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GEOMETRIC ASPECTS OF CLASSICAL NONHOLONOMIC, SCLERONOMIC MECHANICAL SYSTEMS

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Abstract. A classical nonholonomic, scleronomic mechanical system $\sum$(1.1) is considered, whose the evolution equations are (2.6.). We associate to system $\sum$ a canonical semispray S^* on the phases space TM and we use the differential geometry of Lagrange spaces to study the systems $\sum$.

1. INTRODUCTION

We consider classical nonholonomic, scleronomic mechanical systems

$$\sum = (M, L(x, y), F_i(x), Q_\sigma(x, y)) \tag{1.1}$$

where $L(x, y)$ is given by the kinetic energy

$$L(x, y) = g_{ij}(x) y^i y^j, \quad y^i = \frac{dx^i}{dt} \tag{1.2}$$

$g_{ij}(x)$ being the fundamental tensor of a Riemann space $R^n = (M, g_{ij}(x))$, the external forces $F_i(x)$ gives a d-covectors field on the base manifold M and Q_σ give the cinematic constrains $Q_\sigma = a_{\sigma_i} \dot{x}^i \quad (\sigma = m+1, \ldots, n)$.

The space R^n will be named the associated Riemann space of the system Σ. It coincides with the Lagrange space $L^n = (M, L(x, y))$ whose fundamental tensor is $g_{ij}(x)$ depending only by the material points x of Σ.

For the Lagrangian nonholonomic, scleronomic mechanical systems

$$\sum{}' = (M, L(x, y), F_i(x,y), Q_\sigma(x, y)) \tag{1.3}$$

external forces $F_i(x, y)$ determine a d-covariant vector field and

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(1.4) $$F_{ij} = \frac{\partial F_j}{\partial y^i} - \frac{\partial F_i}{\partial y^j}$$

is an antisymmetric d-tensor field, named elicoidal tensor of system Σ.

Also, the functions that determine the constrains of the system (1.3)

(1.5) $$Q_\sigma(x,y) = a_{\sigma_i}(x) y^i, \quad (\sigma = m+1,...,n)$$

are scalars with respect to the changes of the coordinates on TM.

Then $a_{\sigma_i}(x)$ are n-m covector fields on M and

(1.6) $$\sum_{\sigma=m+1}^{n} \lambda^\sigma(x) Q_\sigma(x,y)$$

is also a scalar function on TM. The functions $\lambda^\sigma(x)$ are the Lagrange multipliers.

Some properties of Σ' were investigated by us in [4].

The classical nonholonomic, scleronomic mechanical systems are the particular case of the systems Σ' (1.3) obtained for $L(x,y) = g_{ij}(x) y^i y^j, \quad y^i = \frac{dx^i}{dt}$.

In 1926, Gh.Vrănceanu introducesed the notion of Riemannian nonholonomic space and realized a first geometric model for the nonholonomic, scleronomic mechanical system. He considers as evolution the equations of system, the Lagrange equations:

(1.7) $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right) - \frac{\partial T}{\partial x^i} = \sum_{\sigma=m+1}^{n} \lambda_\sigma a_{\sigma_i}(x) + F_i(x)$$

where $Q_\sigma(x,dx) = a_{\sigma_i}(x) dx^i = 0$ give the kinematic constraints.

In 1928, E. Cartan showed that the equations (1.7) are not sufficient. He gives the geometrization of these systems by fixing the normal distribution to the distribution $Q_\sigma = 0$.

M. Haimovici completed Cartan, supposing that the system of Pfaff equations $Q_\sigma = 0$ has the first derivative system identically null.

Let L* be the Lagrangian

(1.8) $$L^*(x,y) = L(x,y) + \sum_{\sigma=m+1}^{n} \lambda^\sigma(x) Q_\sigma(x,y).$$

The Lagrangians L^* and L are equivalent if corresponding solutions of Lagrange equations

$$(1.9) \qquad \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^i} = 0, \quad \frac{\partial L^*}{\partial x^i} - \frac{d}{dt}\frac{\partial L^*}{\partial y^i} = 0$$

are equal.

2. MAIN RESULT

For nonholonomic, scleronomic mechanical system (1.1), the elycoidal tensor of the system Σ F_{ij} given by (1.4) vanishes

$$(2.1) \qquad F_{ij}(x,y) = 0$$

The Lagrangian $L^*(x,y)$ from (1.8) has the classical form:

$$(2.2) \qquad L^*(x,y) = g_{ij}(x)y^i y^j + \lambda^\sigma(x)a_{\sigma_i}(x)y^i$$

So, we have

Proposition 2.1. *The Lagrangian $L^*(x,y)$ of a classical nonholonomic mechanical system Σ has the form of a Lagrangian from electrodynamics where the electromagnetics potentials $A_i(x)$ are given by*

$$(2.3) \qquad A_i(x) = \lambda^\sigma(x)a_{\sigma_i}(x).$$

Proposition 2.2. *The Euler – Lagrange equations of the Lagrangian $L^*(x,y)$ are given by*

$$(2.4) \qquad \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^i} + \left[\frac{\partial \lambda^\sigma}{\partial x^i}Q_\sigma + \lambda^\sigma\frac{\partial Q_\sigma}{\partial x^i} - \frac{d}{dt}\left(\lambda^\sigma a_{\sigma_i}\right)\right]$$

or

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^i} + \left[\frac{\partial \lambda^\sigma}{\partial x^i}a_{\sigma_i} - \frac{\partial \lambda_\sigma}{\partial x^j}a_{\sigma_i} + \lambda^\sigma\left(\frac{\partial a_{\sigma_i}}{\partial x^i} - \frac{\partial a_{\sigma_i}}{\partial x^j}\right)\right]y^i = 0$$

We know from [4] that the canonical semispray of the system Σ' is a vector field S^* on the phases space

$$(2.5) \qquad S^* = y^i\frac{\partial}{\partial x^i} - 2G^{*i}(x,y)\frac{\partial}{\partial y^i}$$

whose integral curves and the canonical nonlinear connection were investigated by us in [4].

In our case, for system Σ we obtain

Theorem 2.1. *The equations of evolution of a classical nonholonomic mechanical system* Σ *are:*

$$\frac{d^2x^i}{dt^2}+\Gamma^i_{jk}(x)\frac{dx^j}{dt}\frac{dx^k}{dt}=\frac{1}{2}g^{is}\left[F_s(x)+\lambda^\sigma a_{\sigma_s}(x)\right], \tag{2.6}$$

$\Gamma^i_{jk}(x)$ *being the Christoffel symbols.*

Also, we have

Theorem 2.2. *The canonical semispray of the system* Σ *is given by*

$$S^*=y^i\frac{\partial}{\partial x^i}-2G^{*i}(x,y)\frac{\partial}{\partial y^i}, \tag{2.7}$$

where

$$2G^{*i}(x,y)=\Gamma^i_{jk}(x)y^jy^k-\frac{1}{2}\left(F^i(x)+\lambda^\sigma a^i_\sigma(x)\right), \tag{2.8}$$

$\Gamma^i_{jk}(x)$ *are the Christoffel symbols of the associated Riemann space* R^n *and*

$$F^i(x)=g^{ih}(x)F_h(x), a^i_\sigma(x)=g^{ih}(x)a_{\sigma h}(x).$$

The Theorem 2.2 from [4] leads to:

Theorem 2.3. *The integral curves of the semispray* S^* *are the evolution curves of the nonholonomic mechanical system* Σ.

Because the tensor $\dfrac{\partial F^i}{\partial y^j}$ vanishes, we obtain

Theorem 2.4. *The canonical nonlinear connection* N^* *of the system* Σ *coincides with the canonical nonlinear connection N with the coefficients* $N^i_j=\Gamma^i_{jk}(x)y^k$ *of the associated Riemann space*.

So, the nonlinear connection N^* does not depend on the external forces $F_i(x)$ or the nonholonomy forces $Q_\sigma(x,y)$. This property simplifies whole theory, because N^*- canonical metrical connection $C\Gamma\left(N^*\right)$ has the coefficients

$$L^{*i}_{jk}=\Gamma^i_{jk}(x), C^i_{jk}=0. \tag{2.9}$$

So, $C\Gamma\left(N^*\right)$ is a Cartan connection.

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