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COMMON FIXED POINTS IN CONVEX 2-METRIC SPACES

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Abstract. In this paper, we introduce the concept of convex 2-metric spaces and present a generalization of Banach contraction principle in this newly defined space. Our result is a generalization of some well known results of 2-metric spaces. In 1970 Takahashi [4] introduced the notion of convex metric spaces and studied some fixed point theorems for non-expansive mappings in this space. Gahler [1] introduced the concept of 2-metric space alike to metric space. The purpose of this paper is to introduce the concept of convex 2-metric space analogue to convex metric space and obtain a generalization of Banach contract principle in this space.

Let X be a 2-metric space and $I = [0,1]$. A mapping W: $X \times X \times I \to X$ is called convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $a, u \in X$

$$d(u, a, W(x, y, \lambda)) \le \lambda\, d(u, x, a) + (1-\lambda)\, d(u, y, a)$$

A 2-metric space X together with a convex structure W is called a convex 2-metric space.

Definition 1.1. A non-empty subset K of X is said to be convex if

$$W(x, y, \lambda) \in K \text{ for all } (x, y, \lambda) \in K \times K \times I .$$

A 2-Banach space or any convex subset of a 2-Banach space is a convex 2-metric space. Since by Gahler [1]

$$d(a, b, c) = \| b-a, c-a \|$$

now

$$\begin{aligned} d(u, a, \lambda x + (1-\lambda)y) &= \| a-u, (\lambda x + (1-\lambda)y) - a \| \\ &= \| a-u, \lambda x - \lambda a + \lambda a - \lambda y + y-u \| \\ &= \| a-u, \lambda(x-u) + \lambda(u-y) + (y-u) \| \\ &\le \lambda\| a-u, x-u \| + (1-\lambda) \| y-u, a-u \| \\ &= \lambda\, d(u, x, a) + (1-\lambda)\, d(u, y,a) \end{aligned}$$

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Definition 1.2. Let (X, d) be a convex 2-metric space and let K be a convex subset of X. A mapping $I : K \to K$ is said to be W-affine if

$$IW(x, y, \lambda) = W(Ix, Iy, \lambda) \text{ for all } (x, y, \lambda) \in K \times K \times I.$$

Definition 1.3. Let (X,d) be a 2-metric space and let $S, T : X \to X$ be two mappings. S and T are said to be compatible if, whenever $\{x_n\}$ is sequence in X such that $Sx_n, Tx_n \to t \in X$, then

$$d(STx_n, TSx_n, a) \to 0 \text{ for all } a \,\varepsilon\, X.$$

Throughout this paper, we assume that X is complete convex 2-metric space with a convex structure W and K is a non-empty closed convex subset of X.

Theorem. *Let I and T be compatible mappings of K into itself satisfying the following condition:*

$$d(Tx, Ty, a) \le a\, d(Ix, Iy, a) \text{ for all } x, y \text{ and } a \text{ in } K, \text{ where } 0 < a < 1/4 . \qquad (1)$$

If I is W-affine and continuous in K and $T(K) \subset I(K)$, then T and I have a unique common fixed point z in K and T is continuous at z .

Proof. Let $x = x_0$ be an arbitrary point in K. Then, by (1), since $T(K) \subset I(K)$, for an arbitrary point $x_0 \in K$, there exists a point $x \in K$ such that $Ix_1 = Tx$. For $x_1 \in K$ there exists a point $x_2 \in K$ such that $Tx_1 = Ix_2$. Similarly for x_3 we have $Ix_3 = Tx_2$. For r = 1, 2, 3,..., (1) leads to

$$\begin{aligned} d(Tx_r, Ix_r, a) &= d(Tx_r, Tx_{r-1}, a) \\ &\le a\, d(Ix_r, Ix_{r-1}, a) \\ &= a\, d(Tx_{r-1}, Ix_{r-1}, a) < d(Tx_{r-1}, Ix_{r-1}, a) \qquad (2) \end{aligned}$$

From (1), (2), and

$$\begin{aligned} d(Tx_2, Ix_1, a) &= d(Tx_2, Tx, a) \\ &\le a\, d(Ix_2, Ix, a) \\ &\le a[d(Ix_2, Ix_1, a) + d(Ix_1, Ix, a) + d(Ix_2, Ix_1, Ix)] \\ &= a[d(Tx_1, Ix_1, a) + d(Ix_1, Ix, a) + 0] \\ &\le a[d(Tx, Ix, a) + d(Tx, Ix, a)] \\ &= 2\, a\, d(Tx, Ix, a), \quad \text{we have} \qquad (3) \end{aligned}$$

$$d(Ix_2, Ix_1, Ix) = d(Tx_1, Ix_1, Ix) \le d(Tx, Ix, Ix) = 0$$

Letting $z = W(x_2, x_3, 1/2)$, then $z \in K$

$$Iz = W(Ix_2, Ix_3, 1/2) = W(Tx_1, Tx_2, 1/2), \qquad (4)$$

From (2), (3) and (4)

$$\begin{aligned} d(Iz, Ix_1, a) &= d(Ix_1, Iz, a) \qquad (5) \\ &= d(Ix_1, W(Tx_1, Tx_2, 1/2), a) \\ &\le 1/2\, d(Ix_1, Tx_1, a) + 1/2\, d(Ix_1, Tx_2, a) \\ &\le 1/2\, [d(Ix, Tx, a) + 2\, ad(Tx, Ix, a)] \\ &= 1/2[(2a + 1)]\, d(Tx, Ix, a) \end{aligned}$$

and $d(Iz, Ix_2, a) = d(Ix_2, W(Tx_1, Tx_2, 1/2), a)$

$\leq 1/2\ d(Ix_2, Tx_1, a) + 1/2\ d(Ix_2, Tx_2, a)$

$= 1/2\ d(Ix_2, Tx_2, a) \leq 1/2\ d(Ix, Tx, a)$ (6)

From (1), (2), (3) and (6), we have

$$
\begin{aligned}
d(Tz, Iz, a) &= d(Tz, W(Tx_1, Tx_2, 1/2), a) \\
&\leq 1/2\ d(Tz, Tx_1, a) + 1/2\ d(Tz, Tx_2, a) \\
&= 1/2\ d(Tz, Ix_2, a) + 1/2\ d(Tz, Ix_1, a) \\
&\leq 1/2\ [1/2\ d(Ix, Tx, a)] + 1/2(2a+1)\ d(Ix, Tx, a)] \\
&= [1/4 + 1/2(2a+1)]\ d(Ix, Tx, a).
\end{aligned}
$$

Therefore $d(Tz, Iz, a) \leq \lambda d(Ix, Tx, a)$, where

$$\lambda = [1/4 + 1/2\ (2a+1)] \tag{7}$$

Since x is an arbitrary point in K, from (7), it follows that there exists a sequence $\{z_n\}$ in K such that

$d(Tz_0, Iz_0, a) \leq \lambda\ d(Tx_0, Ix_0, a)$,

$d(Tz_1, Iz_1, a) \leq \lambda\ d(Tz_0, Iz_0, a)$,

$d(Tz_n, Iz_n, a) \leq \lambda\ d(Tz_{n-1}, Iz_{n-1}, a)$, which yields that

$d(Tz_n, Iz_n, a) \leq \lambda^{n+1}\ d(Tx_0, Ix_0, a)$ and so we have

$$\lim_{n\to\infty} d(Tz_n, Iz_n, a) = 0 \tag{8}$$

Setting $K_n = \{x \in K : d(Tx, Ix, a) \leq 1/n\}$ for n = 1, 2,..., then (8) shows that $Kn \neq \phi$ for n = 1, 2,...., and $K_1 \supset K_2 \supset k_3$ ..... Obviously, we have $\overline{T}k_n \neq \phi$ and $\overline{T}k_n \supset \overline{T}k_{n+1}$ for n = 1,2,... . From (1), we have

$$
\begin{aligned}
d(Tx, Ty, a) &\leq a\ d(Ix, Iy, a) \\
&\leq a\ d(Ix, Tx, a) + d(Tx, Iy, a) + d(Ix, Iy, Tx)] \\
&\leq a[d(Ix, Tx, a) + d(Ix, Tx, Iy) + d(Tx, Ty, a) \\
&\qquad + d(Tx, Iy, Ty) + d(Ty, Iy, a)]
\end{aligned} \tag{9}
$$

$d(Tx, Ty, a) \leq 4/n\ (a/(1-a))$.

Therefore, we have

$$\lim_{n\to\infty} \operatorname{diam}(\overline{T}K_n) = \lim_{n\to\infty} \operatorname{diam}(TK_n) = 0.$$

By Cantor's intersection theorem, there exists a point u in K such that

$$\bigcap_{n=1}^{\infty} \overline{T}K_n = u.$$

Since $u \in K$, for each n = 1,2,..., there exists a point y_n in TK_n such that $d(y_n, u, a) < 1/n$. Then there exists a point x_n in K_n such that

$d(u, Tx_n, a) < 1/n$ and so $Tx_n \to u$ as $n\to\infty$.

Since $x_n \in K_n$, we have also

$d(Tx_n, Ix_n, a) < 1/n$ and so $Ix_n \to u$ as $n\to\infty$.

Since I is continuous, $ITx_n \to Iu$ and $IIx_n \to Iu$ as $n \to \infty$. Moreover, $d(TIx_n, ITx_n, a) \to 0$ as $n \to \infty$. Since I and T are compatible and $Tx_n, Ix_n \to u$ as $n \to \infty$. Thus we have $TIx_n \to Iu$. Now

$$d(Tu, Iu, a) \le d(Tu, TIx_n, a) + d(TIx_n, Iu, a) + d(TIx_n, Iu, Tu)$$

From (9)

$$d(Tu, Iu, a) \le 4/n\ (a/(1-a)) + d(TIx_n, Iu, a) + d(TIx_n, Iu, Tu)$$

Letting as $n \to \infty$, we have $Tu = Iu$.

Thus $TIu = ITu$ and $TTu = TIu = ITu$ since I and T are compatible. Furthermore, we have

$$d(TTu, TTu, a) \le a\, d(ITu, Iu, a)$$
$$= a\, d(TTu, Tu, a),$$ a contradiction, since $a < 1/4$.

Therefore, $TTu = Tu$. Let $z = Tu = Iu$. Then $Tz = z$ and $Iz = ITz = TIz = Tz = z$. Obviously, z is a unique common fixed point of T and I.

Now to prove T is continuous at z . Let $\{y_n\}$ be a sequence in K such that $y_n \to z$.

Since I is continuous, $Iy_n \to Iz$. By (9)

$$d(Ty_n, Tz, a) \le 4/n \quad a/(1-a).$$

Letting $n \to \infty$, we have $Ty_n = Tz$ and so T is continuous at z. This completes the proof of the theorem.

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