2> = 0$ and $\alpha \in \mathbb{C}^3$, implies that $\alpha = 0$.
It follows that $(a_1 - a_2, \lambda_1, \gamma_2)$ is a basis of the complex vector space $\mathbb{C}^3$ ($\mathbb{C}^3$ is a
complex vector space of dimension 3) and $s_1 > 0$.
Since v_5 is not strictly psh on $\mathbb{C}^3$, then there exists $z \in \mathbb{C}^3$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 \backslash \{0\}$ such that $L(v_5)(z)(\alpha) = |<\alpha/a_1 - a_2>|^2 + s_1^2 |<\alpha/\lambda_1>|^2 |<z/\lambda_1> + \mu_1|^{2s_1-2} + |<\alpha/\gamma_2>|^2 | e^{(<z/\gamma_2>+\delta_2)}|^2 = 0$.
Therefore $s_1 \geq 2$. In fact we take $z \in \mathbb{C}^3$ satisfying the condition $<z/\lambda_1> + \mu_1 = 0$. In this situation we can prove by an algebraic method that there exists $\alpha \in \mathbb{C}^3 \backslash \{0\}$ such that $<\alpha/a_1 - a_2> = <\alpha/\gamma_2> = 0$.
Consequently, in this case we have $s_1 \geq 2$ and $(a_1 - a_2, \lambda_1, \gamma_2)$ is a basis of the complex vector space $\mathbb{C}^3$.
Case 3. $\varphi_1(z) = e^{(<z/\gamma_1>+\delta_1)}$, $\varphi_2(z) = (<z/\lambda_2> + \mu_2)^{s_2}$.

$$L(v_6)(z)(\alpha) = |<\alpha/a_1 - a_2>|^2 \int_{B(0,1)} 1 dm_6(\xi) +$$

$$|<\alpha/\gamma_1>|^2 \int_{B(0,1)} | e^{(<z/\gamma_1>+<\xi/\gamma_1>+\delta_1)}|^2 dm_6(\xi) +$$

$$s_2^2 \ |<\alpha/\lambda_2>|^2 \int_{B(0,1)} |<z/\lambda_2> + <\xi/\lambda_2> + \mu_2|^{2s_2-2} dm_6(\xi),$$

$z, \alpha \in \mathbb{C}^3$.
Note that $\int_{B(0,1)} | e^{(<z/\gamma_1>+<\xi/\gamma_1>+\delta_1)}|^2 dm_6(\xi) > 0$,
$\int_{B(0,1)} |<z/\lambda_2> + <\xi/\lambda_2> + \mu_2|^{2s_2-2} dm_6(\xi) > 0$ and
$\int_{B(0,1)} 1 dm_6(\xi) > 0$.
Therefore $L(v_6)(z)(\alpha) = 0$ if and only if $|<\alpha/a_1 - a_2>|^2 + |<\alpha/\gamma_1>|^2 + s_2^2 |<\alpha/\lambda_2>|^2 = 0$.
Thus v_6 is strictly psh on $\mathbb{C}^3$ if and only if $(a_1 - a_2, \gamma_1, \lambda_2)$ is a basis of the complex vector space $\mathbb{C}^3$.
Now since v_5 is not strictly psh on $\mathbb{C}^3$, then $s_2 \geq 2$. In fact there exists $z \in \mathbb{C}^3$ such that $(<z/\lambda_2> + \mu_2) = 0$. It follows that $L(v_5)(z)(\alpha) = |<\alpha/a_1 - a_2>|^2 + |<\alpha/\gamma_1>|^2 | e^{(<z/\gamma_1>+\delta_1)}|^2$, $\alpha \in \mathbb{C}^3$.
If we take in this situation (by an algebraic method) $<\alpha/a_1 - a_2> = <\alpha/\gamma_1> = 0$ and $\alpha \in \mathbb{C}^3 \backslash \{0\}$, then we have $L(v_5)(z)(\alpha) = 0$ and $\alpha \neq 0$.
Therefore v_5 is not strictly psh on $\mathbb{C}^3$.

4. Concluding remarks

Remark 1. Let $u : \mathbb{C}^n \to \mathbb{R}$ be a function of class C^∞ and psh. Define $u_1(z) = \int_{B(0,1)} u(z+\xi)dm_{2n}(\xi)$, for $z \in \mathbb{C}^n$. Note that if u is strictly psh on $\mathbb{C}^n$, then u_1 is strictly psh on $\mathbb{C}^n$.
Now observe that if u_1 is strictly psh on $\mathbb{C}^n$, we can not conclude that u is strictly psh on $\mathbb{C}^n$. Example. Let $v(z) =| z |^4$, for $z \in \mathbb{C}$. v is a function of class C^∞ and sh on $\mathbb{C}$. v is not strictly sh on $\mathbb{C}$, but v_1 is strictly sh on $\mathbb{C}$, where $v_1(z) = \int_{D(0,1)} | z+\xi |^4 \, dm_2(\xi)$, for $z \in \mathbb{C}$. In fact v_1 is a function of class C^∞ on $\mathbb{C}$. $\frac{\partial^2}{\partial z \partial \overline{z}} \int_{D(0,1)} | z+\xi |^4 \, dm_2(\xi) = 4\int_{D(0,1)} | z+\xi |^2 \, dm_2(\xi) = \pi \mid z \mid^2 + \frac{\pi}{2} > 0$, for all $z \in \mathbb{C}$. But $\frac{\partial^2 v}{\partial z \partial \overline{z}}(0) = 0$.
Remark 2. Let $(a,b) \in \mathbb{C}^N \times \mathbb{C}^N \backslash \{(0,0)\}$, $n, N, k \in \mathbb{N}\backslash\{0\}$. Put $u(z,w) =|< w/a > -f_1(z) |^2 + |< w/b > -f_2(z) |^2$, for $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions and $(z,w) \in \mathbb{C}^n \times \mathbb{C}^N$.
(A) Suppose that u is convex on $\mathbb{C}^n \times \mathbb{C}^N$. Then we have two cases.
(I) Assume that $\{a, b\}$ is a free family on the complex vector space $\mathbb{C}^N$. Then f_1 and f_2 are affine functions on $\mathbb{C}^n$.
(II) Suppose that $\{a, b\}$ is not a free family on $\mathbb{C}^N$. Now by using this paper, we show that f_1 and f_2 have several holomorphic representations.
(B) Let $v(z,w) =|< w/a >^k -f_1(z) |^2 + |< w/b >^k -f_2(z) |^2$ and $v_1(z,w) =| \exp(< w/a >) - f_1(z) |^2 + | \exp(< w/b >) - f_2(z) |^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^N$. Assume that v and v_1 are convex functions on $\mathbb{C}^n \times \mathbb{C}^N$. Analogously, from (A) and the above section 2, we can formulate our main result as the holomorphic representations of f_1 and f_2.

References

[1] J. Abidi, **Sur quelques problèmes concernant les fonctions holomorphes et plurisousharmoniques**, Rend. Circ. Mat. Palermo, 51 (2002), 411-424.
[2] J. Abidi, **Contribution à l'étude des fonctions plurisousharmoniques convexes et analytiques**, Serdica Math. J., 40 (2014), 329-388.
[3] J. Abidi, **The real and complex convexity**, J. Math. Appl., 41 (2018), 123-156.

[4] J. Abidi, **On convex and analytic functions**, Anal. Univ. Oradea. fasc. Math. Tom XXVI, 1 (2019), 107-128.
[5] H. J. Bremermann, **Complex convexity**, Trans. Amer. Math. Soc., 82 (1956), 17-51.
[6] M. Andersson, M. Passare, R. Sigurdsson, **Complex convexity and analytic functionals**, Progress in Mathematics, Vol. 225, Birkhauser, Berlin, 2004.
[7] U. Cegrell, **Removable singularities for plurisubharmonic functions and related problems**, Proc. Lond. Math. Soc., 36 (1978), 310-336.
[8] H. El Mir, **Sur le prolongement des courants positifs fermés**, Acta Math., 153 (1984), 1-45.
[9] G. M. Henkin, J. Leiterer, **Theory of Functions on Complex Manifolds**, Birkhäuser, Boston, Mass., 1984.
[10] L. Hormander, **Notions of convexity**, Birkhauser, Basel- Boston- Berlin, 1994.
[11] M. Klimek, **Pluripotential theory**, Clarendon Press, Oxford, 1991.
[12] S. G. Krantz, **Function theory of several complex variables**, Wiley, New York, 1982.
[13] P. Lelong, **Définition des fonctions plurisousharmoniques**, C. R. Acad. Sci. Paris, 215 (1942), 398-400.
[14] P. Lelong, **Les fonctions plurisousharmoniques**, Ann. Sci. Ecole Norm. Sup., 62 (1945), 301-338.

University Tunis El Manar
Faculty of Sciences of Tunis, Department of Mathematics, 2092 El Manar, Tunis, Tunisia.
E-mail: abidijamel1@gmail.com