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FIXED POINT THEOREMS FOR MAPPINGS SATISFYING A NEW TYPE OF IMPLICIT RELATION

by

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Abstract. *In this paper we introduce a new class al functions* $F : R_+^6 \to R$ *such that the fulfilment at the inequality [5] for* x, y *in* (X, d)*, ensures the existence and the uniqueness of a fixed point.*

1.Introduction

The notion of contractive mappings has been introduced by Banach in [1]. In the last years different types of generalizations of this concept appeared. The connection between types have been studied in different paper, for example [2], [3] [6]-[10].

Let (X, d) be a metric space and $T : (X, d) \to (X, d)$ be a mapping. In essence, T is a generalized contraction if on inequalite of type

(1) $d(Tx, Ty) \le f(d(x, y), d(x, Tx), d(y, Ty).d(x, Ty), d(y, Ty))$ holds for all $x, y \in X$, where $f : R_+^5 \to R$ satisfies some properties or has a special form.

In [4], the prezent author established a class of mappings $F : R_+^6 \to R$ such that the fulfilment of the inequality of the type.

(2) $F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \le 0$ for $x, y \in X$, ensures the existence and the uniqueness of a fixed point for T.

Recently [5], the present author established two classes of mappings $F, G : R_+^6 \to R$
such that the fulfilment of the inequality of type.

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(3) $F\left(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d\left(y,T^2x\right),d(y,Tx)\right)\le 0$ or

(4) $G\left(d(Tx,Ty),d(x,y),d(x,Tx)d(y,Ty),d\left(x,T^2y\right),d(x,Ty)\right)\le 0$ for $x,y\in X$, ensures the existence and the uniqueness of a fixed point for T.

The purpose of this paper is to introduce a new class of mappings $F:R_+^6\to R$ such that the fulfiliment of the inequality of type.

(5) $F\left(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d\left(y,T^2x\right),d\left(Tx,T^2x\right)\right)\le 0$ for $x,y\in X$, ensures the existence and the uniqueness of a fixed point for T.

2. Implicit relation

Let $\mathcal{F}$ be the set of all real continous functions $F(t_1,...,t_6):R_+^6\to R$ satisfying the following conditions:

(Fm): F is nonincreasing in variables t_5 and t_6,

(Fh): There exists $h\in(0,1)$ such that for every $u,v\ge 0$, $F(u,v,v,u,u,u)\le 0$, implies $u\le hv$,

(Fu): $F(t,t,0,0,t,0)>0$, $\forall t>0$

Ex.1. $F(t_1,...,t_6)=t_1-at_2-bt_3-ct_4-dt_5-et_6$, where $a,b,c,d,e\ge 0$ and $0<(\ a+b+c+d+e\ )<1$.

(Fm): Obviously.

(Fh): $F(u,v,v,u,u,u)=u-av-bv-cu-du-eu\le 0$ then $u\le hv$, where $0<h=a+b/1-c-d-e<1$

(Fu): $F(t,t,0,0,t,0)=t(1-a-d)>0;\forall t>0$

Ex.2. $F(t_1,...,t_6)=t_1-c\max\left\{t_2,t_3,t_4,\frac{1}{2}(t_5+t_6)\right\}$, where $0<c<1$.

(Fm): Obviously.

(Fh): Let $u>0,v\ge 0$ and $F(u,v,v,u,u,u)=u-c\max\{u,v\}\le 0$. If $u\ge v$, then $u(1-c)\le 0$, a contradiction then $u<v$ and $u\le hv$, where $0<h=c<1$. If $u=0$, then $u\le hv$.

(Fu): $F(t,t,0,0,t,0)=t(1-c)>0;\forall t>0$.

Ex.3. $F(t_1,...,t_6)=t_1^3-at_1^2t_2-bt_1t_2^2-ct_2t_3t_4-dt_5^2t_6$, where $\mathrm{a,b,c,d}\ge 0$ and $0<\mathrm{a+b+c+d}<1$.

. (F_m): Obviously.

(F_h): Let $u > 0, v > 0$ and $F(u,v,v,u,u,u) = u^3 - au^2v - buv^2 - cuv^2 - du^3 \leq 0$, which implies $u^2(1-d) - auv - v^2(b+c) \leq 0$.

If $b = c = 0$, then $u \leq hv$, where $0 < h = a / 1 - d < 1$.

If $b + c \neq 0$, then $f(t) = (b+c)t^2 + at - (1-d) \geq 0$, where $t = \frac{v}{u}$.

Since $f(1) = (b + c + a + d - 1) < 0$, let $r > 1$ be the root of equation $f(t) = 0$.

Then $f(t) > 0$ for $t > r$ which implies $u \leq hv$, where $h = \frac{1}{r}$.

If $u = 0$, then $u \leq hv$.

(F_u): $F(t,t,0,0,t,0) = t(1-a-b) > 0$; $\forall t > 0$.

3. **Fixed points in complete metric spaces**

Theorem1. Let (X, d) be a metric space and $T : (X, d) \to (X, d)$ be a mapping satisfying the inequality (5) for $x, y \in X$, where F satisfies condition (F_u). Then T has at most one fixed point.

Proof. Suppose that T has two fixed points u and v with $u \neq v$. Then by (5) we have successively.

$F(d(Tu,Tv), d(u,v), d(u,Tu), d(v,Tv), d(v,T^2u), d(Tu,T^2u)) \leq 0$

$F(d(u,v), d(u,v), 0, 0, d(u,v), 0) \leq 0$ a contradiction of (F_u)

Theorem2.[5]. Let (X, d) be a metric space and $T : (X,d) \to (X,d)$ be a mapping such that there exists $h \in [0,1)$ with $d(T^2x, Tx) \leq hd(x,Tx)$ for $\forall x \in X$. Then for $x \in X$ the sequence $\{T^n x\}$ is a Cauchy sequence.

Theorem3. Let (X, d) be a complete metric space and $T : (X,d) \to (X,d)$ a mapping satisfying inequality (5) for every $x, y \in X$, where $F \in \mathcal{F}$.

Proof. Let X be arbitrary in X. We shall show that the sequence defined by $x_{n+1} = T^n x$ is a Cauchy sequence. From (5) for $y = Tx$ we have $F(d(Tx,T^2x), d(x,Tx), d(x,Tx), d(Tx,T^2x), d(Tx,T^2x), d(Tx,T^2x)) \leq 0$

By (F_u) we have $d(T^2x, Tx) \leq hd(x,Tx)$. By Theorem 2 the sequence $x_{n+1} = T^n x$ is a Cauchy sequence. Since (X, d) is complete , there exists $u \in X$ such that $\lim x_n = u$. By (5) we have successively.

$$F(d(Tx_n,Tu), d(x_n,u), d(x_n,Tx_n), d(u,Tu), d(u,T^2x_n) d(Tx_n,T^2x_n)) \leq 0$$

$$F\left(d\left(x_{n+1},Tu\right),d\left(x_n,u\right),d\left(x_u,x_{n+1}\right),d\left(u,T_u\right)d\left(u,x_{n+2}\right),d\left(x_{n+1},.x_{n+2}\right)\right)\leq 0$$

Letting n tend to infinity we have successively:

$$F\left(d\left(u,Tu\right),0,0,d\left(u,Tu\right)0,0\right)\leq 0$$

$$F\left(d\left(u,Tu\right),0,0,d\left(u,Tu\right),d\left(u,Tu\right),d\left(u,Tu\right)\right)\leq 0$$

which implies by $(F\ )$ that $u = Tu$.

By Theorem 1 u is the unique fixed point of T .

Corollary1. Let (X, d) be a complete metric space and $T:(X,d)\to(X,d)$ satisfying one of the following inequality .

(1.1) $d\left(Tx,Ty\right)\leq ad\left(x,y\right)+bd\left(x,Tx\right)+cd\left(y,T\ y\right)+dd\left(y,T^2x\right)+ed\left(Tx,T^2x\right)\leq 0$

where $a, b, c, d, e \geq 0$ and $0 < a + b + c + d + e < 1$;

(1.2)

d

$\left(Tx,Ty\right)\leq c\max\left\{d\left(x,y\right),d\left(x,Tx\right),d\left(y.Ty\right),\frac{1}{2}d\left(y,T^2x\right)+d\left(Tx,T^2x\right)\leq 0\right\}$

where $0 < c < 1$, or;

(1.3) $$\begin{aligned}&d^3\left(Tx,Ty\right)-ad^2\left(Tx,Ty\right)d\left(x,y\right)-bd\left(Tx,Ty\right)d^2\left(x,y\right)-\\&-cd\left(x,y\right)d\left(x,Tx\right)d\left(y,Ty\right)-d^2\left(y,T^2x\right)d\left(Tx,T^2x\right)\leq 0\end{aligned}$$

where $a, b, c, d \geq 0$ and $0 < a + b + c + d < 1$, for all x, y in X. Then T has a unique fixed point .

Proof. The proof is following from Theorem 4 and Ex 1-3.

Remark 1. Let $\mathcal{G}$ be the set of all real continuous functions: $G\,(t_1,...,t_6): R^6{}_+\to R$ satisfying the following conditions:

$\left(G_m\right)$: G is noincreasing in variables t_5 and t_6,

$\left(G_h\right)$: there exists n∈) such that for every $u, v \geq 0$

$G\left(u,v,u,v,u,u\right)\leq 0$ implies $u\leq hv$,

$\left(G_u\right)$: $G\left(t,t,0,0,t,0\right)>0$; $\forall\ t>0$.

Remark 2. The functions F from Ex 1-3 satisfies conditions G_m, G_h and G_u .

Theorem 4. If the inequality

(6) $G\left(d\left(Tx,Ty\right),d\left(x,y\right),d\left(x,Tx\right),d\left(y,Ty\right),d\left(x,T^2y\right),d\left(y,T^2y\right)\right)\leq 0$ holds for all x,y in X, where $G\in\mathcal{G}$, then T has a unique fixed point.

Proof. The proof is similar to the proof of Theorem 3.

4. Fixed points in compact metric spaces.

Let $\widetilde{\mathrm{F}}$ be the set of all real continuous functions $F(t_1,...,t_6): R_+^6 \to R$ satisfying the following conditions:

$(\widetilde{F}_i)$: for every $u \geq 0,\ v > 0,\ F(u,v,v,u,u,u) < 0$ implies $u < v$,

$(\widetilde{F}_u)$: $F(t,t,0,0,t,0) > 0,\ \forall t > 0.$

Remark 3. The functions F from Ex. 1-3 satisfies conditions $(\widetilde{F}_i)$ and $(\widetilde{F}_u)$.

Remark 4. There exists functions $F \in \widetilde{F}$ which is increasing in variables t_5 and t_6.

Ex. 4 $F(t_1,...,t_6) = t_1^3 - c\dfrac{t_2 t_3 t_4}{1 + t_5 + t_6}$, where $0 < c < 1$.

$(\widetilde{F}_u)$ Let $u,v > 0$ and $F(u,v,u,u,u) = u^3 - c\dfrac{v^2 u}{1+2u} < 0$, then $u^2 < \dfrac{c}{1+2u} v^2$, which implies $u < v$. If $u = 0$ and $v > 0$, then $u < v$.

$(\widetilde{F}_u)$: $F(t,t,0,0,t,0) = t^3 > 0, \forall t > 0$.

Theorem 5. Let T be a continuous mapping of the compact metric space (X,d) into itself such that

(7) $F(d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), d(y,T^2x), d(Tx,T^2x)) < 0$ for every $x \neq y$ in X, where $F \in \widetilde{F}$. Then T has a unique fixed point.

Proof. Let $f(x) = d(x,Tx)$ for all $x \in X$. Since T is continuous, f is continuous. There exists a point $z \in X$ such that $f(z) = \inf\{f(u) : x \in X\}$. Suppose that $z \neq Tz$. Then, by (7) for $x = z$ and $y = Tz$ we obtain $F(d(Tz,T^2z), d(z,Tz), d(z,Tz), d(Tz,T^2z), d(Tz,T^2z), d(Tz,T^2z)) < 0$ which implies by $(\widetilde{F}_i)$ $d(Tz,T^2z) < d(z,Tz) = imf\{d(x,Tx) : x \in X\}$. A contradiction Hence, $z = Tz$. From Theorem 1 z is the unique fixed point of T

Corollary 2 Let T be a continuous mapping of the compact metric space (t,d) into itself such that

$d^3(Tx,Ty) < c\dfrac{d(x,y)d(x,Tx)d(y,Ty)}{1 + d(y,T^2x) + d(Tx,T^2x)}$, where $0 < c < 1$, for all $x \neq y$ in X.

Then T has a unique fixed point.

Proof. The proof it follows from Theorem 5 and Ex. 4

Remark 5. A corollary analogous to Corollary 1 is obtained by Ex. 1-3.
Remark 6. A theorem similar to Theorem 4 is obtained for compact metric spaces.

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