"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 19 (2009), No. 1, 5 - 16

THE REMAINDER TERM OF SOME QUADRATURE FORMULAE

ANA MARIA ACU, ARIF RAFIQ, CARMEN VIOLETA MURARU

Abstract. In this paper we give some new quadrature formulae and we derive the estimates for the remainder term. The optimality in the sense of Nikolski of some quadrature formulae is studied. A property of the intermediate point for the optimal quadrature formula is established.

1. Introduction

Let $\mathcal{F}$ be a linear space of real valued functions, defined and integrable on a finite interval $[a,b] \subset \mathbb{R}$ and $I : \mathcal{F} \to \mathbb{R}$ be the integration operator defined by $I[f] = \displaystyle\int_a^b f(x)dx$. For $f \in \mathcal{F}$, one considers the quadrature formula

$$I[f] = \int_a^b f(x)dx = \sum_{i=1}^{n} A_i f(x_i) + \mathcal{R}[f], \tag{1}$$

where $x_i \in [a,b]$, respectively A_i, $i = \overline{0,m}$ are called the nodes, respectively the coefficients of the quadrature formula, and $\mathcal{R}[f]$ is the remainder term.

Keywords and phrases: quadrature rule, optimal quadrature, intermediate point
(2000)Mathematics Subject Classification: 65D30 , 65D32

We denote

$$\begin{aligned} H^n[a,b] &:= \left\{ f \in C^{n-1}[a,b] : f^{(n-1)} \quad \text{absolutely continuous} \right\} \\ W_n^p[a,b] &:= \left\{ f \in H^n[a,b] : \ \left\| f^{(n)} \right\|_p < \infty \right\} \end{aligned}$$

with

$$\begin{aligned} \|f\|_p &:= \left\{ \int_a^b |f(x)|^p \, dx \right\}^{\frac{1}{p}} \quad \text{for} \quad 1 \le p < \infty \\ \|f\|_\infty &:= \sup_{x \in [a,b]} |f(x)| \, . \end{aligned}$$

2. Inequalities for the remainder term of some quadrature formulae

Recently, N. Ujević ([5]) obtained the following result:

Theorem 2.1. [5] *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) and suppose that $\gamma \le f''(t) \le \Gamma$ for all $t \in (a,b)$. Then we have the double inequality:*

$$\frac{3S - \Gamma}{24}(b-a)^2 \le \frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \le \frac{3S - \gamma}{24}(b-a)^2,$$

where $S = \dfrac{f'(b) - f'(a)}{b-a}$.

In this section we generalize this result and we obtain new quadrature formulae. Some double inequalities for the remainder term of these quadrature formulae will be established.

We define a mapping $Q : [a,b]^2 \to \mathbb{R}$,

$$Q(x,t) = \begin{cases} \dfrac{1}{n!}\left[(t-x)^n - (x-a)^n\right], \ \text{if } t \in [a,x), \\ \\ \dfrac{1}{n!}\left[(t-x)^n - (x-b)^n\right], \ \text{if } t \in [x,b]. \end{cases}$$

Theorem 2.2. *Let $n \in \mathbb{N}$, $n > 1$ be an even number. If $f \in H^n[a,b]$, and there exist real numbers γ, Γ such that $\gamma \le f^{(n)}(t) \le \Gamma$, $t \in [a,b]$,* *then we have the following quadrature formula*

$$\begin{aligned}\int_a^b f(t)dt &= \sum_{k=0}^{n-2}\frac{(x-a)^{k+1}}{(k+1)!}f^{(k)}(a)-\sum_{k=0}^{n-2}\frac{(x-b)^{k+1}}{(k+1)!}f^{(k)}(b)\\ &- \frac{1}{n!}\left[(b-x)^n-(x-a)^n\right]f^{(n-1)}(x)+\mathcal{R}_n[f],\\ \mathcal{R}_n[f] &= \int_a^b Q(x,t)f^{(n)}(t)dt,\end{aligned}$$

and the remainder term satisfies the double inequality

$$\begin{aligned}&-\frac{n\gamma}{(n+1)!}N_2-\frac{1}{n!}N_1(S-\gamma)(b-a)\le\mathcal{R}_n[f]\\ &\le -\frac{n\Gamma}{(n+1)!}N_2+\frac{1}{n!}N_1(b-a)(\Gamma-S),\end{aligned}$$

where

$$\begin{aligned}&S=\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a},\ N_1=\left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^n,\\ &N_2=(x-a)^{n+1}+(b-x)^{n+1}.\end{aligned}$$

Proof. We have

$$\begin{aligned}\int_a^b Q(x,t)dt &= \frac{1}{n!}\int_a^x\left[(t-x)^n-(x-a)^n\right]dt\\ &+ \frac{1}{n!}\int_x^b\left[(t-x)^n-(x-b)^n\right]dt\\ &= -\frac{n}{(n+1)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right]=-\frac{n}{(n+1)!}N_2.\end{aligned}$$

Integrating by parts, we have

$$\int_a^b Q(x,t)f^{(n)}(t)dt = \int_a^x \left[\frac{(t-x)^n}{n!} - \frac{(x-a)^n}{n!}\right] f^{(n)}(t)dt$$

$$+\int_x^b \left[\frac{(t-x)^n}{n!} - \frac{(x-b)^n}{n!}\right] f^{(n)}(t)dt$$

$$=\sum_{k=0}^{n-1}(-1)^{n-1-k}\left[\frac{(t-x)^n}{n!} - \frac{(x-a)^n}{n!}\right]^{(n-1-k)} f^{(k)}(t)\Bigg|_a^x + (-1)^n\int_a^x f(t)dt$$

$$+\sum_{k=0}^{n-1}(-1)^{n-1-k}\left[\frac{(t-x)^n}{n!} - \frac{(x-b)^n}{n!}\right]^{(n-1-k)} f^{(k)}(t)\Bigg|_x^b + (-1)^n\int_x^b f(t)dt$$

$$=\frac{1}{n!}\left[(b-x)^n-(x-a)^n\right] f^{(n-1)}(x) - \sum_{k=0}^{n-2}\frac{(x-a)^{k+1}}{(k+1)!}f^{(k)}(a)$$

$$+\sum_{k=0}^{n-2}\frac{(-1)^{k+1}}{(k+1)!}(b-x)^{k+1}f^{(k)}(b) + \int_a^b f(t)dt,$$

and the desired quadrature formula is produced.

From the relations above we obtain

$$\int_a^b Q(x,t)\left[\gamma - f^{(n)}(t)\right]dt = -\mathcal{R}_n[f] - \frac{n\gamma}{(n+1)!}N_2.$$

Since

$$\int_a^b Q(x,t)\left(\gamma - f^{(n)}(t)\right) \le \max|Q(x,t)| \int_a^b \left(f^{(n)}(t)-\gamma\right)dt$$
$$= \max|Q(x,t)| \cdot \left[f^{(n-1)}(b) - f^{(n-1)}(a) - \gamma(b-a)\right]$$
$$= \max|Q(x,t)| \cdot (S-\gamma)(b-a),$$

$$\max|Q(x,t)| = \frac{1}{n!}\max\{(x-a)^n,(b-x)^n\} = \frac{1}{n!}\left(\max\{x-a,b-x\}\right)^n$$
$$= \frac{1}{n!}\left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right)^n = \frac{1}{n!}N_1,$$

we obtain

$$\mathcal{R}_n[f] \ge -\frac{n\gamma}{(n+1)!}N_2 - \frac{1}{n!}N_1(S-\gamma)(b-a).$$

On the other hand we have

$$\int_a^b Q(x,t)\left[f^{(n)}(t)-\Gamma\right]dt = \mathcal{R}_n[f] + \frac{n\Gamma}{(n+1)!}N_2.$$

and

$$\begin{aligned}&\int_a^b Q(x,t)\left[f^{(n)}(t)-\Gamma\right]dt \le \max|Q(x,t)|\cdot\int_a^b \left(\Gamma - f^{(n)}(t)\right)dt\\ &= \max|Q(x,t)|\cdot\left[\Gamma(b-a)-\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\right]\\ &= \max|Q(x,t)|\,(b-a)(\Gamma-S).\end{aligned}$$

From the relations above we obtain

$$\mathcal{R}_n[f] \le -\frac{n\Gamma}{(n+1)!}N_2 + \frac{1}{n!}N_1(b-a)(\Gamma-S).$$

Remark 2.1. *If in Theorem 2.2 we choose $n=2$, we obtain the following quadrature formula*

(2)
$$\int_a^b f(t)dt = (x-a)f(a)+(b-x)f(b)+(b-a)\left(x-\frac{a+b}{2}\right)f'(x)+\mathcal{R}_2[f],$$

and the remainder term satisfies the double inequality

$$-\frac{\gamma}{3}N_2 - \frac{1}{2}N_1(S-\gamma)(b-a) \le \mathcal{R}_2[f] \le -\frac{\Gamma}{3}N_2 + \frac{1}{2}N_1(b-a)(\Gamma-S),$$

where

$$S=\frac{f'(b)-f'(a)}{b-a},\, N_1=\left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^2,\, N_2=(x-a)^3+(b-x)^3.$$

and

$$\gamma \le f''(t) \le \Gamma.$$

For $x=\dfrac{a+b}{2}$ we obtain the inequality from Theorem 2.1 ([5]*).*

If in (2) we choose $x=a$, $x=b$, respectively, sum the results and divide by 2, we obtain the following quadrature formula

$$\int_a^b f(t)dt = \frac{b-a}{2}\left[f(a)+f(b)\right]+\frac{(b-a)^2}{4}\left[f'(b)-f'(a)\right]+\mathcal{R}_2[f], \tag{3}$$

where

$$\frac{(b-a)^3}{6}(\gamma-3S) \le \mathcal{R}_2[f] \le \frac{(b-a)^3}{6}(\Gamma-3S).$$

3. Optimal quadrature formula in the sense of Nikolski

Definition 3.1. *The quadrature formula (1) is called optimal in the sense of Nikolski in the space $\mathcal{F}$, if*

$$\mathcal{E}_n\left(\mathcal{F}, A, X\right) = \sup_{f\in\mathcal{F}} \left|\mathcal{R}_n[f]\right|,$$

attains the minimum value with regard to the coefficients $A = (A_1, \ldots, A_n)$ and the quadrature nodes $X = (x_1, \ldots, x_n)$ for A and X of (1).

In this section we obtain a new quadrature formula and we study its optimality in the sense of Nikolski.

Let $(\Delta_m)_{m\in\mathbb{N}}$ be a division of $[a, b]$, $\Delta_m : a = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = b$ and

$$K(t) = \begin{cases} \frac{1}{n!}\left[(t-x_i)^n - (x_i - x_{i-1})^n\right], & \text{if } t \in [x_{i-1}, x_i),\ i = \overline{1, m-1}, \\ \frac{1}{n!}\left[(t-x_m)^n - (x_m - x_{m-1})^n\right], & \text{if } t \in [x_{m-1}, x_m] \end{cases}$$

Theorem 3.1. *Let $n \in \mathbb{N}$, $n > 1$ be an even number. If $f \in H^n[a, b]$, $n > 1$, then*

$$\int_a^b f(t)dt = \sum_{k=0}^{n-2}\sum_{i=0}^{m-1} A_{k,i} f^{(k)}(x_i) + \sum_{i=0}^{m-1} A_{n-1,i} f^{(n-1)}(x_{i+1}) + \mathcal{R}_n[f], \tag{4}$$

where

$$\mathcal{R}_n[f] = \int_a^b K(t) f^{(n)}(t)dt,$$
$$A_{k,i} = \frac{(x_{i+1}-x_i)^{k+1}}{(k+1)!}, \quad k = \overline{0, n-1}, \quad i = \overline{0, m-1}.$$

Proof. Integrating by parts, we have

$$\int_a^b K(t)f^{(n)}(t)dt = \sum_{i=1}^m \int_{x_{i-1}}^{x_i} \frac{1}{n!}\left[(t-x_i)^n - (x_i - x_{i-1})^n\right] f^{(n)}(t)dt$$
$$= \sum_{i=1}^m \sum_{k=0}^{n-1} \frac{(-1)^{n-1-k}}{n!}\left[(t-x_i)^n - (x_i - x_{i-1})^n\right]^{(n-1-k)} f^{(k)}(t)\Big|_{x_{i-1}}^{x_i}$$
$$+(-1)^n \int_a^b f(t)dt = -\sum_{i=1}^m \sum_{k=0}^{n-2} \frac{(x_i - x_{i-1})^{k+1}}{(k+1)!} f^{(k)}(x_{i-1})$$
$$-\frac{1}{n!}\sum_{i=1}^m (x_i - x_{i-1})^n f^{(n-1)}(x_i) + \int_a^b f(t)dt$$
$$= -\sum_{i=0}^{m-1} \sum_{k=0}^{n-2} \frac{(x_{i+1} - x_i)^{k+1}}{(k+1)!} f^{(k)}(x_i)$$
$$-\frac{1}{n!}\sum_{i=0}^{m-1} (x_{i+1} - x_i)^n f^{(n-1)}(x_{i+1}) + \int_a^b f(t)dt.$$

From this relation we obtain the quadrature formula (4).

Next, we will study the optimality in sense of Nikolski for this quadrature formula.

If $f \in W_n^\infty[a,b]$ for rest term we have the evaluation

$$|\mathcal{R}_n[f]| \le M_n^\infty[f] \int_a^b |K(t)|\, dt,$$

where $M_n^\infty[f] = \sup\limits_{t\in[a,b]} \left|f^{(n)}(t)\right|.$

The quadrature formula (4) is optimal in sense of Nikolski in $W_n^\infty[a,b]$, if $\displaystyle\int_a^b |K_n(t)|\, dt$ attains the minimum value.

Theorem 3.2. *Let $n \in \mathbb{N}$, $n > 1$ be a even number. If $f \in W_n^\infty[a,b]$, then quadrature formula (4), optimal with regard to the error, is*

$$\int_a^b f(t)dt = \sum_{k=0}^{n-2}\sum_{i=0}^{m-1} A_k^* f^{(k)}(x_i^*) + \sum_{i=0}^{m-1} A_{n-1}^* f^{(n-1)}(x_i^*) + \mathcal{R}_n^*[f], \tag{5}$$

where

$$x_i^* = a + \frac{b-a}{m} i, \ i = \overline{0, m}$$

$$A_k^* = \frac{(b-a)^{k+1}}{(k+1)! m^{k+1}}, \ k = \overline{0, n-1}, \tag{6}$$

with $|\mathcal{R}_n^*[f]| \leq \dfrac{n(b-a)^{n+1}}{m^n (n+1)!} M_n^\infty[f]$.

Proof. We will determine the parameters x_i, $i = \overline{1, m-1}$ for which

$$F(x) = \int_a^b |K(t)| \, dt, \quad x = (x_1, \dots, x_{m-1}),$$

attains the minimum value. We have

$$F(x) = \sum_{i=1}^m \int_{x_{i-1}}^{x_i} \frac{1}{n!} \left[(x_i - x_{i-1})^n - (t - x_i)^n \right] dt = \frac{n}{(n+1)!} \sum_{i=1}^m (x_i - x_{i-1})^{n+1}.$$

The optimal nodes constitute the solution of the system

$$\frac{\partial F(x)}{\partial x_k} = \frac{1}{(n-1)!} \left\{ (x_k - x_{k-1})^n - (x_{k+1} - x_k)^n \right\}, \ k = \overline{1, m-1}. \tag{7}$$

From (7) we obtain

$$x_{k+1} - 2x_k + x_{k-1} = 0, \quad k = \overline{1, m-1}. \tag{8}$$

From recurrent relation (8) we obtain

$$x_k = a + \frac{b-a}{m} k, \quad k = \overline{1, m-1}.$$

Because the quadratic form

$$\phi = \sum_{i=1}^{m-1} \sum_{j=1}^{m-1} \frac{\partial^2 F(x)}{\partial x_i \partial x_j} a_i a_j$$

in the stationary point $x = (x_1, x_2, \dots, x_{m-1})$ is positive, namely

$$\phi = \frac{n}{(n-1)!} \frac{(b-a)^{n-1}}{m^{n-1}} \left\{ a_1^2 + \sum_{i=1}^{m-2} (a_i - a_{i+1})^2 + a_{m-1}^2 \right\},$$

then $F(x)$ attains the minimum value for the knots

$$x_i^* = a + \frac{b-a}{m} i, \quad i = \overline{1, m-1}$$

and the coefficients of optimal quadrature formula are given in (6).

Finally, we have

$$F(x^*) = \frac{n(b-a)^{n+1}}{m^n(n+1)!}, \quad \text{where} \quad x^* = (x_1^*, \ldots, x_{m-1}^*)$$

and

$$|\mathcal{R}_n^*[f]| \leq \frac{n(b-a)^{n+1}}{m^n(n+1)!} M_n^\infty[f].$$

4. An intermediate point property in the quadrature formula

In this section we study a property of the intermediate point for the quadrature formula (5). It is well know that if $f : [a,b] \to \mathbb{R}$, $f \in C^n[a,b]$ and n is an even number, then for any $x \in (a,b]$, there exists $c_x \in (a,x)$ such that

$$\begin{aligned} \int_a^x f(t)dt &= \sum_{k=0}^{n-2} \frac{(x-a)^{k+1}}{(k+1)!m^{k+1}} f^{(k)}(a) + \frac{(x-a)^n}{n!m^n} f^{(n-1)}(x) \\ &+ \sum_{k=0}^{n-1}\sum_{i=1}^{m-1} \frac{(x-a)^{k+1}}{(k+1)!m^{k+1}} f^{(k)}\left(a + \frac{x-a}{m} i\right) \\ &- \frac{n(x-a)^{n+1}}{m^n(n+1)!} f^{(n)}(c_x). \end{aligned} \tag{9}$$

Theorem 4.1. *If $f \in C^{2n+1}[a,b]$, n is even and $f^{(n+1)}(a) \neq 0$, then for the intermediate point c_x that appears in formula (9), we have*

$$\lim_{x \to a} \frac{c_x - a}{x - a} = \frac{1}{2}\left\{1 + \frac{1}{m(n+2)}\right\}.$$

Proof. Let $F, G : [a,b] \to \mathbb{R}$ be defined as follows

$$\begin{aligned} F(x) &= \int_a^x f(t)dt - \sum_{k=0}^{n-2} \frac{(x-a)^{k+1}}{(k+1)!m^{k+1}} f^{(k)}(a) - \frac{(x-a)^n}{n!m^n} f^{(n-1)}(x) \\ &- \sum_{k=0}^{n-1}\sum_{i=1}^{m-1} \frac{(x-a)^{k+1}}{(k+1)!m^{k+1}} f^{(k)}\left(a + \frac{x-a}{m} i\right) + \frac{n(x-a)^{n+1}}{m^n(n+1)!} f^{(n)}(a), \\ G(x) &= (x-a)^{n+2}. \end{aligned}$$

We observe that $F(a) = 0$ and for $j = \overline{1, n+1}$ we have

$$F^{(j)}(a) = f^{(j-1)}(a)\left[1 - \frac{1}{m^j} - \frac{1}{m^j}\sum_{i=1}^{m-1}\sum_{k=0}^{j-1}\begin{pmatrix} j \\ k+1 \end{pmatrix} i^{j-1-k}\right] = 0,$$

and

$$F^{(n+2)}(a) = f^{(n+1)}(a)\left\{1 - \frac{1}{m^{n+2}}\sum_{i=1}^{m-1}\sum_{k=0}^{n-1}\binom{n+2}{k+1} i^{n-k+1} - \frac{(n+2)(n+1)}{2m^n}\right\}$$
$$= -\frac{n}{2m^{n+1}}\left\{2m + 1 + nm\right\} f^{(n+1)}(a).$$

Applying L′ Hospital′s rule successively, we obtain

$$\lim_{x\to a}\frac{F(x)}{G(x)} = \lim_{x\to a}\frac{F^{(n+2)}(x)}{G^{(n+2)}(x)} = -\frac{n}{2m^{n+1}(n+2)!}\left\{2m + 1 + nm\right\} f^{(n+1)}(a). \tag{10}$$

On the other hand

$$\begin{aligned} \lim_{x\to a}\frac{F(x)}{G(x)} &= \lim_{x\to a} -\frac{n(x-a)^{n+1}}{m^n(n+1)!}\,\frac{f^{(n)}(c_x) - f^{(n)}(a)}{(x-a)^{n+2}} \\ &= \lim_{x\to a} -\frac{n}{m^n(n+1)!}\,\frac{f^{(n)}(c_x) - f^{(n)}(a)}{c_x - a}\cdot\frac{c_x - a}{x - a} \\ &= -\frac{n}{m^n(n+1)!} f^{(n+1)}(a)\lim_{x\to a}\frac{c_x - a}{x - a}. \end{aligned} \tag{11}$$

From (10) and (11) we obtain

$$\lim_{x\to a}\frac{c_x - a}{x - a} = \frac{1}{2}\left\{1 + \frac{1}{m(n+2)}\right\}.$$

References

[1] G.A.Anastassiou, Ostrowski type inequalities, **Proc. Amer. Math.Soc.**, Vol 123,No 12, 1995, 3775-3781.

[2] P. Cerone, S.S. Dragomir, **Midpoint type rules from an inequalities point of view**, in Analytic-Computational Methods in Applied Mathematics, G.A. Anastassiou (Ed), CRC Press, New York, 2000,135-200.

[3] P. Cerone, S.S. Dragomir, **Trapezoidal type rules from an inequalities point of view**, in Analytic-Compuational Methods in Applied Mathematics, G.A. Anastassiou (Ed), CRC Press, New York, 2000, 65-134.

[4] F. Qi, Z.-L. Wei, Q. Yang, **Generalization and refinements of Hermite-Hadamard′s inequality**, Rocky Mountain J. Math 35(1) (2005), 235-251.

[5] N. Ujević, **Some double integral inequalities and applications**, Acta Math. Univ. Comenianae, Vol. LXXI, 2(2002), 189-199.

[6] N.Ujević, **A generalization of Ostrowski′s inequality and applications in numerical integration**, Appl. Math. Lett., 17(2), 2004, 133-137.

Arif Rafiq
COMSATS Institute of Information Technology
Department of Mathematics
Islamabad, Pakistan
e-mail: arafiq@comsats.edu.pk

Ana Maria Acu,
"Lucian Blaga" University
Department of Mathematics
Sibiu, Romania
e-mail: acuana77@yahoo.com

Carmen Violeta Muraru
University of Bacau
Department of Mathematics
Bacau, Romania
e-mail: carmen_7419@yahoo.com