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STATIONARY POINTS FOR MULTIFUNCTIONS ON THREE METRIC SPACES

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Abstract. In this paper we prove a general unique fixed point theorem for multifunctions on three metric spaces which generalize the main results from [3] and [4].

1. Introduction

Let (X, d) be a complete metric space and $B(X)$ be the set of all nonempty bounded subsets of X.

As in [1] we define the function $\delta(A, B)$ with A and B in $B(X)$by $\delta(A, B) = sup\{d(a, b) : a \in A, b \in B\}$.

If A is consists of a single point a we write $\delta(A, B) = \delta(a, B)$. If B also consists of a single point b, then $\delta(A, B) = d(a, b)$. It follows immediately that $\delta(A, B) = \delta(B, A) \geq 0$ and $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$ for all sets A, B, C in $B(X)$.

If $\delta(A, B) = 0$, then $A = B = \{a\}$.

Now if $\{A_n\}, n = 1, 2, ...$ is a sequence in $B(X)$, we say that it converges to the set A in $B(X)$ if:

(i) each point $a \in A$ is limit of some convergent sequence $\{a_n\}$, where $a_n \in A_n$, $n = 1, 2, ..$;

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(ii) for arbitrary $\varepsilon > 0$, there exists an integer N such that $A_n \subset A_\varepsilon$ for all $n > N$, where A_ε is the union of all open spheres with centers in A of radius ε.

The following Lemma was proved in [1].

Lemma. *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X, d), which converge to the bounded subsets A and B, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

If T is a multifunction of X into $B(X)$, a point $z \in X$ is called a *stationary point* of T if $Tz = \{z\}$.

Recently, Jain and Fisher [2] initiated the study of fixed points for multifunctions in three metric spaces. The present author [4] proved a general fixed point theorem for functions on three metric space satisfying implicit relations.

The following theorem is proved in [3].

Theorem 1 [3]. *Let (X, d_1),(Y, d_2) and (Z, d_3) be complete metric spaces and suppose F is a mapping of X into $B(Y)$, G is a mapping of Y into $B(Z)$ and H is a mapping of Z into $B(X)$ satisfying the following inequalities:*

$$\delta_1^2(HGy, HGFx) \leq c\,max \left\{ \begin{array}{c} d_1(x, HGy)d_2(y, Fx), \delta_2(y, Fx),\ \delta_1(x, HGFx), \\ \delta_1(x, HGFx)\delta_3(Gy, GFx), \delta_3(Gy, GFx)d_1(x, HGy)) \end{array} \right\},$$

$$\delta_2^2(FHz, FHGy) \leq c\,max \left\{ \begin{array}{c} d_2(y, FHz)\delta_3(z, Gy), \delta_3(z, Gy)\delta_2(y, FHGy), \\ \delta_2(y, FHGy)\delta_1(Hz, HGy), \delta_1(Hz, HGy)d_2(y, FHz) \end{array} \right\},$$

and

$$\delta_3^2(GFx, GFHz) \leq c\,max \left\{ \begin{array}{c} d_3(z, GFx)\delta_1(x, Hz), \delta_1(x, Hz)\delta_3(z, GFHz), \\ \delta_3(z, GFHz)\delta_2(Fx, FHz), \delta_2(Fx, FHz)d_3(z, GFx) \end{array} \right\}$$

for all x in X, y in Y and z in Z, where $0 \leq c < 1$.

If at least one of the mappings F, G, H is continuous, then HGF has a unique fixed point u in X, FHG has a unique fixed point v in Y and GFH has a unique fixed point w in Z. Further, $Fu = \{v\}, Gv = \{w\}$ and $Hw = \{u\}$.

In [4], it is denoted by $\mathcal{F}_5$ the set of all continuous functions $F : R_+^5 \to R$ such that there exists $h \in [0,1)$ having the following property: for every $u \geq 0, v \geq 0$ with

$$\text{(a) } F(u,v,0,u,w) \leq 0 \text{ or (b) } F(u,v,u,0,w) \leq 0,$$

we have $u \leq h\,max\{v,w\}$.

Example 1 [4]. $F(t_1,...,t_5) = t_1 - c\,max\{t_2,...,t_5\}$, where $c \in [0,1)$.

Example 2 [4]. $F(t_1,...,t_5) = t_1^2 - c\,max\{t_3t_2, t_2t_4, t_4t_5, t_5t_3\}$, where $c \in [0,1)$.

Example 3 [4]. $F(t_1,...,t_5) = t_1^3 + t_1^2 - (at_1t_2 + bt_1t_3 + ct_1t_4 + dt_5^2)$, where $0 \leq a+b+c+d < 1$.

The following theorem is proved in [4].

Theorem 2 [4]. *Let (X,d), (Y,ρ) and (Z,σ) be complete metric spaces. Assume that T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X, satisfying the inequalities*

$$F(d(RSy,RSTx),\rho(y,Tx),d(x,RSTx),d(x,RSy),\sigma(Sy,STx)) \leq 0,$$
$$F(\rho(TRz,TRSy),\sigma(z,Sy),\rho(y,TRSy),\rho(y,TRz),d(Rz,RSy)) \leq 0\ ,$$

and

$$F(\sigma(STx,STRz),d(x,Rz),\sigma(z,STRz),\sigma(z,STx),\rho(Tx,TRz)) \leq 0$$

for all x in X, y in Y, z in Z, where $F \in \mathcal{F}_5$.

If at least one of the mappings R,S,T is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$ and $Rw = u$.

In this paper we prove a generalization of Theorem 1 which extends Theorem 2 to multivalued mappings.

2. **Main result**

Theorem 3. *Let $(X,d_1),(Y,d_2)$ and (Z,d_3) be complete metric spaces and suppose that F is a mapping of X into $B(Y)$, G is a mapping of Y into $B(Z)$, and H is a mapping of Z into $B(X)$ satisfying the inequalities*

$$(1)\ \Phi(\delta_1(HGy,HGFx),\delta_2(y,Fx),d_1(x,HGy),\delta_1(x,HGFx),\delta_3(Gy,GFx)) \leq 0,$$

$$(2)\ \Phi(\delta_2(FHz,FHGy),\delta_3(z,Gy),d_2(y,FHz),\delta_2(y,FHGy),\delta_1(Hz,HGy)) \leq 0,$$

(3) $\Phi(\delta_3(GFx, GFHz), \delta_1(x, Hz), d_3(z, GFx), \delta_3(z, GFHz), \delta_2(Fx, FHz)) \leq 0,$

for all x in X, y in Y and z in Z where $\Phi \in \mathcal{F}_5$ and is nonincreasing in each of the variables $t_2, ..., t_5$.

If al least one of the mappings F, G, H is continuous, then HGF has a stationary point u in X, FHG has a stationary point v in Y and GFH has a stationary point w in Z. Further, $Fu = \{v\}$, $Gv = \{w\}$ and $Hw = \{u\}$. If in addition

(c) Φ is increasing in variable t_1,

then u is the unique fixed point of HGF, v is the unique fixed point of FHG and w is the unique fixed point of GFH.

Proof. Let $x = x_1$ be an arbitrary point in X. We define the sequences $\{x_n\}$ in X, $\{y_n\}$ in Y and $\{z_n\}$ in Z, inductively, as follows. Choose a point y_1 in Fx_1 and a point z_1 in Gy_1. In general, having choosen x_n in X, y_n in Y and z_n in M, choose $x_{n+1} \in Hz_n$, $y_{n+1} \in Fx_{n+1}$, $z_{n+1} \in Gy_{n+1}$ for $n = 1, 2, ...$

Applying the inequality (1) for $y = y_n$ and $x = x_{n+1}$ we have successively

$$\Phi\left(\begin{array}{c} \delta_1(HGy_n, HGFx_{n+1}), \delta_2(y_n, Fx_{n+1}), d_1(x_{n+1}, HGy_n), \\ \delta_1(x_{n+1}, HGFx_{n+1}), \delta_3(Gy_n, GFx_{n+1}) \end{array}\right) \leq 0,$$

$$\Phi\left(\begin{array}{c} \delta_1(HGy_n, HGFx_{n+1}), \delta_2(Fx_n, Fx_{n+1}), 0, \\ \delta_1(HGy_n, HGFx_{n+1}), \delta_3(GFx_n, GFx_{n+1}) \end{array}\right) \leq 0,$$

which implies by (a)

$$\delta_1(HGy_n, HGFx_{n+1}) \leq h\, max\{\delta_2(Fx_n, Fx_{n+1}), \delta_3(GFx_n, GFx_{n+1})\}.$$

Since $d(x_{n+1}, x_{n+2}) \leq \delta_1(HGy_n, HGy_{n+1}) \leq \delta_1(HGy_n, HGFx_{n+1})$, we obtain

(4) $d(x_{n+1}, x_{n+2}) \leq h\, max\{\delta_2(Fx_n, Fx_{n+1}), \delta_3(GFx_n, GFx_{n+1})\}$

But, from (2) we have successively for $z = z_{n-1}$ and $y = y_n$

$$\Phi\begin{pmatrix} \delta_2(FHz_{n-1}, FHGy_n), \delta_3(z_{n-1}, Gy_n), d_2(y_n, FHz_{n-1}), \\ \delta_2(y_n, FHGy_n), \delta_1(Hz_{n-1}, HGy_n) \end{pmatrix} \leq 0,$$

$$\Phi\begin{pmatrix} \delta_2(FHz_{n-1}, FHGy_n), \delta_3(Gy_{n-1}, Gy_n), 0, \\ \delta_2(FHz_{n-1}, FGHy_n), \delta_1(HGy_{n-1}, HGy_n) \end{pmatrix} \leq 0,$$

which implies by (a) that

$$\delta_2(FHz_{n-1}, FHGy_n) \leq h\, max\{\delta_3(Gy_{n-1}, Gy_n), \delta_2(HGy_{n-1}, HGy_n)\}.$$

Since $\delta_2(Fx_n, Fx_{n+1}) \leq \delta_2(FHz_{n-1}, FHGy_n)$ we have

$$(5)\ \delta_2(Fx_n, Fx_{n+1}) \leq h\, max\{\delta_3(Gy_{n-1}, Gy_n), \delta_1(HGy_{n-1}, HGy_n)\}.$$

Similarly, from (3) we have successively for $x = x_n$ and $z = z_n$

$$\Phi\begin{pmatrix} \delta_3(GFx_n, GFHz_n), \delta_1(x_n, Hz_n), d_3(z_n, GFx_n), \\ \delta_3(z_n, GFHz_n), \delta_2(Fx_n, FHz_n) \end{pmatrix} \leq 0,$$

$$\Phi\begin{pmatrix} \delta_3(GFx_n, GFHz_n), \delta_1(Hz_{n-1}, HGy_n), 0, \\ \delta_3(GFx_n, GFz_n), \delta_2(FHz_{n-1}, FHGy_n) \end{pmatrix} \leq 0,$$

which implies by (a) that

$$\delta_3(GFx_n, GFHz_n) \leq \\ h\, max\{\delta_1(Hz_{n-1}, HGy_n), \delta_2(FHz_{n-1}, FHGy_n)\}.$$

Since $\delta_3(GFx_n, GFx_{n+1}) \leq \delta_3(GFx_n, GFHz_n)$, we have

$$(6)\ \delta_3(GFx_n, GFx_{n+1}) \leq \\ h\, max\{\delta_1(Hz_{n-1}, HGy_n)\}, \delta_2(FHz_{n-1}, FHGy_n)\}.$$

On the other hand, by (2) for $z = z_{n-1}$ and $y = y_n$ we have successively

$$\Phi\begin{pmatrix} \delta_2(FHz_{n-1}, FHGy_n), \delta_3(z_{n-1}, Gy_n), d_2(y_n, FHz_{n-1}), \\ \delta_2(y_n, FGHy_n), \delta_1(Hz_{n-1}, HGy_n) \end{pmatrix} \leq 0,$$

$$\Phi\left(\begin{array}{c}\delta_2(FHz_{n-1}, FHGy_n), \delta_3(Gy_{n-1}, Gy_n), 0, \\ \delta_2(FHz_{n-1}, FHGy_n), \delta_1(HGy_{n-1}, HGy_n)\end{array}\right) \leq 0$$

which implies by (a) that

(7) $\delta_2(FHz_{n-1}, FGy_n) \leq h\, max\{\delta_3(Gy_{n-1}, Gy_n), \delta_1(HGy_{n-1}, HGy_n)\}$

Then by (4),(5),(6) and (7) it follows that

$$\begin{array}{rcl}(8)\ d_1(x_{n+1}, x_{n+2}) & \leq & \delta_1(HGy_n, HGy_{n+1}) \\ & \leq & h^2 max\{\delta_1(HGFx_{n-1}, GHFx_n), \delta_3(GFHz_{n-2}, GFHz_{n-1})\}.\end{array}$$

Similarly, we have

(9) $d_2(y_{n+1}, y_{n+2}) \leq h^2 max\{\delta_1(HGFx_{n-1}, HGFx_n), \delta_3(GFHz_{n-2}, GFHz_{n-1})\}$

and

(10) $d_3(z_{n+1}, z_{n+2}) \leq h^2 max\{\delta_1(HGFx_{n-1}, HGFx_n), \delta_3(GFHz_{n-2}, GFHz_{n-1})\}$.

It now follows easily by induction on using (8), (9) and (10) that

$$d_1(x_{n+1}, x_{n+2}) \leq$$
$$h^{2(n-2)} max\{\delta_1(HGFx_2, HGFx_3), \delta_3(GFHz_1, GFHz_2)\},$$
$$d_2(y_{n+1}, y_{n+2}) \leq$$
$$h^{2(n-2)} max\{\delta_1(HGFx_2, HGFx_3), \delta_3(GFHz_1, GFHz_2)\},$$
$$d_3(z_{n+1}, z_{n+2}) \leq$$
$$h^{2(n-2)} max\{\delta_1(HGFx_2, HGFx_3), \delta_3(GFHz_1, GFHz_2)\}.$$

Then for $r = 1, 2, ...$ and arbitrary $\varepsilon > 0$, we have by (8)

$$(11)\ d(x_{n+1}, x_{n+r+1}) \leq \delta_1(HGy_n, HGFx_{n+r})$$

$\leq \delta_1(HGy_n, HGy_{n+1}) + \delta_1(HGy_{n+1}, HGy_{n+2}) + ...\delta_1(HGy_{n+r-1}, HGFx_{n+r})$

$\leq (h^{2(n-2)} + h^{2(n-1)} + ... + h^{2(n+r-3)}) max\{\delta_1(HGFx_2, HGFx_3), \delta_3(GFHz_1, GFHz_2)\}$
$< \varepsilon$

for n greater that some N, since $0 \leq h < 1$. Therefore, $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d_1) and so has a limit u in X. Similarly, the sequences $\{y_n\}$ and $\{z_n\}$ are also Cauchy sequences with limits v and w in complete metric spaces (Y, d_2) and (Z, d_3), respectively. Further, the inequality (11) gives

$$\delta_1(u, HGFx_n) \leq d_1(u, x_{m+1}) + \delta_1(x_{m+1}, HGFx_n) \leq$$
$$d_1(u, x_{m+1}) + \delta_1(HGFx_m, HGFx_n) < d_1(u, x_{m+1}) + \varepsilon$$

for $m, n > N$. Letting m tend to infinity in the above inequality, it follows that $\delta_1(u, GFx_n) \leq \varepsilon$ for $n > N$ and therefore

$$(12) \lim_{n\to\infty} HGFx_n = \{u\} = \lim_{n\to\infty} HGy_n,$$

since $\varepsilon > 0$ is arbitrary. Similarly,

$$(13) \lim_{n\to\infty} FHGy_n = \{v\} = \lim_{n\to\infty} FHz_n$$

and

$$(14) \lim_{n\to\infty} GFHz_n = \{w\} = \lim_{n\to\infty} GFx_n.$$

Suppose that F is continuous . Then $\lim_{n\to\infty} Fx_n = Fu = \lim_{n\to\infty} y_n = v$. Hence, $Fu = \{v\}$.

By (3) for $x = u$ and $z = z_n$ we have successively

$$\Phi\left(\begin{array}{c} \delta_3(GFu, GFHz_n), \delta_1(u, Hz_n), d_3(z_n, GFu), \\ \delta_3(z_n, GFHz_n), \delta_2(Fu, FHz_n) \end{array}\right) \leq 0,$$

$$\Phi\left(\begin{array}{c} \delta_3(Gv, GFHz_n), \delta_1(u, HGy_n), \delta_3(z_n, Gv), \\ \delta_3(z_n, GFHz_n), \delta_2(v, FHz_n) \end{array}\right) \leq 0.$$

Letting n tend to infinity we obtain $\Phi(\delta_3(Gv, w), 0, \delta_3(Gv, w), 0, 0) \leq 0$.

By (b) it follows that $\delta_3(Gv, w) = 0$, which implies $Gv = \{w\}$.

By (1) for $y = v$ and $x = x_n$ we have successively

$$\Phi\left(\begin{array}{c} \delta_1(HGv, HGFx_n), \delta_2(v, Fx_n), d_1(x_n, HGv), \\ \delta_1(x_n, HGFx_n), \delta_3(Gv, GFx_n) \end{array}\right) \leq 0,$$

$$\Phi\left(\begin{array}{c} \delta_1(Hw, HGFx_n), \delta_2(v, Fx_n), \delta_1(x_n, Hw), \\ \delta_1(x_n, HGFx_n), \delta_3(Gv, GFx_n) \end{array}\right) \leq 0.$$

Letting n tend to infinity we have $\Phi(\delta_1(Hw, u), 0, \delta_1(u, Hw), 0, 0) \leq 0$, which implies by (b) that $\{u\} = Hw$.

By (1) for $y = y_n$ and $x = u$ we have successively

$$\Phi\left(\begin{array}{c} \delta_1(HGy_n, HGFu), \delta_2(y_n, Fu), \delta_1(u, HGy_n), \\ \delta_1(u, HGFu), \delta_3(Gy_n, GFu) \end{array}\right) \leq 0,$$

$$\Phi\left(\begin{array}{c}\delta_1(HGy_n,HGFu),\delta_2(y_n,v),\delta_1(u,HGy_n),\\ \delta_1(u,HGFu),\delta_3(GFx_n,Gv)\end{array}\right)\leq 0$$

Letting n tend to infinity we have

$$\Phi(\delta_1(u,HGFu),0,0,\delta_1(u,HGFu),0)\leq 0,$$

which implies by (a) that $HGFu=\{u\}$.
Similarly, by (2) for $z=z_n$ and $y=v$ we obtain $\{v\}=FHGv$.
By (3) for $x=x_n$ and $z=w$ we obtain that $GFHw=\{w\}$.

To prove uniqueness,we will suppose that HGF has a second fixed point u' so that $u'\in HGFu'$. Then there exist $v'\in Fu$ and $u'\in Hw'$. Using inequality (2), we have successively

$$\Phi\left(\begin{array}{c}\delta_2(FHw',FHGv'),\delta_3(w',Gv'),d_2(v',FHw'),\\ \delta_2(v',FHGv'),\delta_1(Hw',HGv')\end{array}\right)\leq 0,$$

$$\Phi\left(\begin{array}{c}\delta_2(FHw',FHGv'),\delta_3(GFu',GFu'),0,\\ \delta_2(FHw',FHGv'),\delta_1(HGv',HGv')\end{array}\right)\leq 0,$$

which implies by (a) that

$$\delta_2(FHw',FHGv')\leq h\,max\{\delta_3(GFu',GFu'),\delta_1(HGv',HGv')\}.$$

Since $\delta_2(FHw',FHw')\leq\delta_3(FHw',FHGv')$ it follows that

(15) $\delta_2(FHw',FHw')\leq h\,max\{\delta_3(GFu',GFu'),\delta_1(HGv',HGv')\}$.

Similarly, applying (3), we get

(16) $\delta_3(GFu',GFu')\leq h\,max\{\delta_1(HGv',HGv'),\delta_2(FHw',FHw')\}$

and using (1), we have

(17) $\delta_1(HGv',HGv')\leq h\,max\{\delta_2(FHw',FHw'),\delta_3(GFu',GFu')\}$.

From (15) and (16) we obtain

(18) $\delta_2(FHw',FHw')\leq h\,\delta_1(HGv',HGv')$.

Similarly, (16) and (17) give

(19) $\delta_3(GFu',GFu')\leq h\,\delta_2(FHw',FHw')$

and (15) and (17) yield

$$(20)\ \delta_1(HGv', HGv') \leq h\,\delta_3(GFu', GFu').$$

Now, it follows from (20), (19) and (18) that

$$\delta_1(HGv', HGv') \leq h\,\delta_3(GFu', GFu') \leq$$
$$h^2\delta_2(FHw', FHw')\delta_1(HGv', HGv'),$$

which implies that $\delta_1(HGv', HGv') = 0$.

Similarly, $\delta_2(FHw', FHw') = 0$ and $\delta_3(GFu', GFu') = 0$.

Since $\delta_1(u', Hw') \leq \delta_1(HGv', HGv')$, $\delta_2(v', Fu') \leq \delta_2(FHw'FHw')$ and $\delta_3(w', Gv') \leq \delta_3(GFu', GFu')$, it follows that $Fu' = \{v'\}, Gv' = \{w'\}$ and $Hw' = \{u'\}$. Further, $HGFu' = HGv' = Hw' = \{u'\}$, $FHGv' = FHw' = Fu' = \{v'\}$, $GFHw' = GFu' = Gv' = \{w'\}$.

Using (1) and (c) we have successively

$$\Phi\begin{pmatrix} \delta_1(HGv, HGFu'), \delta_2(v, Fu'), d_1(u', HGv), \\ \delta_1(u', HGFu'), \delta_3(Gv, GFu') \end{pmatrix} \leq 0,$$

$$\Phi(d_1(u, u'), d_2(v, v'), d_1(u, u'), 0, d_3(w, w') \leq 0,$$

which implies by (b) that

$$(21)\ d_1(u, u') \leq h\,max\{d_2(v, v'), d_3(w, w')\}.$$

Similarly, applying (2), we have

$$(22)\ d_2(v, v') \leq h\,max\{d_3(w, w'), d_1(u, u')\}$$

and using (3), we obtain

$$(23)\ d_3(w, w') \leq h\,max\{d_1(u, u'), d_2(v, v')\}.$$

Now (21) and (22) yield

$$(24)\ d_1(u, u') \leq h\,d_3(w, w').$$

Similarly, (22) and (23) imply

$$(25)\ d_2(v, v') \leq h\,d_1(u, u'),$$

while (21) and (23) imply

$$(26)\ d_3(w, w') \leq h\,d_2(v, v').$$

From (24), (25) and (26) it follows that

$$d_1(u, u') \leq h^3 \, d_1(u, u'),$$

hence $d_1(u, u') = 0$, i.e. $u = u'$.
Similarly $d_2(v, v') = 0$, i.e. $v = v'$ and $d_3(w, w') = 0$, i.e. $w = w'$. This completes the proof of Theorem 3.

References

[1] B.Fisher, **Common fixed point of mappings and set valued mappings**, Rostok Math.Kolloq, 8(1981), 68-77.

[2] S.Jain and B.Fisher, **A related fixed point theorem for three metric spaces**, Hacettepe J.Math. Stat. 31(2002), 19-24.

[3] R.K.Jain and S.Jain, **A related fixed point theorem for set valued mappings on three metric spaces**, to appear.

[4] V.Popa, **Two general fixed point theorems on three complete metric spaces**, Novi Sad J.Math., 30(2000),43-50.

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