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SOME MINIMAL HELICOIDAL SURFACES IN MINKOWSKI SPACE $\mathbb{R}^3_1$

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Abstract. A helicoidal surface is a surface obtained by rotating a curve around an axis and simultaneously translating the curve along that axis. In this paper we identify some minimal surfaces inside of three classes of helicoidal surfaces in the Minkowski space $\mathbb{R}^3_1$.

1. Preliminaries

Let $\mathbb{R}^3$ be a 3 - dimensional real vector space.

Definition 1.1. *The 3 - dimensional Minkowski space is the pair* $(\mathbb{R}^3, \langle\cdot,\cdot\rangle_1)$, *denoted by* $\mathbb{R}^3_1$, *where the pseudo - inner product* $\langle\cdot,\cdot\rangle_1$ *is given by*

$$\langle x, y\rangle_1 = -x_1y_1 + x_2y_2 + x_3y_3 \tag{1.1}$$

for every $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, *or*

$$\langle x, y\rangle_1 = x^t\eta y \tag{1.2}$$

where $\eta = diag\,(-1, 1, 1)$.

It is easy to verify that $B = \{\xi_1 = (1, 0, 0),\ \xi_2 = (0, 1, 0),\ \xi_3 = (0, 0, 1)\}$ is an orthonormal base of $\mathbb{R}^3_1$.

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The rotations about spacelike and timelike axes form the so-called *Lorentz group* $SO(2,1)$. We quote from [1] the following part of

Proposition 1.1. *The rotations around the timelike axis ξ_1, the spacelike axes ξ_2, ξ_3, are respectively determined by the following rotational matrices:*

i) $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{pmatrix}$ if the axis is ξ_1;

ii) $R = \begin{pmatrix} \cosh v & 0 & \sinh v \\ 0 & 1 & 0 \\ \sinh v & 0 & \cosh v \end{pmatrix}$ if the axis is ξ_2;

iii) $R = \begin{pmatrix} \cosh v & \sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{pmatrix}$ if the axis is ξ_3.

In each case, R is a Lorentz transformation that preserves the corresponding axis, i.e., R satisfies the following conditions:

$$\begin{cases} R\xi_k = \xi_k, \text{ for all } \xi_k \in \{\xi_1, \xi_2, \xi_3, \xi_1 \pm \xi_2, \xi_1 \pm \xi_3\} \\ R\eta R^t = \eta \\ \det R = 1 \end{cases} \tag{1.3}$$

If we take, for example, the axis determined by the timelike vector $\xi_1 = (1, 0, 0)$, then the curve α will be in the plane $O\xi_1\xi_2$ or $O\xi_1\xi_3$.

In the first case $(\alpha \subset (O\xi_1\xi_2))$, $\alpha(u) = (f(u), g(u), 0)$. Taking the parameter u on the rotational axis (ξ_1), the curve has the form: $\alpha(u) = (u, a(u), 0)$ and the equation of helicoidal surface is:

$$X(u, v) = (u + b(v), a(u)\cos v, a(u)\sin v).$$

In generally, this is not a regular surface. The regularity condition implies $a'(u)a(u) \neq 0$. This condition doesn't take place if $a(u) = 0$, which means that the rotational curve is intersecting the rotational axis or if $a'(u) = 0$. In the following we will eliminate this situation, so we will suppose that we rotate an arc of curve which doesn't intersect the rotational axis.

Definition 1.2. *The regular surface of equation*

$$X(u, v) = (u + b(v), a(u)\cos v, a(u)\sin v) \tag{1.4}$$

with $a'(u) \neq 0$ is called ${}^{1,2}H_1$ - helicoidal surface.

In the second case, let $\alpha(u) = (u, 0, a(u))$ be a curve in the plane $O\xi_1\xi_3$ and let $\beta(v) = (b(v), 0, 0)$ an arbitrary vector. Rotating this curve around axis ξ_1 and translating in the same time with vector $\beta(v)$, taking into acount Proposition 1.1 we get:

Definition 1.3. *The regular surface of equation*

$$X(u, v) = (u + b(v), -a(u)\sin v, a(u)\cos v) \tag{1.5}$$

with $a'(u) \neq 0$ *is called* ${}^{1,3}H_1$ *- helicoidal surface.*

The change of parameter $v \mapsto v + \dfrac{\pi}{2}$ in ${}^{1,2}H_1$ - helicoidal surface, equivalent with dephasing of rotational angle with $\dfrac{\pi}{2}$, lead to a ${}^{1,3}H_1$ - helicoidal surface (with another vector β, translated) and conversely, starting from a ${}^{1,3}H_1$ - helicoidal surface, making the change of variable $v \mapsto v - \dfrac{\pi}{2}$, we obtain a ${}^{1,2}H_1$ - helicoidal surface.

Similarly, if the rotational axis is the spacelike vector $\xi_2 = (0, 1, 0)$, then we have the following surfaces.

Definition 1.4. *The regular surface of equation*

$$X(u, v) = (a(u)\cosh v, u + b(v), a(u)\sinh v) \tag{1.6}$$

with $a'(u) \neq 0$ *is called* ${}^{1,2}H_2$ *- helicoidal surface.*

Definition 1.5. *The regular surface of equation*

$$X(u, v) = (a(u)\sinh v, u + b(v), a(u)\cosh v) \tag{1.7}$$

with $a'(u) \neq 0$ *is called* ${}^{2,3}H_2$ *- helicoidal surface.*

For the axis determined by the spacelike vector $\xi_3 = (0, 0, 1)$, we have:

Definition 1.6. *The regular surface of equation*

$$X(u, v) = (a(u)\cosh v, a(u)\sinh v, u + b(v)) \tag{1.8}$$

with $a'(u) \neq 0$ *is called* ${}^{1,3}H_3$ *- helicoidal surface.*

Definition 1.7. *The regular surface of equation*

$$X(u, v) = (a(u)\sinh v, a(u)\cosh v, u + b(v)) \tag{1.9}$$

with $a'(u) \neq 0$ *is called* ${}^{2,3}H_3$ *- helicoidal surface.*

Since the coordinates ξ_2 and ξ_3 have a symmetric role, there is no difference (just a symmetry) between ${}^{1,2}H_2$ - helicoidal surfaces and ${}^{1,3}H_3$ - helicoidal surfaces, respectively between ${}^{2,3}H_2$ - helicoidal surfaces and ${}^{2,3}H_3$ - helicoidal surfaces.

We recall

Theorem 1.1 ([1]). *Let $X(u,v)$ be a surface in $\mathbb{R}^3_1$. The mean curvature H of this surface is given by:*

$$H = \frac{1}{2}\frac{GL + EN - 2FM}{EG - F^2} \tag{1.10}$$

where E, F, G, the coefficients of the first fundamental form, are given by

$$E = \langle X_u, X_u\rangle_1, F = \langle X_u, X_v\rangle_1, G = \langle X_v, X_v\rangle_1 \tag{1.11}$$

and L, M, N, the coefficients of the second fundamental form, are given by

$$\begin{cases} L = \dfrac{1}{\sqrt{EG-F^2}}\det(X_u, X_v, X_{uu}) \\ M = \dfrac{1}{\sqrt{EG-F^2}}\det(X_u, X_v, X_{uv}) \\ N = \dfrac{1}{\sqrt{EG-F^2}}\det(X_u, X_v, X_{vv}) \end{cases} \tag{1.12}$$

2. Main results

In the following we shall look for minimal surfaces ($H = 0$) in each of these classes. Thus we have

Proposition 2.1. *Let S be a ${}^{1,2}H_1$ - helicoidal surface given by (1.4). The mean curvature of surface S is:*

$$H = \frac{1}{2}\frac{\omega_-\eta_+ a(u) - 2\gamma - \sigma_-\theta}{(\omega_- a^2(u) - \gamma)^{3/2}}$$

where

$$\varepsilon = \begin{cases} 1, & \textit{if } S \textit{ is timelike} \\ -1, & \textit{if } S \textit{ is spacelike} \end{cases} \tag{2.1}$$

$$\begin{gathered} \omega_- = a'^2(u) - 1, \sigma_- = a^2(u) - b'^2(v), \\ \eta_+ = a(u) + a'(u)b''(v), \gamma = a'^2(u)b'^2(v), \\ \theta = a(u)a''(u) \end{gathered} \tag{2.2}$$

Proof. Since for a ${}^{1,2}H_1$ - helicoidal surface we have:

$$X_u = (1, a'(u)\cos v, a'(u)\sin v), X_v = (b'(v), -a(u)\sin v, a(u)\cos v),$$

it follows, successively, the coeficients of the first fundamental form:

$$E = \langle X_u, X_u\rangle_1 = a'^2(u) - 1,$$
$$F = \langle X_u, X_v\rangle_1 = -b'(v),$$
$$G = \langle X_v, X_v\rangle_1 = a^2(u) - b'^2(v),$$

from where

$$EG - F^2 = a^2(u)(a'^2(u) - 1) - a'^2(u)b'^2(v);$$

the coeficients of the second fundamental form are:

$$L = -\frac{a(u)a''(u)}{\sqrt{a^2(u)(a'^2(u) - 1) - a'^2(u)b'^2(v)}},$$

$$M = -\frac{a'^2(u)b'(v)}{\sqrt{a^2(u)(a'^2(u) - 1) - a'^2(u)b'^2(v)}},$$

$$N = \frac{a^2(u) + a(u)a'(u)b''(v)}{\sqrt{a^2(u)(a'^2(u) - 1) - a'^2(u)b'^2(v)}}$$

and the mean curvature is:

$$H = \tfrac{1}{2}[(a'^2(u) - 1)a(u)(a(u) + a'(u)b''(v)) - 2a'^2(u)b'^2(v) -$$
$$-(a^2(u) - b'^2(v))a(u)a''(u)]/2[(a^2(u)(a'^2(u) - 1) - a'^2(u)b'^2(v))^{3/2}].$$

For

$$\omega_- = a'^2(u) - 1, \sigma_- = a^2(u) - b'^2(v),$$
$$\eta_+ = a(u) + a'(u)b''(v),$$
$$\gamma = a'^2(u)b'^2(v), \theta = a(u)a''(u)$$

the mean curvature become:

$$H = \frac{1}{2}\frac{\omega_-\eta_+ a(u) - 2\gamma - \sigma_-\theta}{(\omega_- a^2(u) - \gamma)^{3/2}}$$

and so the proof is completed. □

Corollary 2.1. *If for a* ${}^{1,2}H_1$ *- helicoidal surface,* $a(u)$ *and* $b(v)$*, with* $a'(u) \neq \pm 1$*, satisfy the relation*

$$b''(v) = p(u)b'^2(v) + q(u) \tag{2.3}$$

where
(2.4)
$$p(u) = \frac{2a'^2(u) - a(u)a''(u)}{a(u)a'(u)(a'^2(u)-1)}, q(u) = \frac{a^3(u)a''(u) - a^2(u)(a'^2(u)-1)}{a(u)a'(u)(a'^2(u)-1)}$$
then the $^{1,2}H_1$ *- helicoidal surface is minimal.*

Proof. From
$$\begin{aligned} b''(v) &= p(u)b'^2(v) + q(u) \\ &= \frac{2a'^2(u) - a(u)a''(u)}{a(u)a'(u)(a'^2(u)-1)} b'^2(v) + \frac{a^3(u)a''(u) - a^2(u)(a'^2(u)-1)}{a(u)a'(u)(a'^2(u)-1)} \end{aligned}$$
we get
$$\begin{aligned} &a(u)a'(u)(a'^2(u)-1)b''(v) - (2a'^2(u) - a(u)a''(u))b'^2(v) - \\ &-a^3(u)a''(u) - a^2(u)(a'^2(u)-1) = 0 \end{aligned}$$
relation replaced in the mean curvature formula lead to $H = 0$. □

Remark 2.1. *Simultaneously,* $p(u) = 0$ *and* $q(u) = 0$ *can't take place; indeed, from* $p(u) = 0$ *and* $q(u) = 0$ *it follows*
$$\begin{cases} 2a'^2(u) - a(u)a''(u) = 0 \\ a^3(u)a''(u) - a^2(u)(a'^2(u)-1) = 0 \end{cases}.$$
The second equation lead to:
$$a(u)a''(u) - a'^2(u) + 1 = 0$$
from where, using the first equation we get $a'^2(u) + 1 = 0$*, equation with no solution.*

Corollary 2.2. *If for a* $^{1,2}H_1$ *- helicoidal surface, we have between* $a(u) = Au + B$*,* $A \neq \pm 1$ *and* $b(v) = Cv + D$*,* $B \neq 0$ *a relation of form*
$$(A^2 - 1)(Au + B)^2 - 2A^2C^2 = 0$$
then the surface is minimal.

Proof. Replacing $a''(u) = 0$ and $b''(v) = 0$ in the expresion of mean curvature of a $^{1,2}H_1$ - helicoidal surface we obtain:
$$H = \frac{1}{2} \frac{(a'^2(u) - 1)a^2(u) - 2a'^2(u)b'^2(v)}{(a^2(u)(a'^2(u)-1) - a'^2(u)b'^2(v))^{3/2}}$$
From the condition $(a'^2(u) - 1)a^2(u) = 2a'^2(u)b'^2(v)$ if follows:
$$H = \frac{1}{2} \frac{0}{a'^3(u)b'^3(v)} = 0,$$

so, the surface is minimal. $\square$

Proposition 2.2. *Let S be a spacelike ${}^{2,3}H_2$ - helicoidal surface. The mean curvature of surface S are:*

$$H = \frac{1}{2}\frac{2\gamma + \sigma_- a(u)a''(u) - \omega_+\eta_- a(u)}{(\gamma - \omega_+ a^2(u))^{3/2}}, \tag{2.5}$$

where

$$\varepsilon = \begin{cases} 1, & \textit{if } S \textit{ is timelike} \\ -1, & \textit{if } S \textit{ is spacelike} \end{cases} \tag{2.6}$$

and

$$\begin{gathered} \omega_+ = a'^2(u) + 1, \sigma_- = a^2(u) - b'^2(v), \\ \eta_- = a(u) - a'(u)b''(v), \gamma = a'^2(u)b'^2(v), \\ \theta = a(u)a''(u) \end{gathered} \tag{2.7}$$

Proof. Successively we have:

$$\begin{aligned} E &= 1 + a'^2(u), \\ F &= b'(v), \\ G &= b'^2(v) - a^2(u), \end{aligned}$$

$$\begin{aligned} EG - F^2 &= -a^2(u) - a'^2(u)a^2(u) + a'^2(u)b'^2(v) \\ &= a'^2(u)b'^2(v) - a^2(u)(1 + a'^2(u)), \end{aligned}$$

$$L = -\frac{a(u)a''(u)}{\sqrt{a'^2(u)b'^2(v) - a^2(u)(1 + a'^2(u))}},$$

$$M = -\frac{a'^2(u)b'(v)}{\sqrt{a'^2(u)b'^2(v) - a^2(u)(1 + a'^2(u))}},$$

$$N = \frac{a(u)a'(u)b''(v) - a^2(u)}{\sqrt{a'^2(u)b'^2(v) - a^2(u)(1 + a'^2(u))}},$$

from where, the mean curvature of surface S is:

$$\begin{aligned} H = {} & \frac{1}{2}[(1 + a'^2(u))(a(u)a'(u)b''(v) - a^2(u)) + 2a'^2(u)b'^2(v) - \\ & -(b'^2(v) - a^2(u))a(u)a''(u)]/[(a'^2(u)b'^2(v) - a^2(u)(1 + a'^2(u)))^{3/2}] \end{aligned}$$

For

$$\begin{gathered} \omega_+ = a'^2(u) + 1, \sigma_- = a^2(u) - b'^2(v), \\ \eta_- = a(u) - a'(u)b''(v), \gamma = a'^2(u)b'^2(v), \\ \theta = a(u)a''(u) \end{gathered}$$

the mean curvature become:

$$H = \frac{1}{2}\frac{-\omega_+\eta_- a(u) + 2\gamma + \sigma_- a(u)a''(u)}{(\gamma - \omega_+ a^2(u))^{3/2}}.$$

□

Corollary 2.3. *If for a spacelike* ${}^{2,3}H_2$ *- helicoidal surface,* $a(u)$ *and* $b(v)$*, with* $a'(u) \neq \pm 1$*, satisfy the relation*

$$b''(v) = P(a)b'^2(v) + Q(a) \tag{2.8}$$

where

(2.9)

$$P(a) = \frac{a(u)a''(u) - 2a'^2(u)}{(1 + a'^2(u))a(u)a'(u)}, Q(a) = \frac{a^2(u)(1 + a'^2(u)) - a^3(u)a''(u)}{(1 + a'^2(u))a(u)a'(u)}$$

then, the ${}^{2,3}H_2$ *- helicoidal surfaces are minimal.*

Proof. From hypothesis:

$$b''(v) = \frac{a(u)a''(u) - 2a'^2(u)}{(1 + a'^2(u))a(u)a'(u)}b'^2(v) + \frac{a^2(u)(1 + a'^2(u)) - a^3(u)a''(u)}{(1 + a'^2(u))a(u)a'(u)} \Leftrightarrow$$
$$b''(v)(1 + a'^2(u))a(u)a'(u) = b'^2(v)(a(u)a''(u) - 2a'^2(u)) +$$
$$+ a^2(u)(1 + a'^2(u)) - a^3(u)a''(u),$$

relation that lead to $H = 0$. □

Corollary 2.4. *There are no minimal* ${}^{2,3}H_2$ *- helicoidal surfaces for which* $a(u) = Au + B$ *and* $b(v) = Cv + D$*,* $b'(v) \neq 0$*.*

Proof. From the mean curvature formula, for $a''(u) = 0 = b''(v)$ we have:

$$H = \frac{1}{2}\frac{-(1 + a'^2(u))a^2(u) + 2a'^2(u)b'^2(v)}{(a'^2(u)b'^2(v) - a^2(u)(1 + a'^2(u)))^{3/2}}$$

If $-(1 + a'^2(u))a^2(u) + 2a'^2(u)b'^2(v) = 0$, $b'(v) \neq 0$, then:

$$H = \frac{1}{2}\frac{0}{(-a'^2(u)b'^2(v))^{3/2}}$$

imposible. □

Proposition 2.3. *Let* S *be a spacelike* ${}^{1,3}H_3$ *- helicoidal surface. The mean curvature of surface* S *is:*

$$H = \frac{1}{2}\frac{2\gamma + \sigma_-\theta - \eta_-\omega_+ a(u)}{(\gamma - \omega_+ a^2(u))^{3/2}} \tag{2.10}$$

where

$$\begin{gathered}\omega_+ = 1 + a'^2(u), \sigma_- = a^2(u) - b'^2(v),\\ \eta_- = a(u) - a'(u)b''(v), \gamma = a'^2(u)b'^2(v),\\ \theta = a(u)a''(u)\end{gathered} \tag{2.11}$$

Proof. Since

$$\begin{aligned} E &= 1 + a'^2(u),\\ F &= b'(v),\\ G &= b'^2(v) - a^2(u),\end{aligned}$$

$$\begin{aligned} EG - F^2 &= -a^2(u) + a'^2(u)b'^2(v) - a'^2(u)a^2(u)\\ &= -a^2(u)(a'^2(u) + 1) + a'^2(u)b'^2(v),\end{aligned}$$

$$\begin{aligned} L &= -\frac{a(u)a''(u)}{\sqrt{a'^2(u)b'^2(v) - a^2(u)(a'^2(u)+1)}},\\ M &= -\frac{a'^2(u)b'(v)}{\sqrt{a'^2(u)b'^2(v) - a^2(u)(a'^2(u)+1)}},\\ N &= \frac{a^2(u) + a(u)a'(u)b''(v)}{\sqrt{a'^2(u)b'^2(v) - a^2(u)(a'^2(u)+1)}},\end{aligned}$$

we have the mean curvature:

$$\begin{aligned} H \;=\; & \frac{1}{2}[(1 + a'^2(u))a(u)(a'(u)b''(v) - a(u)) + 2a'^2(u)b'^2(v) -\\ & -(b'^2(v) - a^2(u))a(u)a''(u)]/[(a'^2(u)b'^2(v) - a^2(u)(a'^2(u)+1))^{3/2}].\end{aligned}$$

For

$$\begin{gathered}\omega_+ = 1 + a'^2(u), \sigma_- = a^2(u) - b'^2(v),\\ \eta_- = a(u) - a'(u)b''(v), \gamma = a'^2(u)b'^2(v),\\ \theta = a(u)a''(u)\end{gathered}$$

the expresion of mean curvature become:

$$H = \frac{1}{2}\frac{2\gamma + \sigma_-\theta - \eta_-\omega_+ a(u)}{(\gamma - \omega_+ a^2(u))^{3/2}}.$$

□

Corollary 2.5. *If for a spacelike ${}^{1,3}H_3$ - helicoidal surface, $a(u)$ and $b(v)$, with $a'(u) \neq \pm 1$, satisfy the relation*

$$b''(v) = P(a)b'^2(v) + Q(a) \tag{2.12}$$

where

$$P(a) = \frac{a(u)a''(u) - 2a'^2(u)}{(1 + a'^2(u))a(u)a'(u)},$$
$$Q(a) = \frac{a^2(u)(1 + a'^2(u)) - a^3(u)a''(u)}{(1 + a'^2(u))a(u)a'(u)}$$

then, the surface is maximal.

Proof. From

$$b''(v) = \frac{a(u)a''(u) - 2a'^2(u)}{(1 + a'^2(u))a(u)a'(u)} b'^2(v) + \frac{a^2(u)(1 + a'^2(u)) - a^3(u)a''(u)}{(1 + a'^2(u))a(u)a'(u)}$$

it follows

$$b''(v)(1 + a'^2(u))a(u)a'(u) = b'^2(v)(a(u)a''(u) - $$
$$-2a'^2(u)) + a^2(u)(1 + a'^2(u)) - a^3(u)a''(u)$$

from where, $H = 0$. $\square$

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