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ON THE MINIMAL WEAK UPPER GRADIENT OF A BANACH-SOBOLEV FUNCTION ON A METRIC SPACE

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Abstract. We prove that every function belonging to a Sobolev-type space $N^{1,B}(X)$ on a metric measure space X has a $B-$weak upper gradient in B that is pointwise minimal $\mu-$almost everywhere, provided that the Banach function space B has a strictly convex and strictly monotone norm. This result generalizes corresponding known results involving Lebesgue spaces $B = L^p(X)$, $p > 1$ [16] or, more general, Orlicz spaces $B = L^{\Psi}(X)$ [17] with a strictly convex Young function Ψ satisfying a Δ_2-condition.

1. Introduction and preliminaries

In what follows, (X, d, μ) is a metric measure space, where the outer measure μ is Borel regular, positive and finite on balls. In the extensions of first-order calculus to metric measure space there is a substitute for the length of the gradient, namely the upper gradient. A Borel measurable function $g : X \to [0, +\infty]$ is said to be an *upper gradient* of a function $u : X \to \mathbb{R}$ if for every compact, rectifiable path $\gamma : [0, 1] \to X$

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$$|u(\gamma(1)) - u(\gamma(0))| \leq \int_\gamma g ds. \tag{1.1}$$

In the Euclidean setting, the length of the gradient of a function of class C^1 on a domain of $\mathbb{R}^n$ is an upper gradient of that function.

Since upper gradients are not stable neither under changes on a set of zero measure, nor under limits, a more flexible tool generalizing the notion of upper gradient was needed in the development of analysis on metric measure spaces. A weak form of the notion of upper gradient, the concept of p-weak upper gradient was first defined by Koskela and MacManus in [9], by using the $p-$modulus of curve families. The notion of $p-$weak upper gradient turned out to be a flexible tool, which is essential in defining and studying the Sobolev-type spaces on metric measure spaces introduced by Shanmugalingam [14], [15], known as the Newtonian spaces $N^{1,p}(X)$, $p \geq 1$. In [17] Tuominen introduced and studied an important generalization of $N^{1,p}(X)$, the notion of Orlicz-Sobolev space $N^{1,\Psi}(X)$, where the Lebesgue space $L^p(X)$ involved in the definitions of $p-$modulus, $p-$weak upper gradient and $N^{1,p}(X)$ is replaced by an Orlicz space $L^\Psi(X)$.

We generalized Orlicz-Sobolev spaces on metric measure spaces in [12], by introducing a Sobolev-type space based on weak upper gradients, where the role of the Orlicz spaces is played by an abstract Banach function space $(B, \|\cdot\|_B)$. For the definition and properties of Banach function spaces, see [1]. The norm $\|\cdot\|_B$ of the Banach function space B is monotone by definition, i.e. $0 \leq g \leq f$ $\mu-$a.e. in X implies $\|g\|_B \leq \|f\|_B$ for every $f, g \in B$. We will say that the norm $\|\cdot\|_B$ is *strictly monotone* if for every $f, g \in B$ with $0 \leq g \leq f$ $\mu-$a.e. in X, $\|g\|_B = \|f\|_B$ implies $g = f$ $\mu-$a.e.

The notion of $B-$modulus introduced in [12] is a generalization of $\Psi-$modulus from [17], that in turn generalizes the well-known concept of $p-$modulus in metric measure spaces [5]. Let Γ_{rec} be the family of all rectifiable curves in X. The $B-$*modulus of a family* Γ of curves in X is defined by $M_B(\Gamma) = \inf \|\rho\|_B$,where the infimum is taken over all Borel functions $\rho : X \to [0, +\infty]$ with $\int_\gamma \rho\, ds \geq 1$ for all rectifiable curves γ in X. Even for a general Banach function space B, the $B-$modulus share many properties of the $p-$modulus, in particular

M_B is an outer measure on the family of all curves in X, as it is proved in [12]. A $B-$weak upper gradient of a function $u : X \to \mathbb{R}$ is a Borel measurable function $g : X \to [0, +\infty]$ such that (1.1) holds for all compact, rectifiable paths $\gamma : [0, 1] \to X$ except for a curve family with zero $B-$modulus. For every function $u : X \to \mathbb{R}$ we will denote by G_u the family of all $B-$weak upper gradients $g \in B$ of u in X.

The set $\widetilde{N}^{1,B}(X)$ formed from the functions $u \in B$ for which G_u is non-empty is a linear subspace in the space of real functions defined on X. The functional $\|u\|_{1,B} := \|u\|_B + \inf\{\|g\|_B : g \in G_u\}$ is a seminorm on $\widetilde{N}^{1,B}(X)$. The Sobolev-type space $N^{1,B}(X)$ is the quotient normed space of $\widetilde{N}^{1,B}(X)$ with respect to the equivalence relation defined by: $u \sim v$ if $\|u - v\|_{1,B} = 0$.

It is natural to look for assumptions on the Banach function space B implying that the infimum $\inf\{\|g\|_B : g \in G_u\}$ is attained for some $g = g_u$ whenever every $u \in N^{1,B}(X)$, which simplifies the definition of the norm $\|\cdot\|_{1,B}$ to $\|u\|_{1,B} := \|u\|_B + \|g_u\|_B$.

The following result is a substitute for Mazur's lemma in Sobolev-type spaces on metric measure spaces and generalizes [17, Theorem 4.17], that in turn is a generalization of [15, Lemma 4.11].

Lemma 1. [12, Theorem 1] *Let $(u_j)_{j\geq 1}$ be a sequence of functions in B and $(g_j)_{j\geq 1}$ be a sequence in B of corresponding $B-$weak upper gradients. Assume that $u_j \to u$ and $g_j \to g$ weakly in B, for some $u, g \in B$. Then there are sequences $(\widetilde{u}_j)_{j\geq 1}$ and $(\widetilde{g}_j)_{j\geq 1}$ of convex combinations*

$$\widetilde{u}_j = \sum_{k=j}^{n_j} \lambda_{kj} u_k,\ \widetilde{g}_j = \sum_{k=j}^{n_j} \lambda_{kj} g_k,$$

where $\lambda_{kj} \geq 0$, $\sum_{k=j}^{n_j} \lambda_{kj} = 1$, such that $\widetilde{u}_j \to u$ and $\widetilde{g}_j \to g$ in B. In addition, g is a $B-$weak upper gradient of u.

For $B = L^p(X)$ with $1 < p < \infty$, it was shown by Shanmugalingam in [16, Corollary 3.7] that every function $u \in N^{1,p}(X)$ has is a $p-$weak upper gradient $g_u \in L^p(X)$ such that $g_u(x) \leq g(x)$ for $\mu-$almost every $x \in X$, whenever g is a $p-$weak upper gradient of u. Any such $p-$weak upper gradient g_u is called a minimal $p-$weak upper gradient of u, and obviously satisfies $\|g_u\|_{L^p(X)} = \inf\left\{\|g\|_{L^p(X)} : g \in G_u\right\}$. Note that the existence of another type of minimal upper gradient has been previously proved by Cheeger in [3]. In the classical case $u \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a domain, it is known that $g_u = \|\nabla u\|$ $\mu-$a.e., where ∇u is the distributional gradient of u. This property extends from the Euclidean setting to the setting of Riemannian manifolds.

Minimal $p-$weak upper gradients play an essential role in nonlinear potential theory on metric measure spaces [7], [16], [8]. In nonlinear potential theory a central problem is the minimization, under various conditions, of the $p-$Dirichlet energy integral of a function $u \in N^{1,p}_{loc}(\Omega)$ on $\Omega' \subset\subset \Omega$, which is defined by

$$\int_{\Omega'} g_u^p d\mu.$$

In the case when $B = L^{\Psi}(X)$ Tuominen has proved in [17, Theorem 6.6, Theorem 6.7, Lemma 6.8] several results on the existence of $\Psi-$weak upper gradients minimizing the $\Psi-$integral or the $L^{\Psi}-$norm. We recall that a Young function $\Psi : [0,\infty) \to [0,\infty)$ is said to satisfy *the* Δ_2-*condition* if there is a constant $C_2 > 0$ such that

$$\Psi(2t) \leq C_2 \Psi(t) \tag{1.2}$$

for every $t \in [0,\infty)$. A Young function satisfying the Δ_2-condition is also said to be *doubling.*

We consider below an arbitrary function $u \in N^{1,\Psi}(X)$. Let us denote by $\widetilde{G}_u$ the set of all $\Psi-$weak upper gradients g of u such that $\int_X \Psi(g)\, d\mu < +\infty$. In general $\widetilde{G}_u \subset G_u$ and $\widetilde{G}_u = G_u$ if the Young function Ψ is doubling. Assuming that Ψ is doubling, it is shown that the infimum $I(u) := \inf \left\{ \int_X \Psi(g)\, d\mu : g \in \widetilde{G}_u \right\}$ is attained for every $u \in N^{1,\Psi}(X)$. Assuming that Ψ is strictly increasing, it is proved that $I(u) = \int_X \Psi(g_u)\, d\mu$ with $g_u \in \widetilde{G}_u$ implies $\|g_u\|_{L^{\Psi}(X)} = \inf \left\{ \|g\|_{L^{\Psi}(X)} : g \in \widetilde{G}_u \right\}$ and $g_u \leq g$ $\mu-$almost everywhere in X for all $g \in \widetilde{G}_u$. Moreover, in the case when Ψ is doubling it is proved that $\|g_u\|_{L^{\Psi}(X)} = \inf \left\{ \|g\|_{L^{\Psi}(X)} : g \in G_u \right\}$ implies $g_u \leq g$ $\mu-$almost everywhere in X for all $g \in G_u$ [17, Theorem 6.11]. Here $\|\cdot\|_{L^\Psi(X)}$ is the Luxemburg norm on $L^\Psi(X)$.

The purpose of this paper is to provide some assumptions on the Banach function space B which are sufficient to imply, for every $u \in N^{1,\Psi}(X)$, that the infimum $\inf\{\|g\|_B : g \in G_u\}$ is attained, respectively that there is $g_u \in G_u$ such $g_u \leq g$ $\mu-$almost everywhere in X for all $g \in G_u$. Our assumptions have to be general enough to hold for $B = L^p(X)$ whenever $1 < p < \infty$, as well as for $B = L^\Psi(X)$ whenever the Young function Ψ is doubling and strictly increasing.

We need some basic notions describing geometric properties of Banach spaces (see [2]).

A Banach space $(V, \|\cdot\|)$ is said to be *strictly convex* (or *rotund*) if $x \neq y$ and $\|x\| = \|y\| = 1$, where $x, y \in V$ together imply that $\|x + y\| < 2$. The Banach space $(V, \|\cdot\|)$ is strictly convex if and only if $x \neq 0$ and $y \neq 0$ and $\|x + y\| = \|x\| + \|y\|$ together imply that $x = cy$ for some constant $c > 0$. Recall that a Banach space $(V, \|\cdot\|)$ is said to be *uniformly convex* (or *uniformly rotund*) if for every $\varepsilon > 0$ there is $\delta > 0$ such that for any $x, y \in V$ with $\|x\| < 1$, $\|y\| < 1$ and $\|x + y\| > 2 - \delta$ we have $\|x - y\| < \varepsilon$. An alternative way to speak about a of strictly (or uniformly) convex Banach space $(V, \|\cdot\|)$ is to say that V has a strictly convex (respectively, uniformly convex) norm. Every uniformly convex Banach space is strictly convex. The converse holds in finite-dimensional Banach spaces, but not in general. By the Milman–Pettis theorem, every uniformly convex space is reflexive. In general, there is no implication between strictly convexity and reflexivity. We will use a well-known result from functional analysis, stating that every convex and closed non-empty subset of a strictly convex Banach space has an element of smallest norm

We prove that every $u \in N^{1,B}(X)$ has a $B-$weak upper gradient $g_u \in B$ minimizing the $B-$norm, provided that the Banach function space B is strictly convex. If in addition the norm of is strictly monotone, it follows that there exists a $B-$weak upper gradient $g_u \in B$ of u in X such that $g_u \leq g$ $\mu-$a.e. in X, whenever $g \in B$ is a $B-$weak upper gradient of u in X. Our results on the existence of a minimal $B-$ weak upper gradient generalize the known results from [16] and [17].

2. Main results

The following characterization of a $B-$weak upper gradient is well-known for $B = L^p(X)$, $p \geq 1$ (see [10, Lemmas 3.1 and 3.3]).

Lemma 2. *Let $u : X \to \mathbb{R}$ and let $g \in B$ be a Borel measurable non-negative function. For each compact rectifiable curve γ parameterized by arc length define $h(s) = u(\gamma(s))$, $s \in [0, l(\gamma)]$.*

a) Assume that for $B-$almost every curve $\gamma \in \Gamma_{rec}$ the function h is absolutely continuous on $[0, l(\gamma)]$ and

$$|h'(s)| \leq g(\gamma(s)) \text{ for almost every } s \in [0, l(\gamma)]. \tag{2.1}$$

Then g is a $B-$weak upper gradient of u.

b) Conversely, if g is a $B-$weak upper gradient of u, then (2.1) holds for $B-$almost every curve $\gamma \in \Gamma_{rec}$

Proof. a) For fixed $\gamma \in \Gamma_{rec}$ denote $l := l(\gamma)$ and let $x = \gamma(0)$ and $y = \gamma(l)$ be the endpoints of γ. We have

$$|u(x) - u(y)| = |h(0) - h(1)| \leq \int_0^l |h'(s)|\, ds \leq \int_0^l g(\gamma(s))ds = \int_\gamma g\ ds.$$

Hence $|u(x) - u(y)| \leq \int_\gamma g\ ds$ for $B-$almost every curve $\gamma \in \Gamma_{rec}$ with endpoints x, y, therefore g is a $B-$weak upper gradient of u.

b) Assume that g is a $B-$weak upper gradient of u. Since $g \in B$, it follows that $u \in ACC_B(X)$, by [12, Proposition 3 (a)]. Moreover, for $B-$almost every curve $\gamma \in \Gamma_{rec}$ the function h is absolutely continuous on $[0, l]$, $\int_\gamma g\ ds < \infty$ and the following inequality holds for every $s_1 \leq s_2$ in $[0, l]$:

$$|u(\gamma(s_1)) - u(\gamma(s_2))| \leq \int_{\gamma|_{[s_1, s_2]}} g\ ds.$$

Being absolutely continuous on $[0, l]$, the function h is differentiable $\mathcal{L}^1-$almost everywhere on $[0, l]$. By Lebesgue differentiation theorem, for $\mathcal{L}^1-$almost every $s_0 \in [0, l]$ we have

$$\lim_{s \to s_0} \frac{1}{|s - s_0|} \left| \int_{s_0}^{s} g(\gamma(t))\ dt \right| = g\left(\gamma\left(s_0\right)\right). \tag{2.2}$$

Suppose that h is differentiable at s_0 and (2.2) holds. Since $|h(s_0) - h(s)| = |u(\gamma(s_0)) - u(\gamma(s))| \leq \left| \int_{s_0}^{s} g(\gamma(t))\ dt \right|$, it follows that

$$|h'(s_0)| \leq g(\gamma(s_0)).$$

∎

The following property, showing that we can paste two $B-$weak upper gradients of a function, is a generalization of [17, Lemma 4.10] (a result stated without proof).

Lemma 3. *Assume that $g_1, g_2 \in B$ are two $B-$weak upper gradients of a function $u : X \to \mathbb{R}$ and that $F \subset X$ is a Borel set. Then the function $\rho = g_1 \chi_{X \setminus F} + g_2 \chi_F$ is also a $B-$weak upper gradient of u in X.*

Proof. Since F is a Borel set and g_1, g_2 are Borel measurable functions, it follows that the function ρ is also Borel measurable. By Lemma 2 (b) and by the subadditivity of $B-$modulus, there exists a family of curves $\Gamma_0 \subset \Gamma_{rec}$ with $M_B(\Gamma_0) = 0$ such that for every $\gamma \in \Gamma_{rec} \setminus \Gamma_0$

$$\left|(u \circ \gamma)'\right| \leq g_k \circ \gamma\ ,\ k = 1, 2.$$

$\mathcal{L}^1-$almost everywhere on $[0, l(\gamma)]$. Note that $\left|(u \circ \gamma)'(s)\right| \leq g_1(\gamma(s))\chi_{X \setminus F}(\gamma(s)) + g_2(\gamma(s))\chi_F(\gamma(s))$ for $\mathcal{L}^1-$almost every $s \in [0, l(\gamma)]$. Then ρ is a $B-$weak upper gradient of u in X, by Lemma 2 (a). ∎

For every function $u : X \to \mathbb{R}$ we denote by G_u the family of all $B-$weak upper gradients $g \in B$ of u in X.

Lemma 4. *For every function $u \in N^{1,B}(X)$ the set $G_u \subset B$ is convex and closed.*

Proof. If G_u is empty, there is nothing to prove. Assume that G_u is non-empty. The set G_u is convex, since $|u(x)) - u(y)| \leq \int\limits_\gamma g_k ds$ for $k = 1, 2$ implies

$$|u(x)) - u(y)| \leq \int\limits_\gamma (\lambda_1 g_1 + \lambda_2 g_2)\, ds$$

for every $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$, by the subadditivity of the $B-$modulus.

Assume that $g_j \in G_u$ for $j \geq 1$ and $g_j \to g$ in B. Next we apply to the sequences $u_j = u$, $j \geq 1$ and g_j, $j \geq 1$ the substitute of Mazur's lemma in $N^{1,B}(X)$, Lemma 1. It follows that $g \in B$ is a $B-$weak upper gradient of u in X. This proves that G_u is closed in the normed space B. ∎

Lemma 5. *If the Banach function space B is strictly convex, then every $u \in N^{1,B}(X)$ has a $B-$weak upper gradient $g_u \in B$ minimizing the $B-$norm, that is satisfying $\|g_u\|_B = \inf \{\|g\|_B : g \in G_u\}$.*

Proof. Let $u \in N^{1,B}(X)$. By Lemma 4, the non-empty set G_u is convex and closed. Since B is a strictly convex Banach space, every convex and closed non-empty subset of B has an element of smallest norm. ∎

Theorem 1. *Assume that the Banach function space B has a strictly convex and strictly monotone norm. Then for every $u \in N^{1,B}(X)$ there exists a $B-$weak upper gradient $g_u \in B$ of u in X such that $g_u \leq g$ $\mu-$a.e. in X, whenever $g \in B$ is a $B-$weak upper gradient of u in X.*

Proof. Let $u \in N^{1,B}(X)$. By Lemma 5 there exists $g_u \in G_u$ such that $\|g_u\|_B = \inf \{\|g\|_B : g \in G_u\}$.

Fix arbitrarily $g \in G_u$. We will prove that $g_u \leq g$ $\mu-$a.e. in X, i.e. the set $F := \{x \in X : g_u(x) > g(x)\}$ has measure zero. Note that F is a Borel set, since g_u and g are Borel measurable functions. By Lemma 3, the function $\rho = g_u \chi_{X \setminus F} + g \chi_F$ belongs to G_u.

Since g_u is an element of smallest norm in G_u, we have $\|g_u\|_B \leq \|\rho\|_B$. On the other hand, $\rho(x) \leq g_u(x)$ for every $x \in X$, hence by the monotonicity of the norm in a Banach function space, $\|\rho\|_B \leq \|g_u\|_B$. It follows that $\|\rho\|_B = \|g_u\|_B$.

Since the norm $\|\cdot\|_B$ is strictly monotone, from $\rho \leq g_u$ $\mu-$a.e. in X and $\|\rho\|_B = \|g_u\|_B$ it follows that $\rho = g_u$ $\mu-$a.e. in X. Then $(g - g_u)\,\chi_F = 0$ $\mu-$a.e. in X, but $g - g_u < 0$ on F, hence $\mu(F) = 0$, q.e.d. ∎

Corollary 1. *Let Ψ be a Young function that is doubling and strictly convex. For every $u \in N^{1,\Psi}(X)$ there exists a $\Psi-$weak upper gradient $g_u \in L^{\Psi}(X)$ of u in X such that $g_u \leq g$ $\mu-$a.e. in X, whenever $g \in L^{\Psi}(X)$ is a $\Psi-$weak upper gradient of u in X.*

Proof. Assume that $(X, \mathcal{A}, \mu)$ is a of a measure space, where the measure μ is atomless. It was proved in [6] that if the Young function Ψ is doubling and strictly convex, then the Orlicz space $L^{\Psi}(X)$ is strictly convex. (The converse holds if $\mu(X) = +\infty$. For $\mu(X) < +\infty$, the Orlicz space $L^{\Psi}(X)$ is strictly convex if and only if Ψ strictly convex, vanishing only at the origin and (1.2) is satisfied for large t).

A Young function Ψ is convex and increasing, therefore Ψ is strictly increasing if Ψ is strictly convex.

If the Young function Ψ is strictly increasing and doubling, then the Luxemburg norm on $L^{\Psi}(X)$ is strictly monotone. Let $f, g \in L^{\Psi}(X)$, such that $0 \leq g \leq f$ $\mu-$a.e. in X and $\|f\|_{L^{\Psi}(X)} = \|g\|_{L^{\Psi}(X)} =: N$. If $N = 0$, then $f = g = 0$ $\mu-$a.e. in X. If $N > 0$, then by [13, Proposition 1.2.11], we have $\int\limits_X \Psi\left(\frac{f}{N}\right) d\mu = \int\limits_X \Psi\left(\frac{g}{N}\right) d\mu = 1$. The function $\varphi := \Psi\left(\frac{f}{N}\right) - \Psi\left(\frac{g}{N}\right)$ is nonnegative, by the monotonicity of Ψ and $\int\limits_X \varphi \, d\mu = 0$, therefore $\varphi = 0$ $\mu-$a.e. in X. It follows that $f = g$ $\mu-$a.e. in X, since Ψ is injective. In our case the measure μ is atomless and the Young function Ψ is doubling and strictly convex. Then $L^{\Psi}(X)$ has a strictly convex and strictly monotone norm. By Theorem 1 the claim follows, taking into account that for $B = L^{\Psi}(X)$ the notion $B-$weak upper gradient is called $\Psi-$weak upper gradient. ∎

Remark 1. *Corollary 1 is a consequence of* [17, Theorem 6.6, Theorem 6.7, Lemma 6.8, Theorem 6.11]

Corollary 2. [16, Corollary 3.7] *Let $1 < p < \infty$. Then every function $u \in N^{1,p}(X)$ has is a $p-$weak upper gradient $g_u \in L^p(X)$ such that $g_u(x) \leq g(x)$ for $\mu-$almost every $x \in X$, whenever g is a $p-$weak upper gradient of u.*

Proof. For $\Psi(t) = \frac{t^p}{p}$, $t \in [0, \infty)$, the corresponding Orlicz space is a Lebesgue space: $L^{\Psi}(X) = L^p(X)$. The Young function Ψ is doubling. Since $p > 1$, Ψ is strictly convex. The claim follows by Corollary 1. ∎

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