"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 19 (2009), No. 1, 131 - 138

THE SPEED OF CONVERGENCE OF THE RIEMANN SUMS WITH APPLICATIONS TO GAMMA FUNCTION

CRISTINEL MORTICI

Abstract. The purpose of this paper is to establish some results about the convergence speed of the Riemann sums and to use them to give some properties related with Gamma function. A new proof of the Stirling's formula is given, then we pass to the continuous case, using the Croft's lemma.

1. Introduction

Let $f : [0, 1] \to \mathbb{R}$ be a Riemann integrable function.

One of the most known formula for estimating the definite integral of the function f uses the Riemann sums, since

$$\int_0^1 f(x)\,\mathrm{d}\,x = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

We study here the speed of convergence of this sequence of Riemann sums, then we apply these results to establish some properties of the Gamma function. The Stirling's formula

$$n! \approx \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n}$$

Keywords and phrases: Riemann sums, speed of convergence, Croft's lemma, Gamma function, Stirling's formula.
(2000)Mathematics Subject Classification: 26A42, 33B15, 33D05

is an approximation for the factorial function, which provides good results for large values of n. It was first discovered by Abraham de Moivre in the form

$n! \approx [\text{constant}] \cdot n^{n+\frac{1}{2}} \cdot \mathrm{e}^{-n}$

(with missing constant), while Stirling's contribution consisted of showing that the constant is $\sqrt{2\pi}$.

We establish the Stirling's formula in continuous form, by rewriting the multiplication formula of Gauss as a Riemann sum.

2. An estimate of speed of convergence of Riemann sums

We give the following

Theorem 2.1. *Let $f : [0,1] \to \mathbb{R}$ be a three times continuously differentiable function with f''' bounded. Then:*

$$\lim_{n\to\infty} n\left(\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\,\mathrm{d}\,x\right) = \frac{f(1)-f(0)}{2} \tag{2.1}$$

and

$$\lim_{n\to\infty} n\left[n\left(\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\,\mathrm{d}\,x\right) - \frac{f(1)-f(0)}{2}\right] = \frac{f'(1)-f'(0)}{12}. \tag{2.2}$$

Proof. By Taylor's formula, for every $x \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$, $1 \le k \le n$, there exists $a \in \left(x, \frac{k}{n}\right)$ such that

$$f\left(\frac{k}{n}\right) - f(x) = f'\left(\frac{k}{n}\right)\left(\frac{k}{n} - x\right) - \frac{1}{2} f''\left(\frac{k}{n}\right)\left(\frac{k}{n} - x\right)^2 + \frac{1}{6} f'''(a)\left(\frac{k}{n} - x\right)^3.$$

If we assume that $m \le f''' \le M$, for some real numbers $m < M$, we get

$$f\left(\frac{k}{n}\right) - f(x) \le f'\left(\frac{k}{n}\right)\left(\frac{k}{n} - x\right) - \frac{1}{2} f''\left(\frac{k}{n}\right)\left(\frac{k}{n} - x\right)^2 + \frac{M}{6}\left(\frac{k}{n} - x\right)^3.$$

By integration on $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ with respect to x, we deduce that

$$\frac{1}{n} f\left(\frac{k}{n}\right) - \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,\mathrm{d}\,x \le f'\left(\frac{k}{n}\right)\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\frac{k}{n} - x\right)\mathrm{d}\,x -$$

$$-\frac{1}{2}f''\left(\frac{k}{n}\right)\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\frac{k}{n}-x\right)^2\mathrm{d}\,x+\frac{M}{6}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\frac{k}{n}-x\right)^3\mathrm{d}\,x,$$

or

$$\frac{1}{n}f\left(\frac{k}{n}\right)-\int_{\frac{k-1}{n}}^{\frac{k}{n}}f(x)\,\mathrm{d}\,x\leq\frac{1}{2n^2}f'\left(\frac{k}{n}\right)-\frac{1}{6n^3}f''\left(\frac{k}{n}\right)+\frac{M}{24n^4}.$$

By summation from $k=1$ to $k=n$, we obtain

$$\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-\int_0^1 f(x)\,\mathrm{d}\,x\leq\frac{1}{2n^2}\sum_{k=1}^{n}f'\left(\frac{k}{n}\right)-\frac{1}{6n^3}\sum_{k=1}^{n}f''\left(\frac{k}{n}\right)+\frac{M}{24n^3},$$

then, by multiplying with n, we deduce that

(2.3)
$$n\left(\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-\int_0^1 f(x)\,\mathrm{d}\,x\right)\leq\frac{1}{2n}\sum_{k=1}^{n}f'\left(\frac{k}{n}\right)-\frac{1}{6n^2}\sum_{k=1}^{n}f''\left(\frac{k}{n}\right)+\frac{M}{24n^2}.$$

Now, by similar computations, starting from the inequality

$$f\left(\frac{k}{n}\right)-f(x)\geq f'\left(\frac{k}{n}\right)\left(\frac{k}{n}-x\right)-\frac{1}{2}f''\left(\frac{k}{n}\right)\left(\frac{k}{n}-x\right)^2+\frac{m}{6}\left(\frac{k}{n}-x\right)^3,$$

(because $f'''\geq m$), we obtain

(2.4)
$$n\left(\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-\int_0^1 f(x)\,\mathrm{d}\,x\right)\geq\frac{1}{2n}\sum_{k=1}^{n}f'\left(\frac{k}{n}\right)-\frac{1}{6n^2}\sum_{k=1}^{n}f''\left(\frac{k}{n}\right)+\frac{m}{24n^2}.$$

If we study carefully the relations (2.3)-(2.4) and take into account that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f'\left(\frac{k}{n}\right)=\int_0^1 f'(x)\,\mathrm{d}\,x=f(1)-f(0),$$

then (2.1) is proved.

Further, by subtracting $\dfrac{f(1)-f(0)}{2}$ from (2.3), then multiplying by n, we obtain

$$n\left[n\left(\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-\int_0^1 f(x)\,\mathrm{d}\,x\right)-\frac{f(1)-f(0)}{2}\right]\leq$$

$$\leq\frac{1}{2}\cdot n\left(\frac{1}{n}\sum_{k=1}^{n}f'\left(\frac{k}{n}\right)-\int_0^1 f'(x)\,\mathrm{d}\,x\right)-\frac{1}{6n}\sum_{k=1}^{n}f''\left(\frac{k}{n}\right)+\frac{M}{24n}.$$

From (2.4), we deduce in a similar way that

$$n\left[n\left(\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-\int_0^1 f(x)\,\mathrm{d}\,x\right)-\frac{f(1)-f(0)}{2}\right]\geq$$

$$\geq\frac{1}{2}\cdot n\left(\frac{1}{n}\sum_{k=1}^{n}f'\left(\frac{k}{n}\right)-\int_0^1 f'(x)\,\mathrm{d}\,x\right)-\frac{1}{6n}\sum_{k=1}^{n}f''\left(\frac{k}{n}\right)+\frac{m}{24n}.$$

Now, using the limit (2.1), we obtain

$$\lim_{n\to\infty}n\left(\frac{1}{n}\sum_{k=1}^{n}f'\left(\frac{k}{n}\right)-\int_0^1 f'(x)\,\mathrm{d}\,x\right)=\frac{f'(1)-f'(0)}{2}$$

and the theorem is completely proved, taking into account that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f''\left(\frac{k}{n}\right)=\int_0^1 f''(x)\,\mathrm{d}\,x=f'(1)-f'(0).\square$$

3. Applications to the Gamma function

We apply now the Theorem 2.1 to establish some new results regarding the famous Gamma function

$$\Gamma(x)=\int_0^\infty t^{x-1}\,\mathrm{e}^{-t}\,\mathrm{d}\,t\quad,\qquad x>0.$$

Let us define the function $f:[0,1]\to\mathbb{R}$, by the formula

$$f(x)=\ln\Gamma(a+x),$$

where $a>0$ is arbitrarily fixed. The idea is to use the folowing multiplication Gauss formula (*e.g.,* [1, 3, 5])

$$\Gamma\left(a+\frac{1}{n}\right)\Gamma\left(a+\frac{2}{n}\right)\cdot\ldots\cdot\Gamma\left(a+\frac{n}{n}\right)=a\left(2\pi\right)^{\frac{n-1}{2}}n^{\frac{1}{2}-na}\Gamma\left(na\right)$$

as a Riemann sum associated to the function f.

Theorem 3.1. *For every $a>0$, we have*

$$\lim_{n\to\infty}\frac{\sqrt[n]{\Gamma\left(na\right)}}{n^a}=\frac{1}{\sqrt{2\pi}}\exp\left(\int_a^{a+1}\ln\Gamma\left(t\right)\mathrm{d}\,t\right).$$

Proof. By using a simple change of variable, we have

$$\int_a^{a+1}\ln\Gamma\left(t\right)\mathrm{d}\,t=\int_0^1\ln\Gamma\left(a+x\right)\mathrm{d}\,x=$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln \Gamma\left(a + \frac{k}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \ln\left(\prod_{k=1}^{n} \Gamma\left(a + \frac{k}{n}\right)\right) =$$

$$= \lim_{n\to\infty} \frac{1}{n} \ln\left[a\,(2\pi)^{\frac{n-1}{2}}\, n^{\frac{1}{2}-na}\Gamma\,(na)\right] = \lim_{n\to\infty} \ln\left[\sqrt[n]{a}\,(2\pi)^{\frac{n-1}{2n}}\, n^{\frac{1}{2n}-a}\,\sqrt[n]{\Gamma(na)}\right] =$$

$$= \lim_{n\to\infty} \ln\left[\sqrt{2\pi}\cdot\frac{\sqrt[n]{\Gamma(na)}}{n^a}\right]$$

and the conclusion follows by consider the exponential. □

Further, by using the Raabe's formula (*e.g.*, [5, 6])

$$\int_a^{a+1} \ln\Gamma(t)\,\mathrm{d}\,t = a\ln a - a + \frac{1}{2}\ln 2\pi,$$

we can obtain from the Theorem 3.1 the following

Corollary 3.1. *For every $a > 0$, we have*

$$\lim_{n\to\infty} \frac{\sqrt[n]{\Gamma\,(na)}}{n^a} = \left(\frac{a}{\mathrm{e}}\right)^a.$$

After a simple transformation, we obtain

$$\lim_{n\to\infty} \frac{[\Gamma\,(na)]^{\frac{1}{na}}}{na} = \frac{1}{\mathrm{e}}. \tag{3.1}$$

If we denote by $g : (0,\infty) \to \mathbb{R}$, the continuous function

$$g(x) = \frac{[\Gamma(x)]^{\frac{1}{x}}}{x},$$

then from the limit (3.1) it results that for all $a > 0$, we have $\lim\limits_{n\to\infty} g(na) = \mathrm{e}^{-1}$.

We can apply now the Croft's lemma (*e.g.*, [4]) to deduce that the function g has the same limit at infinity, that is

$$\lim_{x\to\infty} \frac{[\Gamma(x)]^{\frac{1}{x}}}{x} = \frac{1}{\mathrm{e}}.$$

We use now the limit (2.1) to prove the continuous form of the Stirling's formula.

Theorem 3.2. *We have*

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}} = 1. \tag{3.2}$$

Proof. The limit (2.1) from the Theorem 2.1 in the case of the smooth function $f(x) = \ln \Gamma(a + x)$ becomes

$$\lim_{n\to\infty} n \left[\frac{1}{n}\sum_{k=1}^{n} \ln\Gamma\left(a+\frac{k}{n}\right) - \int_0^1 \ln\Gamma(a+x)\,\mathrm{d}\,x\right] = \frac{\ln\Gamma(a+1)-\ln\Gamma(a)}{2}.$$

Using again the Raabe's formula, we obtain

$$\lim_{n\to\infty}\left(\ln\left[a\,(2\pi)^{\frac{n-1}{2}}\, n^{\frac{1}{2}-na}\Gamma\,(na)\right] - na\ln a + na - \frac{n}{2}\ln 2\pi\right) = \frac{1}{2}\ln a,$$

or equivalently,

$$\lim_{n\to\infty} \frac{\Gamma(na)\,\mathrm{e}^{na}}{(na)^{na-\frac{1}{2}}} = \sqrt{2\pi}.$$

Consider now the continuous function $h : (0, \infty) \to \mathbb{R}$, given by

$$h(x) = \frac{\Gamma(x)\,\mathrm{e}^{x}}{x^{x-\frac{1}{2}}}.$$

For every $a > 0$, we have $\lim\limits_{n\to\infty} h(na) = \sqrt{2\pi}$, so with Croft's lemma, it results that $\lim\limits_{x\to\infty} h(x) = \sqrt{2\pi}$. This limit is in fact the conclusion, where we have only to replace $\Gamma(x) = \Gamma(x + 1)/x$. □

Finally, by applying the limit (2.2), we can state the following

Theorem 3.3. *We have*

$$\lim_{x\to\infty}\left(\frac{\Gamma(x+1)}{(\frac{x}{\mathrm{e}})^{x}\cdot\sqrt{2\pi x}}\right)^{x} = \sqrt[12]{\mathrm{e}}. \tag{3.3}$$

By consider the logarithm, we obtain

$$\lim_{x\to\infty} x\left(\frac{\Gamma(x+1)}{(\frac{x}{\mathrm{e}})^{x}\cdot\sqrt{2\pi x}} - 1\right) = \frac{1}{12},$$

from which we deduce the approximation formula

$$\Gamma(x+1) \approx \left(1+\frac{1}{12x}\right)\cdot\left(\frac{x}{\mathrm{e}}\right)^{x}\cdot\sqrt{2\pi x}. \tag{3.4}$$

Thus for all positive integers n, we have the approximations

$$n! \approx \left(1+\frac{1}{12n}\right)\left(\frac{n}{\mathrm{e}}\right)^{n}\sqrt{2\pi n}, \tag{3.5}$$

which is stronger than the Stirling's formula, as it results also from the following table:

	$n!$	Stirling	Formula 3.5
$n = 10$	3628800	3.5987×10^6	3.6287×10^6
$n = 15$	1307674368000	1.3004×10^{12}	1.3077×10^{12}
$n = 30$	$265252859812191058636308 \times 10^7$	2.6452×10^{32}	2.6525×10^{32}

Proof of the Theorem 3.3. By using the limit (2.2), we have

$$\lim_{n\to\infty} n\left[\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - n\int_0^1 f(x)\,\mathrm{d}\,x - \frac{f(1)-f(0)}{2}\right] =$$

$$= \lim_{n\to\infty} n\left[\left(\ln\left[a\,(2\pi)^{\frac{n-1}{2}}\,n^{\frac{1}{2}-na}\Gamma\,(na)\right] - na\ln a + na - \frac{n}{2}\ln 2\pi\right) - \frac{1}{2}\ln a\right] =$$

$$= \lim_{n\to\infty} n\left[\ln a + \frac{n-1}{2}\ln(2\pi) + \left(\frac{1}{2} - na\right)\ln n + \right.$$

$$\left. + \ln\Gamma(na) - na\ln a + na - \frac{n}{2}\ln 2\pi - \frac{1}{2}\ln a\right] =$$

$$= \lim_{n\to\infty} n\left[\ln\Gamma(na) + na + \left(\frac{1}{2} - na\right)\ln na - \frac{1}{2}\ln 2\pi\right] =$$

$$= \lim_{n\to\infty} n\cdot\ln\frac{\Gamma(na)\,\mathrm{e}^{na}}{na^{na-\frac{1}{2}}\cdot\sqrt{2\pi}} = \lim_{n\to\infty}\ln\left(\frac{\Gamma(na+1)\,\mathrm{e}^{na}}{na^{na}\cdot\sqrt{2\pi na}}\right)^n.$$

As we proved, this limit is equal to

$$\frac{f'(1)-f'(0)}{12} = \frac{1}{12}\left(\frac{\Gamma'(a+1)}{\Gamma(a+1)} - \frac{\Gamma'(a)}{\Gamma(a)}\right) = \frac{1}{12a}.$$

The last equality follows from the well known recurrence relation

$$\psi(x+1) = \psi(x) + \frac{1}{x} \quad , \qquad x > 0$$

where $\psi(x) = \dfrac{\mathrm{d}}{\mathrm{d}\,x}\ln\Gamma(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function (*e.g.*, [1], [3], [5]).

If we multiply by a, then by taking the exponential, we obtain

$$\lim_{n\to\infty}\left(\frac{\Gamma(na+1)\,\mathrm{e}^{na}}{na^{na}\cdot\sqrt{2\pi na}}\right)^{na} = \sqrt[12]{\mathrm{e}} \tag{3.6}$$

and the conclusion follows by the Croft's lemma. ∎

4. Conclusions

The approximation formula (3.5), which is the first approximation of the Stirling's series (*e.g.,* [2])

(4.1)

$$n! \approx \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n}\left(1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{51840n^3}-\frac{571}{2488320n^4}+...\right),$$

was obtained by using the first three derivatives of the function f in the proof of the Theorem 2.1.

Now remark that, at least theoretically, if we consider in the proof of the Theorem 2.1 additional terms in the Taylor's formula, then we can obtain more terms of the approximation (4.1).

References

[1] Abramowitz, M., Stegun, I.A., **Handbook of Mathematical Functions with Formulas**, Graphs, and Mathematical Tables. New York. Dover 1972.

[2] Arfken, G., **Stirling's Series**. In: Mathematical Methods for Physicists. 555-559, Orlando. Academic Press 1985.

[3] Artin, E., **The Gamma Function**. New York. Holt, Rinehart and Winston 1964.

[4] Croft, H.T., **A question of limits**. Eureka 20, 11-13 (1957).

[5] Y. C. Li, **A note on an identity of the gamma function and Stirling's formula**, Real. Anal. Exchange, vol. 32, no. 1 (2006), 267-272.

[6] Nielsen, N. **Die Gammafunktion**. Chelsea. Reprint 1965. Original publication by B.G.Teubner 1906.

Valahia University of Târgovişte
Department of Mathematics
Bd. Unirii 18, 130082 Târgovişte, ROMANIA
email: cmortici@valahia.ro