

UNIVERSITATEA DIN BACĂU
STUDII ȘI CERCETĂRI ȘTIINȚIFICE
Seria: MATEMATICĂ
Nr. 15(2005), pag. 1-9

**A RELATED FIXED POINT THEOREM FOR
TWO PAIRS OF MAPPINGS ON TWO
COMPLETE METRIC SPACES WITHOUT
CONTINUITY**

by

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Abstract. *A related fixed point theorem for two pairs of mappings on two complete metric spaces without continuity is obtained.*

1. Introduction

In the following, we give a new related fixed point theorem. The first related fixed point theorem was the following, see [1].

Theorem 1. *Let (X, d_1) and (Y, d_2) be complete metrics spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities*

$$\begin{aligned}d_2(Tx, TSy) &\leq c \max\{d_1(x, Sy), d_2(y, Tx), d_2(y, TSy)\}, \\d_1(Sy, STx) &\leq c \max\{d_2(y, Tx), d_1(x, Sy), d_1(x, STx)\}\end{aligned}$$

for all x in X and y in Y , where $0 \leq c < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y . Further, $Tz = w$ and $Sw = z$.

Related fixed point theorems were later extended to two pairs of mappings on metric spaces, see for example [2], where the following related fixed point theorem was proved.

Key words and phrases: complete metric space, common fixed point.

(2000) Mathematics Subject Classification: 54H25

Theorem 2. Let (X, d) and (Y, ρ) be complete metric spaces, let A, B be mappings of X into Y and let S, T be mappings of Y into X satisfying the inequalities

$$d(SAx, TBx') \leq c \max\{d(x, x'), d(x, Sx), d(x', TBx'), \rho(Ax, Bx')\},$$

$$\rho(BSy, ATy') \leq c \max\{\rho(y, y'), \rho(y, BSy), \rho(y', ATy'), d(Sy, Ty')\}$$

for all x, x' in X and y, y' in Y , where $0 \leq c < 1$. If one of the mappings A, B, S and T is continuous, then SA and TB have a unique common fixed point u in X and BS and AT have a unique common fixed point v in Y . Further, $Au = Bu = v$ and $Sv = Tv = u$.

For further related fixed point theorems, see [3] to [7].

2. Main result

We prove now the following related fixed point theorem without continuity.

Theorema 3. Let (X, d) and (Y, ρ) be complete metric spaces, let A, B be mappings of X into Y and let S, T be mappings of Y into X satisfying the inequalities

$$(2.1) \quad d(SAx, TBx') \leq c \frac{f(x, x', y, y')}{h(x, x', y, y')},$$

$$(2.2) \quad \rho(BSy, ATy') \leq c \frac{g(x, x', y, y')}{h(x, x', y, y')}$$

for all x, x' in X and y, y' in Y for which $h(x, x', y, y') \neq 0$, where

$$f(x, x', y, y') = \max\{d(x, x')\rho(y', ATy'), d(Sy, TBx')d(x, Sy)$$

$$d(Sy, Ty')d(SAx, Ty'), d(x, Ty')\rho(y, Ax)\},$$

$$g(x, x', y, y') = \max\{d(x, Sy)\rho(y, y'), d(x', TBx')\rho(y', Ax),$$

$$d(SAx, Ty')\rho(Ax, Bx'), \rho(Ax, ATy')d(SAx, Sy)\},$$

$$h(x, x', y, y') = \max\{\rho(BSy, ATy'), d(x, SAx), d(Sy, TBx'), \rho(Bx', ATy')\}$$

and $0 \leq c < 1$. Then SA and TB have a unique common fixed point u in X and BS and AT have a unique common fixed point v in Y . Further, $Au = Bu = v$ and $Sv = Tv = u$.

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Proof. Let x_0 be an arbitrary point in X , let

$$Ax_0 = y_1, \quad Sy_1 = x_1, \quad Bx_1 = y_2, \quad Ty_2 = x_2, \quad Ax_2 = y_3$$

and in general let

$$Sy_{2n-1} = x_{2n-1}, \quad Bx_{2n-1} = y_{2n}, \quad Ty_{2n} = x_{2n}, \quad Ax_{2n} = y_{2n+1}$$

for $n = 1, 2, \dots$

We will first of all suppose that for some n

$$\begin{aligned} h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) &= \max\{\rho(BSy_{2n-1}, ATy_{2n}), d(x_{2n}, SAx_{2n}), \\ &\quad d(Sy_{2n-1}, TBx_{2n-1}), \rho(Bx_{2n-1}, ATy_{2n})\} \\ &= \max\{\rho(y_{2n+1}, y_{2n}), d(x_{2n}, x_{2n+1}), \\ &\quad d(x_{2n-1}, x_{2n}), \rho(y_{2n}, y_{2n+1})\} \\ &= 0. \end{aligned}$$

Then putting $x_{2n-1} = x_{2n} = x_{2n+1} = u$ and $y_{2n} = y_{2n+1} = v$, we see that

$$BSv = ATv = v, SAu = u, Sv = TBu = u, Bu = Atv = v$$

from which it follows that

$$Au = v, Tv = u.$$

Similarly, $h(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n}) = 0$ for some n implies that there exists u in X and v in Y such that

$$(2.3) \quad SAu = TBu = u, Bsv = ATv = v, Au = Bu = v, Sv = Tv = u.$$

We will now suppose that

$$h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) \neq 0 \neq h(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n})$$

for all n .

Applying inequality (2.1), we get

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(SAx_{2n}, TBx_{2n-1}) \\ &\leq c \frac{f(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})}{h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})} \\ &= cd(x_{2n-1}, x_{2n}) \frac{\max\{\rho(y_{2n+1}, y_{2n}), d(x_{2n-1}, x_{2n}), d(x_{2n+1}, x_{2n})\}}{\max\{\rho(y_{2n+1}, y_{2n}), d(x_{2n+1}, x_{2n}), d(x_{2n-1}, x_{2n})\}}, \end{aligned}$$

from which it follows that

$$(2.4) \quad d(x_{2n+1}, x_{2n}) \leq c \max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n+1}, y_{2n})\}$$

Using inequality (2.1) again, we get

$$\begin{aligned} d(x_{2n-1}, x_{2n}) &= d(SAx_{2n-2}, TBx_{2n-1}) \\ &\leq c \frac{f(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2})}{h(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2})} \\ &= cd(x_{2n-1}, x_{2n-2}) \frac{\max\{\rho(y_{2n-2}, y_{2n-1}), d(x_{2n-1}, x_{2n}), d(x_{2n-1}, x_{2n-1})\}}{\max\{\rho(y_{2n}, y_{2n-1}), d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n})\}}, \end{aligned}$$

from which it follows that

$$(2.5) \quad d(x_{2n-1}, x_{2n}) \leq c \max\{d(x_{2n-2}, x_{2n-1}), \rho(y_{2n-1}, y_{2n})\}$$

Similarly, on using inequality (2.2) we have

$$\begin{aligned} \rho(y_{2n}, y_{2n+1}) &= d(BSy_{2n-1}, ATy_{2n}) \\ &\leq c \frac{g(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})}{h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})} \end{aligned}$$

where

$$\begin{aligned} g(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) &= \max\{d(x_{2n}, x_{2n-1})\rho(y_{2n-1}, y_{2n}), \\ &d(x_{2n-1}, x_{2n})\rho(y_{2n+1}, y_{2n})\} \end{aligned}$$

We then have either

$$g(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) = d(x_{2n-1}, x_{2n}) \max\{\rho(y_{2n-1}, y_{2n}), \rho(y_{2n+1}, y_{2n})\}$$

or

$$g(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) = \rho(y_{2n+1}, y_{2n}) \max\{d(x_{2n-1}, x_{2n}), d(x_{2n+1}, x_{2n})\}$$

Further,

$$\begin{aligned} h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) &= \max\{\rho(y_{2n+1}, y_{2n}), d(x_{2n+1}, x_{2n}), d(x_{2n-1}, x_{2n})\} \\ &= \max\{\rho(y_{2n+1}, y_{2n}), d(x_{2n-1}, x_{2n})\} \end{aligned}$$

on using inequality (2.4). It follows that either

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$$\rho(y_{2n}, y_{2n+1}) \leq c \max\{\rho(y_{2n-1}, y_{2n}), \rho(y_{2n}, y_{2n+1})\} = c\rho(y_{2n-1}, y_{2n})$$

or

$$\rho(y_{2n}, y_{2n+1}) \leq c \max\{d(x_{2n-1}, x_{2n}), d(x_{2n+1}, x_{2n})\} = cd(x_{2n-1}, x_{2n})$$

and so

$$(2.6) \quad \rho(y_{2n}, y_{2n+1}) \leq c \max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n-1}, y_{2n})\}$$

Using inequality (2.2) again, we get

$$\begin{aligned} d(y_{2n}, y_{2n-1}) &= d(BSy_{2n-1}, ATy_{2n-2}) \\ &\leq c \frac{g(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2})}{h(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2})}, \end{aligned}$$

where

$$\begin{aligned} g(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2}) &= \max\{d(x_{2n-2}, x_{2n-1})\rho(y_{2n-1}, y_{2n-2}), \\ &\quad d(x_{2n-1}, x_{2n-2})\rho(y_{2n-1}, y_{2n})\} \\ &\leq \max\{d(x_{2n-2}, x_{2n-1})\rho(y_{2n-1}, y_{2n-2}), \\ &\quad \rho(y_{2n-1}, y_{2n})\rho(y_{2n-2}, y_{2n-1}), \\ &\quad d(x_{2n-1}, x_{2n-2})\rho(y_{2n-1}, y_{2n})\} \end{aligned}$$

on using inequality (2.5). We then have either

$$\begin{aligned} g(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2}) &= \\ d(x_{2n-2}, x_{2n-1}) \max\{\rho(y_{2n-1}, y_{2n-2}), \rho(y_{2n-1}, y_{2n})\} & \end{aligned}$$

or

$$g(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2}) = \rho(y_{2n-1}, y_{2n-2}) \max\{d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n})\}.$$

Further

$$\begin{aligned} h(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2}) &= \max\{\rho(y_{2n}, y_{2n-1}), d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n})\} \\ &= \max\{\rho(y_{2n+1}, y_{2n}), d(x_{2n-1}, x_{2n})\} \end{aligned}$$

or using inequality (2.5). It follows that either

$$\rho(y_{2n}, y_{2n-1}) \leq c \max\{\rho(y_{2n-1}, y_{2n}), \rho(y_{2n}, y_{2n+1})\} = c\rho(y_{2n-1}, y_{2n})$$

or

$$\rho(y_{2n}, y_{2n+1}) \leq c \max\{d(x_{2n-1}, x_{2n}), d(x_{2n+1}, x_{2n})\} = cd(x_{2n-1}, x_{2n})$$

and so

$$(2.7) \quad \rho(y_{2n}, y_{2n+1}) \leq c \max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n-1}, y_{2n})\}.$$

From inequalities (2.4) to (2.7), we obtain

$$(2.8) \quad d(x_n, x_{n+1}) \leq c^n \max\{d(x_0, x_1), \rho(y_0, y_1)\},$$

$$(2.9) \quad \rho(y_n, y_{n+1}) \leq c^n \max\{d(x_0, x_1), \rho(y_0, y_1)\}.$$

Since $0 < c < 1$, it follows from inequalities (2.8) and (2.9) that $\{x_n\}$ is a Cauchy sequence in X with a limit u and $\{y_n\}$ is a Cauchy sequence in Y with a limit v .

We now have

$$(2.10) \quad \lim_{n \rightarrow \infty} f(u, x_{2n-1}, Au, y_{2n}) = d^2(SAu),$$

$$(2.11) \quad \lim_{n \rightarrow \infty} g(u, x_{2n-1}, Au, y_{2n}) = d(u, SAu)\rho(Au, v),$$

$$(2.12) \quad \lim_{n \rightarrow \infty} h(u, x_{2n-1}, Au, y_{2n}) = \max\{\rho(v, BSu), d(u, SAu)\}.$$

If

$$(2.13) \quad \max\{\rho(v, BSu), d(u, SAu)\} = 0,$$

then

$$(2.14) \quad SAu = u, \quad BSu = v, \quad Bu = v.$$

If

$$(2.15) \quad \max\{\rho(v, BSu), d(u, SAu)\} \neq 0,$$

then we have on using equations (2.10) and (2.12)

$$\begin{aligned} d(SAu, v) &= \lim_{n \rightarrow \infty} d(SAu, TBx_{2n-1}) \\ &\leq \lim_{n \rightarrow \infty} c \frac{f(u, x_{2n-1}, Au, y_{2n})}{h(u, x_{2n-1}, Au, y_{2n})} \\ &\leq cd(SAu, u) \end{aligned}$$

and so $SAu = u$, since $c < 1$.

Further, using inequality (2.2) and equations (2.11) and (2.12), we get

$$\begin{aligned} \rho(BSu, v) &= \lim_{n \rightarrow \infty} \rho(BSu, ATy_{2n}) \\ &\leq \lim_{n \rightarrow \infty} c \frac{g(u, x_{2n-1}, Au, y_{2n})}{h(u, x_{2n-1}, Au, y_{2n})} \\ &= 0 \end{aligned}$$

and so $BSu = v$, contradicting equation (2.15). Therefore equations (2.13) and (2.14) must hold.

Now suppose that $Tv \neq u$. Then

$$(2.16) \quad \lim_{n \rightarrow \infty} f(x_{2n}, u, v, v) = d^2(u, Tv),$$

$$(2.17) \quad \lim_{n \rightarrow \infty} f(x_{2n}, u, v, v) = \max\{d(u, Tv), \rho(v, ATv)\} \neq 0.$$

Using inequality (2.1) and equations (2.16) and (2.17) we have

$$\begin{aligned} d(u, Tv) &= \lim_{n \rightarrow \infty} d(SAx_{2n}, TBu) \\ &\leq \lim_{n \rightarrow \infty} c \frac{f(x_{2n}, u, y_{2n-1}, v)}{h(x_{2n}, u, y_{2n-1}, v)} \\ &= cd(u, Tv), \end{aligned}$$

a contradiction. Hence $Tv = u = TBu$.

Now suppose that $Au \neq v$, then

$$(2.18) \quad \lim_{n \rightarrow \infty} f(u, u, Au, v) = 0,$$

$$(2.19) \quad \lim_{n \rightarrow \infty} h(u, u, Au, v) = \rho(Au, v) \neq 0.$$

Using inequality (2.2) and equations (2.18) and (2.19), we get

$$\begin{aligned} \rho(Au, v) &= \rho(BSAu, ATBu) \\ &\leq c \frac{f(u, u, Au, v)}{h(u, u, Au, v)} \\ &= 0. \end{aligned}$$

Therefore $Au = Bu = v$ and equations (2.3) follow again.

To prove the uniqueness, suppose that SA and TB have a second distinct common fixed point u' so that $Au \neq Bu'$. Then,

$$(2.20) \quad f(u, u', v, Bu') = d^2(u, u'),$$

$$(2.21) \quad h(u, u', v, Bu') = \max\{d(u, u'), \rho(Au, Bu')\} \neq 0$$

Using inequality (2.1) and equations (2.20) and (2.21) we get

$$\begin{aligned} d(u, u') &= d(SAu, TBu') \\ &\leq c \frac{f(u, u', v, v)}{h(u, u', v, v)} \\ &\leq cd(u, u'), \end{aligned}$$

a contradiction. Therefore u is unique.

We can prove similarly that v is the unique common fixed point of BS and AT .

This completes the proof of the theorem.

Corollary 1. Let A, B, S and T be self mappings on the complete metric space (X, d) satisfying the inequalities

$$d(SAx, TBy) \leq c \frac{f(x, y)}{h(x, y)},$$

$$d(BSx, ATy) \leq c \frac{g(x, y)}{h(x, y)}$$

for all x, y in X for which $h(x, y) \neq 0$, where

$$f(x, y) = \max \{d(x, y)\rho(y, ATy), d(Sx, TBy)d(x, Sx), \\ d(Sx, Ty)d(SAx, Ty), d(x, Ty)d(x, Ax)\}$$

$$g(x, y) = \max \{d(x, Sx)d(x, y), d(y, TBy)d(y, Ax), \\ d(SAx, Ty)d(Ax, By), d(Ax, ATy)d(SAx, Sx)\}$$

$$h(x, y) = \max \{d(BSx, ATy), d(x, SAx), d(Sx, TBy), d(By, ATy)\}$$

and $0 \leq c < 1$. Then SA and TB have a unique common fixed point u and BS and AT have a unique common fixed point v . Further, $Au = Bu = v$ and $Sv = Tv = u$.

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