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Well – posedness of fixed point problem in orbitally complete metric spaces

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Abstract

The purpose of this paper is to prove that for mappings satisfying a new type of implicit relation in orbitally complete metric spaces fixed point problem is well-posed. The result of Theorem 3 generalizes [6, Theorem 6] and other results.

Key words and phrases: well- posedness, orbitally complete space, implicit relation, fixed point.
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1. Introduction

In 1974 Ćirić [2] has first introduced orbitally complete metric spaces.

Definition1. Let T be a self mapping of a metric space (X, d). If for all x in X every Cauchy sequence of the orbit O_X (T) ={x, Tx, T_X^2,…} is convergent in X , then the metric space (X, d) is said T-orbitally complete .

Remark 1. Every complete metric space is T-orbitally complete for any T: X → X. An orbitally complete space may not be complete metric space (Example [6,], Example 1 [15]).

The notion of well-posedness of a fixed point problem has evoked much interest to several mathematicians (see for example [3], [6], [10]).

Definition 2 [3]. Let (X, d) be a metric space, f: (X, d) →(X, d) be a mapping. The fixed point problem of *f* is said to be well-posed if:

(*i*) *f* has a unique fixed point x_0 in x,

(*ii*) For any sequence $\{x_n\}$ cu X with d (x_n, fx_n)→0 as n→ ∞ we have d(x_n, x_0) → 0 as n → ∞ .

Recently, the well-posedness of fixed point problem for certain type of mappings have been investigated in [3], [6] and [10].

The notion of contractive mapping has been introduced by Banach in [1]. During the last thirty years different types of generalization of this concept appeared. The connection between them have been studied in different papers [5], [7], [11]-[14].

Let (X, d) be a metric space and T: (X, d) → (X, d) be a mapping .In essence is a generalized contraction if an inequality of type

(1) d (Tx, Ty) ≤ f(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx))

holds for x, y ∈ X, where f : R_t^5 →R satisfies some properties or has a special form.

In [8], the present author established a class of mappings F: R^6 →R such as the fulfillment of the inequality of type .

(2) F(d (Tx, Ty),d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) ≤ 0 for all x, y ∈ X , ensures the existence and uniqueness of a fixed point of T .

Recently [9], the present author established two classes of mappings F, G: R_t^6 →R such that the fulfillment of the inequality of type

(3) F (d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(y, T^2 x),d(y, Tx)) ≤ 0 or

(4) G(Tx, Ty),d(x, y),d(x, Tx),d(y, Ty),d(x, T^2 y),d(x, Ty)) ≤ 0 for x, y X ensures the existence and the uniqueness of a fixed point for T.

The purpose of this paper is to introduce a new class of mappings F: R_t^6 → R such that the fulfillment of the inequality of type.

(5) F(d(Tx, Ty) ,d(x, y),d(x, Tx),d(y, Ty),d(y, Tx),d(Tx, T^2 x)) ≤ 0

for x, y in X ensures the existence and the uniqueness of a fixed point for T and to prove for mappings satisfying are implicit relation of type (5) in are orbitally complete metric space that fixed point problem is well-posed.

2. Implicit relations

Let F $(t_1,...,t_6)$: R_t^6 →R be a continuous function we define the following properties:

(Fm): F is non-increasing in variable t_6.

(Fh): There exists h∈ (0, 1) such that for every u ≥ 0, v ≥ 0 with F (u, v, v, u, 0, u) ≤ 0 we have u ≤ hv.

(Fu) : F(t, t, 0, 0, t, 0) > 0 for every t > 0.

(Fp): There exists p ∈ (0, 1) such that for every u ≥ 0, v ≥ 0, w ≥ 0 with F(u, v ,0, w, v, 0) ≤ 0 we have u ≤ p max{v, w}.

Example1. F $(t_1,...,t_6) = t_1 -$ c $\max\{t_2,t_3,t_4,\frac{1}{2}(t_5+t_6)\}$, where $c \in (0,1)$.

(Fm): Obviously.

(Fh): Let u > 0, v ≥0 and F(u, v, v, u, 0, u) = u-c max{u, v} ≤ 0 . If u ≥ v then u (1-c) ≤, a contradiction. Then u < v and u ≤ hv, where 0 < h = c < 1. If u = 0, then u ≤ hv.

(Fu) : F(t, t, 0, 0, t, 0) = t(1-c) > 0 , t > 0 .
(Fp) : Let u > 0 , v ≥ 0 ,w ≥ 0 and F(u, v, 0, w, v, 0) =u-c max{v,0,w} ≤ 0 . Then u ≤ h max {v, w}, where 0 < h =c < 1.

Example 2. F $(t_1,...,t_6) = t_1^2 - a\max\{t_2^2, t_3^2, t_4^2\} - b\ t_5 t_6$, where a, b > 0 and a < 1.

(Fm): Obviously.

(Fh): Let u > 0, v ≥ 0 and F (u, v, v, u, 0, u) =u^2-a max {u^2, v^2} ≤ 0 . If u ≥ v, then u^2(1-a) ≤ 0, a contradiction. Then u < v and u ≤ hv, where 0 < h =$\sqrt{a}$ < 1 . If u =0, then u ≤ hv.

(Fu) : F(t,t,0,0,t,0) = t^2(1-a) > 0 , ∀ t >0 .

(Fp): Let u ≥ 0, v ≥ 0, w ≥ 0 and F (u, v, 0, w, v, 0) = u^2-a max { v^2, w^2 } ≤ 0. If $u \ge \max\{v,w\}$, then $u^2(1-a) \le 0$, a contradiction. Thus $u < \max\{v,w\}$ and $u \le h\max\{v,w\}$, where 0 < h =$\sqrt{a}$ < 1 . If u=0, then $u \le h\max\{v,w\}$.

Example3. F $(t_1, ..., t_6) = t_1^2 - t_1(\mathrm{a}t_2 + \mathrm{b}t_3 + \mathrm{c}t_4) - \mathrm{d}\ t_5 t_6$, where a > 0 ; b, c, d ≥ 0 and a+b+c < 1 .

(Fm): Obviously.

(Fh): Let u > 0, v ≥ 0 and F(u, v, v u, 0, u) = u^2-u(av + bv +cu) ≤ 0 . Then u ≤ hv where 0 < h = $\frac{a+b}{1-c}$ <1. If u = 0, then u ≤ hv.

(Fu) : F(t, t, 0, 0, t, 0) = t^2(1-a) > 0, ∀ t > 0.

(Fp): Let u > 0, v ≥ 0 and F (u. v, 0, w, v, 0) = u^2-u (av + cw) ≤ 0, then u ≤ p max {v, w}, where 0< p = a + c < 1. If u = 0, then u ≤ p max{v, w}.

Example 4. F $(t_1, .., t_6)$= t_1^2 -a t_2^2 - $\frac{bt_5t_6}{1-t_3+t_4}$, where 0 < a < 1 and b ≥ 0.

(Fm): Obviously.

(Fh): Let u ≥ 0, v ≥ 0 and F (u, v, v, u, 0, u) = u^2-a v^2≤ 0. Then u ≤ hv, where 0 < h =$\sqrt{a}$ < 1.

(Fu): F(t, t, 0, 0, t,0) = t^2(1-a) > 0,∀ t > 0.

(Fp): Let u ≥ 0, v ≥ 0, w ≥ 0 and F (u, v, 0, w, v, 0) = u^2-av^2≤ 0 which implies u ≤ $\sqrt{a}$ v ≤$\sqrt{a}$ max {v, w). Hence u ≤ p max {v, w}, where 0 < p =$\sqrt{a}$< 1.

Example 5. F $(t_1, ..., t_6)$= t_1- a_1t_2- a_2t_3 - a_3t_4- a_4t_5- a_5t_6, where $a_1 \ge 0$, ..., $a_5 \ge 0$ and 0 < $a_1 + a_2 + a_3 + a_4 + a_5$ <1.

(Fm): Obviously.

(Fh): Let u ≥ 0, v ≥ 0 and F (u, v, v, u, 0, v) = u- a_1 v - a_2 v - a_3 w- a_4 v ≤ 0 then u ≤ hv where $0 < h = \dfrac{a_1 + a_2}{1 - a_3 - a_5} < 1$.

(Fu) : F(t, t, 0, 0, t, 0) = (1 –($a_1 + a_4$)) t >0 , ∀ t> 0.

(Fp): Let u >0, v ≥ 0, w ≥ 0 and F(u, v, 0, w, v, 0) = u- a_1 v - a_3 w - a_4 v ≤ 0. If $u \geq \max\{v, w\}$, then $u(1 - a_1 - a_3 - a_4) \leq 0$, a contradiction. Then $u < \max\{v, w\}$, which implies u ≤ p max{v, w}, where 0 < p =1-($a_1 + a_3 + a_4$)< 1. If $u = 0$, then $u \leq p \max\{v, w\}$.

3. Main results

Theorem 1. Let (X, d) be a metric space and T : (X, d) →(X, d) be a mapping satisfying the inequality (5) for all x, y in X, where F satisfies condition (Fu). Then T has at most one fixed point.

Proof. Suppose that T has two fixed points u and v with u ≠ v. Then by (5) we have successivelly F(d(Tu, Tv), d(u, v) , d(u, Tu) , d(v, Tv) , d(v, Tu) , d(Tu, T^2u)) ≤ 0 , F(d(u, v), d(u, v), 0, 0,d (u, v), 0) ≤ 0, a contradiction with (Fu).

Lemma1. (Popa [9]) Let (X, d) be a metric space and T: (X, d) → (X, d) be a mapping such that there exists h ∈ (0, 1) with d(T^2x, Tx) ≤ h d (x, Tx) for every x ∈X. Then for every x ∈X the sequence {T^nx}, u = 0, 1, 2..., is a Cauchy sequence.

Theorem2. Let (X, d) be a metric space and T: (X, d) → (X, d) be a mapping. If X is T – orbitally complete and T satisfies conditions (Fm), (Fh), (Fu) and inequality (5) for all x, y in X, then T has a unique fixed point.

Proof. Let x be arbitrary in X. We show that the sequence defined by x_{n+1} = T^nx is a Cauchy sequence. From (5) for y = Tx we have F(d(Tx, T^2x), d(x, Tx) , d(x, Tx) , d(Tx, T^2x), 0, d(Tx, T^2x)) ≤ 0.

By (Fh) we have d (Tx, T^2x) ≤ h d (x, Tx). By Lemma 1 the sequence {T^nx} is a Cauchy sequence. Since $x_n \in O_X$(T) and X is T-orbitally complete, the sequence {x_n} is convergent in X and there exists u∈X such that *lim* x_n= u. By (5) we have successively F(d(Tx_n, Tu) d(x_n,u), d(x_n, Tx_n), d(u, Tu), d(u, Tx_n) ,d(Tx_n, T^2 x_n) ≤ 0 and

F(d(x_{n+1}u,Tu), d(x_n,u) ,d(x_n, x_{n+1}), d(u,Tu), d(u, x_{n+1}), d(x_{n+1}, x_{n+2}))≤ 0.

Letting n tend to infinity we have successively

F(d(u, Tu), 0, 0, d(u, Tu), 0, 0) $\leq$ 0.
F(d(u, Tu), 0, 0, d(u, Tu), 0, d(u, Tu)) $\leq$ 0 wich implies by (Fh) that d(u, Tu) = 0. Hence u = Tu. By Theorem 1 u is the unique fixed point of T .

Theorem3. Let T: (X, d) $\rightarrow$ (X, d) be a mapping where (X, d) is T-orbitally complete. If F satisfies conditions (Fm), (Fh), (Fu) and (Fp) and inequality (5) for all x, y in X, then the fixed point problem is well-posed.

Proof. By Theorem *2*, T has a unique fixed point x_0, i.e. $x_0 = Tx_0$. Let $\{x_n\}$ be a sequence in X such that d $(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$. Then by (5) we have successively

$F(d(Tx_0, Tx_n), d(x_0, x_n), d(x_0, Tx_0), d(x_n, Tx_n), d(x_n, Tx_0), d(Tx_0, T^2 x_0))$ ≤ 0 and

$F(d(x_0, Tx_n), d(x_0, x_n), 0, d(x_n, Tx_n), d(x_n, x_0), 0) \leq 0$.

By (Fp) we have:

$d(x_0, Tx_n) \leq p \max\{d(x_0, x_n), d(x_n, Tx_n)\} \leq p\,[d(x_0, x_n) + d(x_n, Tx_n)]$.

Therefore

$d(x_0, x_n) \leq d(x_0, Tx_n) + d(Tx_n, x_n) \leq p[d(x_0, x_n) + d(x_n, Tx_n)] + d(x_n, Tx_n)$,

which implies $d(x_0, x_n) \leq \frac{1+p}{1-p} d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$. This proves the theorem.

Corollary 1. (Lahiri and Dos [6]). Let T: (X, d) $\rightarrow$(X, d) be a mapping , where (X,d) is a complete metric space. If F satisfies conditions (Fu), (Fh), (Fw), (Fp) and inequality (5) holds for all x,y in X, then the fixed-point problem is well-posed.

Let T: (X, d) $\rightarrow$(X, d) be a mapping such that there exists $K \in (0, \frac{1}{2})$ such that for all x, y $\in$ X, d(Tx, Ty) $\leq$ K [d(x, Tx) +d(y, Ty)].

Kannan [4] proved that if (X,d) is complete then T has a unique fixed point.

Corollary 2. (Lahiri and Dos [6]). The fixed point problem for Kannan's map in a complete metric space X is well -posed.

Proof. The proof follows by Remark 1, Theorem 3 and Example 5 for $a_2 = a_3 = k$ and $a_1 = a_4 = a_5 = 0$.

Remark2. By Theorem3 and Example 1-5 we obtain new results.

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