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BOUNDEDNESS OF MULTILINEAR SINGULAR INTEGRAL OPERATOR WITH NON-SMOOTH KERNEL ON L^p SPACES WITH VARIABLE EXPONENT

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Abstract: In this paper, the boundedness for some multilinear operators related to some singular integral operator with non-smooth kernel on L^p spaces with variable exponent is obtained by using a sharp estimate of the multilinear operators.

1. Introduction

Due to the development of the singular integral operators and their commutators, multilinear singular integral operators have been well studied(see [1-5][17-19]). In the last years, a theory of L^p spaces with variable exponent has been developed because of its connections with some questions in fluid dynamics, calculus of variations, differential equations and elasticity(see [6][8-9][13-14][16] and their references).

Karlovich and Lerner study the boundedness of the commutators of singular integral operators on L^p spaces with variable exponent(see [13]). The main purpose of this paper is to introduce some multilinear operator related to some singular integral operator with non-smooth kernel and prove the boundedness for the multilinear operator on L^p spaces with variable exponent by using a sharp estimate of the multilinear operator.

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2. Preliminaries and Results

In this paper, we study some multilinear singular integral operators with non-smooth kernels as follows.

Definition 1. A family of operators $D_t, t > 0$ is said to be an "approximation to the identity" if, for every $t > 0$, D_t can be represented by the kernel $a_t(x, y)$ in the following sense:

$$D_t(f)(x) = \int_{R^n} a_t(x, y)f(y)dy$$

for every $f \in L^p(R^n)$ with $p \geq 1$, and $a_t(x, y)$ satisfies:

$$|a_t(x, y)| \leq h_t(x, y) = Ct^{-n/2}s(|x-y|^2/t),$$

where s is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon}s(r^2) = 0$$

for some $\epsilon > 0$.

Definition 2. A linear operator T is called a singular integral operator with non-smooth kernel if T is bounded on $L^2(R^n)$ and associated with a kernel $K(x, y)$ such that

$$T(f)(x) = \int_{R^n} K(x, y)f(y)dy$$

for every continuous function f with compact support, and for almost all x not in the support of f. In addition, we assume that

(1) There exists an "approximation to the identity" $\{B_t, t > 0\}$ such that TB_t has associated kernel $k_t(x, y)$ and there exist $c_1, c_2 > 0$ so that

$$\int_{|x-y|>c_1t^{1/2}} |K(x, y) - k_t(x, y)|dx \leq c_2 \text{ for all } y \in R^n.$$

(2) There exists an "approximation to the identity" $\{A_t, t > 0\}$ such that A_tT has associated kernel $K_t(x, y)$ which satisfies

$$|K_t(x, y)| \leq c_4t^{-n/2} \text{ if } |x-y| \leq c_3t^{1/2},$$

and

$$|K(x, y) - K_t(x, y)| \leq c_4t^{\delta/2}|x-y|^{-n-\delta} \text{ if } |x-y| \geq c_3t^{1/2},$$

for some $c_3, c_4 > 0, \delta > 0$.

Let m_j be the positive integers($j = 1, \cdots, l$), $m_1 + \cdots + m_l = m$ and b_j be the functions on R^n ($j = 1, \cdots, l$). Set, for $1 \le j \le m$,

$$R_{m_j+1}(b_j; x, y) = b_j(x) - \sum_{|\alpha| \le m_j} \frac{1}{\alpha!} D^\alpha b_j(y)(x - y)^\alpha.$$

The multilinear operator associated to T is defined by

$$T^b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j; x, y)}{|x - y|^m} K(x, y) f(y) dy.$$

Note that when $m = 0$, T^b is just the multilinear commutator generated by T and b (see [18][19]), while when $m > 0$, T^b is a non-trivial generalization of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors(see [1-4]). In [12], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In [19], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator. In [10][15], the boundedness of the singular integral operator with non-smooth kernel are obtained. In [7], the boundedness of the commutator associated to the singular integral operator with non-smooth kernel are obtained. Our works are motivated by these papers. The main purpose of this paper has twofold, first, we establish a sharp estimate for the multilinear integral operator T^b, and second, we prove the boundedness for the multilinear operator on L^p spaces with variable exponent by using the sharp estimate. In Section 4, we will give some applications of the theorems in this paper.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f and $\delta > 0$, the sharp function of f is defined by

$$f_\delta^\#(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |f(y) - f_Q|^\delta dy \right)^{1/\delta},$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that(see [11][20])

$$f_\delta^\#(x) \approx \sup_{Q \ni x} \inf_{c \in C} \left(\frac{1}{|Q|} \int_Q |f(y) - c|^\delta dy \right)^{1/\delta}.$$

We write that $f^\# = f_\delta^\#$ if $\delta = 1$. We say that f belongs to $BMO(R^n)$ if $f^\#$ belongs to $L^\infty(R^n)$ and define $||f||_{BMO} = ||f^\#||_{L^\infty}$. Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)| dy;$$

For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and

$$M^k(f)(x) = M(M^{k-1}(f))(x) \text{ when } k \geq 2.$$

The sharp maximal function $M_A(f)$ associated with the "approximation to the identity" $\{A_t, t > 0\}$ is defined by

$$M_A^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - A_{t_Q}(f)(y)| dy,$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q. For $0 < \delta < \infty$, we denote $M_{A,\delta}^{\#}(f)$ by

$$M_{A,\delta}^{\#}(f) = [M_A^{\#}(|f|^{\delta})]^{1/\delta}.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary function associated to Φ, we denote that the Φ-average by, for a function f,

$$||f||_{\Phi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1 \right\}$$

and the maximal function associated to Φ by

$$M_{\Phi}(f)(x) = \sup_{Q \ni x} ||f||_{\Phi,Q}.$$

The Young functions to be used in this paper are $\Phi(t) = t(1 + logt)^r$ and $\tilde{\Phi}(t) = exp(t^{1/r})$, the corresponding average and maximal functions denoted by $|| \cdot ||_{L(logL)^r,Q}$, $M_{L(logL)^r}$ and $|| \cdot ||_{expL^{1/r},Q}$, $M_{expL^{1/r}}$. Following [21-22], we know the generalized Hölder's inequality:

$$\frac{1}{|Q|} \int_Q |f(y)g(y)| dy \leq ||f||_{\Phi,Q} ||g||_{\tilde{\Phi},Q}$$

and the following inequality, for $r, r_j \geq 1, j = 1, \cdots, l$ with $1/r = 1/r_1 + \cdots + 1/r_l$, and any $x \in R^n$, $b \in BMO(R^n)$,

$$||f||_{L(logL)^{1/r},Q} \leq M_{L(logL)^{1/r}}(f) \leq CM_{L(logL)^l}(f) \leq CM^{l+1}(f),$$

$$||b - b_Q||_{expL^r,Q} \leq C||b||_{BMO},$$

$$|b_{2^{k+1}Q} - b_{2Q}| \leq Ck||b||_{BMO}.$$

The non-increasing rearrangement of a measurable function f on R^n is defined by

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in R^n : |f(x)| > \lambda\}| \leq t\} \quad (0 < t < \infty).$$

For $\lambda \in (0,1)$ and a measurable function f on R^n, the local sharp maximal function of f is defined by

$$M_\lambda^{\#}(f)(x) = \sup_{Q \ni x} \inf_{c \in C} ((f-c)\chi_Q)^*(\lambda|Q|).$$

Let $p : R^n \to [1,\infty)$ be a measurable function. Denote by $L^{p(\cdot)}(R^n)$ the set of all Lebesgue measurable functions f on R^n such that $m(\lambda f, p) < \infty$ for some $\lambda = \lambda(f) > 0$, where

$$m(f,p) = \int_{R^n} |f(x)|^{p(x)} dx.$$

The set $L^{p(\cdot)}(R^n)$ becomes a Banach spaces with respect to the following norm

$$||f||_{L^{p(\cdot)}} = \inf\{\lambda > 0 : m(f/\lambda, p) \leq 1\}.$$

Denote by $M(R^n)$ the sets of all measurable functions $p : R^n \to [1,\infty)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(R^n)$ and the following holds

$$1 < p_- = ess \inf_{x \in R^n} p(x), \qquad ess \sup_{x \in R^n} p(x) = p_+ < \infty. \tag{1}$$

In recent years, the boundedness of classical operators on spaces $L^{p(\cdot)}(R^n)$ have attracted a great attention (see [4-7],[10],[19] and their references).

We shall prove the following theorems.

Theorem 1. Let T be a singular integral operator with non-smooth kernel as in Definition 2 and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then there exists a constant $C > 0$ such that for any $f \in L_0^\infty(R^n)$, $0 < \delta < 1$ and $\tilde{x} \in R^n$,

$$M_{A,\delta}^{\#}(T^b(f))(\tilde{x}) \leq C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{BMO} \right) M^{l+1}(f)(\tilde{x}).$$

Theorem 2. Let T be a singular integral operator with non-smooth kernel as in Definition 2, $p(\cdot) \in M(R^n)$ and $D^\alpha b_j \in BMO(R^n)$ for all

α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then T^b is bounded on $L^{p(\cdot)}(R^n)$, that is

$$||T^b(f)||_{L^{p(\cdot)}} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) ||f||_{L^{p(\cdot)}}.$$

3. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1.([3]) Let b be a function on R^n and $D^\alpha b \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then

$$|R_m(b; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2.([11, p.485]) Let $0 < p < q < \infty$. We define for any function $f \ge 0$ and $1/r = 1/p - 1/q$,

$$||f||_{WL^q} = \sup_{\lambda>0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, \; N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p}/||\chi_E||_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$||f||_{WL^q} \le N_{p,q}(f) \le (q/(q-p))^{1/p} ||f||_{WL^q}.$$

Lemma 3.[19] Let $r_j \ge 1$ for $j = 1, \cdots, l$, we denote $1/r = 1/r_1 + \cdots + 1/r_l$. Then

$$\frac{1}{|Q|} \int_Q |f_1(x) \cdots f_l(x) g(x)| dx \le ||f||_{expL^{r_1},Q} \cdots ||f||_{expL^{r_l},Q} ||g||_{L(logL)^{1/r},Q}.$$

Lemma 4.([14]) Let $p : R^n \to [1, \infty)$ be a measurable function satisfying (1). Then $L_0^\infty(R^n)$ is dense in $L^{p(\cdot)}(R^n)$.

Lemma 5.([14]) Let $f \in L^1_{loc}(R^n)$ and g be a measurable function satisfying

$$|\{x \in R^n : |g(x)| > \alpha\}| < \infty \quad \text{for all} \quad \alpha > 0.$$

Then

$$\int_{R^n} |f(x)g(x)| dx \le C_n \int_{R^n} M^{\#}_{\lambda_n}(f)(x) M(g)(x) dx.$$

Lemma 6.([14][15]) Let $\{A_t, t > 0\}$ be an "approximation to the identity", $\delta > 0$, $0 < \lambda < 1$ and $f \in L^{\delta}_{loc}(R^n)$. Then

$$M^{\#}_{\lambda}(f)(x) \le (1/\lambda)^{1/\delta} M^{\#}_{A,\delta}(f)(x).$$

Lemma 7.([13]) Let $p : R^n \to [1, \infty)$ be a measurable function satisfying (1). If $f \in L^{p(\cdot)}(R^n)$ and $g \in L^{p'(\cdot)}(R^n)$ with $p'(x) = p(x)/(p(x) - 1)$. Then fg is integrable on R^n and

$$\int_{R^n} |f(x)g(x)|dx \le C||f||_{L^{p(\cdot)}}||g||_{L^{p'(\cdot)}}.$$

Lemma 8.([13]) Let $p : R^n \to [1, \infty)$ be a measurable function satisfying (1). Set

$$||f||'_{L^{p(\cdot)}} = \sup\left\{\int_{R^n} |f(x)g(x)|dx : f \in L^{p(\cdot)}(R^n), g \in L^{p'(\cdot)}(R^n)\right\}.$$

Then $||f||_{L^{p(\cdot)}} \le ||f||'_{L^{p(\cdot)}} \le C||f||_{L^{p(\cdot)}}$.

Lemma 9.([10][15]) Let T be a singular integral operator with non-smooth kernel as in Definition 2. Then T is bounded on $L^p(R^n)$ for every $1 < p < \infty$ and bounded from $L^1(R^n)$ to $WL^1(R^n)$.

Lemma 10. Let $\{A_t, t > 0\}$ be an "approximation to the identity" and $b \in BMO(R^n)$. Then, for every $f \in L^p(R^n), p > 1$ and $\tilde{x} \in R^n$,

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|}\int_Q |A_{t_Q}((b - b_Q)f)(x)|dx \le C||b||_{BMO} M^2(f)(\tilde{x}),$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q.

Proof. We write, for any cube Q with $\tilde{x} \in Q$,

$$\begin{aligned}
& \frac{1}{|Q|}\int_Q |A_{t_Q}((b - b_Q)f)(x)|dx \\
\le\ & \frac{1}{|Q|}\int_Q\int_{R^n} h_{t_Q}(x, y)|(b(y) - b_Q)f(y)|dydx \\
\le\ & \frac{1}{|Q|}\int_Q\int_{2Q} h_{t_Q}(x, y)|(b(y) - b_Q)f(y)|dydx \\
& + \sum_{k=1}^{\infty} \frac{1}{|Q|}\int_Q\int_{2^{k+1}Q\setminus 2^kQ} h_{t_Q}(x, y)|(b(y) - b_Q)f(y)|dydx \\
=\ & I_1 + I_2.
\end{aligned}$$

We have, by the generalized Hölder's inequality,

$$\begin{aligned} I_1 &\le C\frac{1}{|Q||2Q|}\int_Q\int_{2Q}|(b(y)-b_Q)f(y)|dydx \\ &\le C||b-b_Q||_{expL,2Q}||f||_{L(logL),2Q} \\ &\le C||b||_{BMO}M^2(f)(\tilde{x}). \end{aligned}$$

For I_2, notice for $x\in Q$ and $y\in 2^{k+1}Q\backslash 2^kQ$, then $|x-y|\ge 2^{k-1}t_Q$ and $h_{t_Q}(x,y)\le C\frac{s(2^{2(k-1)})}{|2^{k-1}Q|}$, then

$$\begin{aligned} I_2 &\le C\sum_{k=1}^{\infty}s(2^{2(k-1)})\frac{1}{|Q|^2}\int_Q\int_{2^{k+1}Q}|(b(y)-b_Q)f(y)|dydx \\ &\le C\sum_{k=1}^{\infty}2^{kn}s(2^{2(k-1)})\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|(b(y)-b_Q)f(y)|dy \\ &\le C\sum_{k=1}^{\infty}2^{kn}s(2^{2(k-1)})||b-b_Q||_{expL,2^{k+1}Q}||f||_{L(logL),2^{k+1}Q} \\ &\le C\sum_{k=1}^{\infty}2^{(k-1)n}s(2^{2(k-1)})||b||_{BMO}M^2(f)(\tilde{x}) \\ &\le C||b||_{BMO}M^2(f)(\tilde{x}), \end{aligned}$$

where the last inequality follows from
$\sum_{k=1}^{\infty}2^{(k-1)n}s(2^{2(k-1)})\le C\sum_{k=1}^{\infty}2^{-(k-1)\varepsilon}<\infty$
for some $\epsilon>0$. This completes the proof.

Proof of Theorem 1. It suffices to prove for $f\in C_0^{\infty}(R^n)$ and some constant C_0, the following inequality holds:

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q\left||T^b(f)(x)|^{\delta}-|A_{t_Q}T^b(f)(x)|^{\delta}\right|dx\right)^{1/\delta} \\ &\le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M^{l+1}(f)(\tilde{x}). \end{aligned}$$

Without loss of generality, we may assume $l=2$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Let $\tilde{Q}=5\sqrt{n}Q$ and $\tilde{b}_j(x)=b_j(x)-\sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^{\alpha}b_j)_{\tilde{Q}}x^{\alpha}$, then $R_{m_j}(b_j;x,y)=R_{m_j}(\tilde{b}_j;x,y)$ and $D^{\alpha}\tilde{b}_j=$

$D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f = f\chi_{\tilde{Q}} + f\chi_{R^n\setminus\tilde{Q}} = f_1 + f_2$,

$$
\begin{aligned}
& T^b(f)(x) \\
= & \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m} K(x,y) f_1(y) dy \\
- & \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1} D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m} K(x,y) f_1(y) dy \\
- & \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2} D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K(x,y) f_1(y) dy \\
+ & \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K(x,y) f_1(y) dy \\
+ & \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m} K(x,y) f_2(y) dy
\end{aligned}
$$

$$
\begin{aligned}
& = T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, \cdot)}{|x-\cdot|^m} f_1\right) \\
& - T\left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2; x, \cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m} f_1\right) \\
& - T\left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1; x, \cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right) \\
& + T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right) \\
& + T\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, \cdot)}{|x-\cdot|^m} f_2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& A_{t_Q}T^b(f)(x) \\
= & \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m} K_t(x,y) f_1(y)dy \\
- & \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1} D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m} K_t(x,y) f_1(y)dy \\
- & \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2} D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K_t(x,y) f_1(y)dy \\
+ & \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K_t(x,y) f_1(y)dy \\
+ & \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m} K_t(x,y) f_2(y)dy \\
= & A_{t_Q}T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, \cdot)}{|x-\cdot|^m} f_1\right)
\end{aligned}
$$

$$
\begin{aligned}
& -A_{t_Q}T\left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2; x, \cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m} f_1\right) \\
& -A_{t_Q}T\left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1; x, \cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right) \\
& +A_{t_Q}T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right) \\
& +A_{t_Q}T\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, \cdot)}{|x-\cdot|^m} f_2\right).
\end{aligned}
$$

Then

$$\begin{aligned}
& \left[\frac{1}{|Q|}\int_Q \left||T^b(f)(x)|^\delta - |A_{t_Q}T^b(f)(x)|^\delta\right| dx\right]^{1/\delta} \\
\leq\ & \left[\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - A_{t_Q}T^b(f)(x)\right|^\delta dx\right]^{1/\delta} \\
\leq\ & \left[\frac{C}{|Q|}\int_Q \left|T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_1\right)\right|^\delta dx\right]^{1/\delta} \\
+\ & \left[\frac{C}{|Q|}\int_Q \left|T\left(\sum_{|\alpha_1|=m_1} \frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m} f_1\right)\right|^\delta dx\right]^{1/\delta} \\
+\ & \left[\frac{C}{|Q|}\int_Q \left|T\left(\sum_{|\alpha_2|=m_2} \frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right)\right|^\delta dx\right]^{1/\delta} \\
+\ & \left[\frac{C}{|Q|}\int_Q \left|T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \int_Q \frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right)\right|^\delta dx\right]^{1/\delta}
\end{aligned}$$

$$
\begin{aligned}
&+ \left[\frac{C}{|Q|}\int_Q\left|A_{t_Q}T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_1\right)\right|^\delta dx\right]^{1/\delta}\\
&+ \left[\frac{C}{|Q|}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}f_1\right)\right|^\delta dx\right]^{1/\delta}\\
&+ \left[\frac{C}{|Q|}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\right|^\delta dx\right]^{1/\delta}\\
&+ \left[\frac{C}{|Q|}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\right|^\delta dx\right]^{1/\delta}\\
&+ \left[\frac{C}{|Q|}\int_Q\left|(T-A_{t_Q}T)\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_2\right)\right|^\delta dx\right]^{1/\delta}\\
&:= I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9.
\end{aligned}
$$

Now, let us estimate I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8 and I_9, respectively. First, for $x\in Q$ and $y\in\tilde{Q}$, by Lemma 1, we get

$$R_m(\tilde{b}_j;x,y)\le C|x-y|^m\sum_{|\alpha_j|=m}||D^{\alpha_j}b_j||_{BMO}.$$

By Lemma 2 and the weak type (1,1) of T (Lemma 9), we obtain

$$
\begin{aligned}
I_1 &\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_{R^n}|T(f_1)(x)|^\delta dx\right)^{1/\delta}\\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)|Q|^{-1}\frac{||T(f_1)\chi_Q||_{L^\delta}}{|Q|^{1/\delta-1}}
\end{aligned}
$$

$$\begin{aligned}
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)|Q|^{-1}||T(f_1)||_{WL^1}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)|\tilde{Q}|^{-1}||f_1||_{L^1}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M(f)(\tilde{x}).
\end{aligned}$$

For I_2, we get, by Lemma 2 and the generalized Hölder's inequality,

$$\begin{aligned}
I_2 &\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{b}_1f_1)(x)|^{\delta}dx\right)^{1/\delta}\\
&\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1}\frac{||T(D^{\alpha_1}\tilde{b}_1f_1)\chi_Q||_{L^{\delta}}}{|Q|^{1/\delta-1}}\\
&\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1}||T(D^{\alpha_1}\tilde{b}_1f_1)||_{WL^1}\\
&\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}|\tilde{Q}|^{-1}||D^{\alpha_1}\tilde{b}_1f_1||_{L^1}\\
&\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}||D^{\alpha_1}b_1-(D^{\alpha}b_1)_{\tilde{Q}}||_{expL,\tilde{Q}}||f||_{L(logL),\tilde{Q}}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M^2(f)(\tilde{x}).
\end{aligned}$$

For I_3, similar to the proof of I_2, we get

$$I_3\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)M^2(f)(\tilde{x}).$$

Similarly, for I_4, taking $r_1, r_2 \geq 1$ such that $1/\delta = 1/r_1 + 1/r_2$, we obtain, by Lemma 3 and the generalized Hölder's inequality,

$$\begin{aligned}
I_4 &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 f_1)(x)|^\delta dx \right)^{1/\delta} \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1} \frac{||T(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 f_1)\chi_Q||_{L^\delta}}{|Q|^{1/\delta-1}} \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1} ||T(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 f_1)||_{WL^1} \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1} ||D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 f_1||_{L^1} \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \prod_{j=1}^{2} \left|\left| D^{\alpha_j} b_j - (D^{\alpha_j} b_j)_{\tilde{Q}} \right|\right|_{expL^{r_j}, \tilde{Q}} \cdot ||f||_{L(logL)^{1/r}, \tilde{Q}} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) M^3(f)(\tilde{x}).
\end{aligned}$$

For I_5, I_6, I_7 and I_8, by Lemma 10, we get

$$\begin{aligned}
& I_5 + I_6 + I_7 + I_8 \\
\leq\ & C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) \frac{1}{|Q|} \int_Q |A_{t_Q} T(f_1)(x)| dx \\
+\ & C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} b_2||_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_Q |A_{t_Q} T(D^{\alpha_1}\tilde{b}_1 f_1)(x)| dx \\
+\ & C \sum_{|\alpha_1|=m_1} ||D^{\alpha_1} b_1||_{BMO} \sum_{|\alpha_2|=m_2} \frac{1}{|Q|} \int_Q |A_{t_Q} T(D^{\alpha_2}\tilde{b}_2 f_1)(x)| dx \\
+\ & C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|} \int_Q |A_{t_Q} T(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 f_1)(x)| dx \\
\leq\ & C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) M^3(f)(\tilde{x}).
\end{aligned}$$

For I_9, we write

$$
\begin{aligned}
& (T - A_{t_Q}T)\left(\frac{\prod_{j=1}^{2} R_{m_j+1}(\tilde{b}_j; x, \cdot)}{|x-\cdot|^m} f_2\right) \\
= & \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m}(K(x,y) - K_t(x,y))f_2(y)dy \\
= & \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m}(K(x,y) - K_t(x,y))f_2(y)dy \\
- & \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!}\int_{R^n} \frac{D^{\alpha_1}\tilde{b}_1(y)(x-y)^{\alpha_1} R_{m_2}(\tilde{b}_2; x, y)}{|x-y|^m}(K(x,y) - K_t(x,y))f_2(y)dy \\
- & \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!}\int_{R^n} \frac{D^{\alpha_2}\tilde{b}_2(y)(x-y)^{\alpha_2} R_{m_1}(\tilde{b}_1; x, y)}{|x-y|^m}(K(x,y) - K_t(x,y))f_2(y)dy \\
+ & \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!}\int_{R^n} \frac{D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} \\
& \times(K(x,y) - K_t(x,y))f_2(y)dy \\
= & I_9^{(1)} + I_9^{(2)} + I_9^{(3)} + I_9^{(4)}.
\end{aligned}
$$

By Lemma 1 and the following inequality (see [20])

$$|b_{Q_1} - b_{Q_2}| \le C\log(|Q_2|/|Q_1|)||b||_{BMO} \text{ for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$
\begin{aligned}
|R_m(\tilde{b}; x, y)| &\le C|x-y|^m \sum_{|\alpha|=m}(||D^\alpha b||_{BMO} + |(D^\alpha b)_{\tilde{Q}(x,y)} - (D^\alpha b)_{\tilde{Q}}|) \\
&\le Ck|x-y|^m \sum_{|\alpha|=m} ||D^\alpha b||_{BMO}.
\end{aligned}
$$

Note that $|x-y| \geq d = t^{1/2}$ and $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, by the conditions on K and K_t, we obtain

$$\begin{aligned}
|I_9^{(1)}| &= \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{\prod_{j=1}^2 |R_{m_j}(\tilde{b}_j;x,y)|}{|x-y|^m} |K(x,y)-K_t(x,y)||f(y)|dy \\
&\leq C\prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO}\right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2 \frac{d^\delta}{|x_0-y|^{n+\delta}} |f(y)|dy \\
&\leq C\prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO}\right) \sum_{k=1}^{\infty} k^2 2^{-\delta k} \frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|dy \\
&\leq C\prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO}\right) M(f)(\tilde{x}).
\end{aligned}$$

For $I_9^{(2)}$, we get, by the generalized Hölder's inequality,

$$\begin{aligned}
|I_9^{(2)}| &\leq C\left(\sum_{|\alpha_2|=m_2} ||D^{\alpha_2} b_2||_{BMO}\right) \\
&\quad \times \sum_{|\alpha_1|=m_1} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{kd^\delta}{|x_0-y|^{n+\delta}} |D^{\alpha_1}\tilde{b}_1(y)||f(y)|dy \\
&\leq C\left(\sum_{|\alpha_2|=m_2} ||D^{\alpha_2} b_2||_{BMO}\right) \\
&\quad \times \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} k2^{-\delta k} ||D^{\alpha_1} b_1 - (D^{\alpha_1} b_1)_{\tilde{Q}}||_{expL,2^k\tilde{Q}} ||f||_{L(logL),2^k\tilde{Q}} \\
&\leq C\prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO}\right) M^2(f)(\tilde{x}).
\end{aligned}$$

Similarly,

$$|I_9^{(3)}| \leq C\prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO}\right) M^2(f)(\tilde{x}).$$

For $I_9^{(4)}$, taking $r, r_1, r_2 \geq 1$ such that $1/r = 1/r_1 + 1/r_2$, then, by Lemma 3 and the generalized Hölder's inequality,

$$\begin{aligned}
&|I_9^{(4)}| \\
\leq\ & C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{d^\delta}{|x_0-y|^{n+\delta}} |D^{\alpha_1}\tilde{b}_1(y)||D^{\alpha_2}\tilde{b}_2(y)||f(y)|dy \\
\leq\ & C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=1}^{\infty} \prod_{j=1}^{2} ||D^{\alpha_j}b_j - (D^{\alpha_j}b_j)_{\tilde{Q}}||_{expL^{r_j}, 2^k\tilde{Q}} ||f||_{L(logL)^{1/r}, 2^k\tilde{Q}} \\
\leq\ & C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) M^3(f)(\tilde{x}).
\end{aligned}$$

Thus

$$|I_9| \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) M^3(f)(\tilde{x}).$$

This completes the proof of Theorem 1.

Proof of Theorem 2. By Lemmas 4-7, we get, for $f \in L_0^\infty(R^n)$ and $g \in L^{p'(\cdot)}(R^n)$,

$$\begin{aligned}
\int_{R^n} |T^b(f)(x)g(x)|dx \ &\leq\ C \int_{R^n} M_{\lambda_n}^{\#}(T^b(f))(x)M(g)(x)|dx \\
&\leq\ C \int_{R^n} M_{A,\delta}^{\#}(T^b(f))(x)M(g)(x)dx \\
&\leq\ C \int_{R^n} M^{l+1}(f)(x)M(g)(x)dx \\
&\leq\ C||M^{l+1}(f)||_{L^{p(\cdot)}}||M(g)||_{L^{p'(\cdot)}} \\
&\leq\ C||f||_{L^{p(\cdot)}}||g||_{L^{p'(\cdot)}},
\end{aligned}$$

thus, by Lemma 8,

$$||T^b(f)||_{L^{p(\cdot)}} \leq ||f||_{L^{p(\cdot)}}.$$

This completes the proof of Theorem 2.

4. Applications

In this section we shall apply Theorems 1 and 2 of this paper to the holomorphic functional calculus of linear elliptic operators. First, we review some definitions regarding the holomorphic functional calculus (see [10][15]). Given $0 \le \theta < \pi$. Define

$$S_\theta = \{z \in C : |\arg(z)| \le \theta\} \bigcup \{0\}$$

and its interior by S_θ^0. Set $\tilde{S}_\theta = S_\theta \setminus \{0\}$. An closed operator L on some Banach space E is said to be of type θ if its spectrum $\sigma(L) \subset S_\theta$ and for every $\nu \in (\theta, \pi]$, there exists a constant C_ν such that

$$|\eta|||(\eta I - L)^{-1}|| \le C_\nu, \quad \eta \notin \tilde{S}_\theta.$$

For $\nu \in (0, \pi]$, let

$$H_\infty(S_\mu^0) = \{f : S_\theta^0 \to C : \ f \ \text{ is holomorphic and } ||f||_{L^\infty} < \infty\},$$

where $||f||_{L^\infty} = \sup\{|f(z)| : z \in S_\mu^0\}$. Set

$$\Psi(S_\mu^0) = \left\{ g \in H_\infty(S_\mu^0) : \exists s > 0, \exists c > 0 \text{ such that } |g(z)| \le c\frac{|z|^s}{1 + |z|^{2s}} \right\}.$$

If L is of type θ and $g \in H_\infty(S_\mu^0)$, we define $g(L) \in L(E)$ by

$$g(L) = -(2\pi i)^{-1} \int_\Gamma (\eta I - L)^{-1} g(\eta) d\eta,$$

where Γ is the contour $\{\xi = re^{\pm i\phi} : r \ge 0\}$ parameterized clockwise around S_θ with $\theta < \phi < \mu$. If, in addition, L is one-one and has dense range, then, for $f \in H_\infty(S_\mu^0)$,

$$f(L) = [h(L)]^{-1}(fh)(L),$$

where $h(z) = z(1+z)^{-2}$. L is said to have a bounded holomorphic functional calculus on the sector S_μ, if

$$||g(L)|| \le N||g||_{L^\infty}$$

for some $N > 0$ and for all $g \in H_\infty(S_\mu^0)$.

Now, let L be a linear operator on $L^2(R^n)$ with $\theta < \pi/2$ so that $(-L)$ generates a holomorphic semigroup e^{-zL}, $0 \le |\arg(z)| < \pi/2 - \theta$. Applying Theorem 6 of [8], we get

Theorem 3. Assume the following conditions are satisfied:

(i) The holomorphic semigroup e^{-zL}, $0 \le |\arg(z)| < \pi/2 - \theta$ is represented by the kernels $a_z(x, y)$ which satisfy, for all $\nu > \theta$, an upper bound

$$|a_z(x, y)| \le c_\nu h_{|z|}(x, y)$$

for $x, y \in R^n$, and $0 \leq |\arg(z)| < \pi/2 - \theta$, where $h_t(x,y) = Ct^{-n/2}s(|x-y|^2/t)$ and s is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon}s(r^2) = 0.$$

(ii) The operator L has a bounded holomorphic functional calculus in $L^2(R^n)$, that is, for all $\nu > \theta$ and $g \in H_\infty(S^0_\mu)$, the operator $g(L)$ satisfies

$$||g(L)(f)||_{L^2} \leq c_\nu ||g||_{L^\infty} ||f||_{L^2}.$$

Then, for $p(\cdot) \in M(R^n)$, $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$, the multilinear operator $g(L)^b$ associated to $g(L)$ and b_j satisfies: $g(L)^b$ is bounded on $L^{p(\cdot)}(R^n)$, that is

$$||g(L)^b(f)||_{L^{p(\cdot)}} \leq C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) ||f||_{L^{p(\cdot)}}.$$

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