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THE GENERALIZED ČEBYŠEV TYPE INEQUALITY

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Abstract. A generalization of Pečarić's extension to the Montgomery's identity has been derived. The generalization is applicable for any weight functions. The generalized Čebyšev type inequality has also been obtained.

1. Introduction

The Čebyšev type inequality is given by [2] as:

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty \tag{1}$$

where the functions $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous functions with bounded first derivatives.

The function $T(f,g)$ is defined as:

(2)

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$$

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and $\|.\|_{\infty}$ is the norm in $L_{\infty}[a,b]$ defined as $\|p\|_{\infty} = ess\sup_{t\in[a,b]} |p(t)|$.

In [3], Grüss established the following inequality

$$|T(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{3}$$

where m, M, n and N are real numbers satisfying the conditions,

$$-\infty < m \leq f(x) \leq M < \infty,\ -\infty < n \leq g(x) \leq N < \infty,\ \text{ for all } x \in [a,b].$$

Some new Čebyšev type inequalities has been derived by Pachpatte using the weighted Montgomery identity given by Pečarić's [8] in [[6], [7]].

In this paper we have generalized the Pečarić's work for Montgomery's identity. This generalization has been derived by using the weight function that need not be a probability density function and can be useful in deriving the Čebyšev type inequalities for any absolutely continuous function.

2. Main results

Let $f : [a,b] \to \mathbb{R}$ be absolutely continuos function on $[a,b]$, then from [4] the Montgomery type identity holds:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \int_a^b P(x,t) f'(t)\,dt, \tag{4}$$

where $P(x,t)$ is the Peano kernel defined by:

$$P(x,t) = \begin{cases} \dfrac{t-a}{b-a}, a \leq t \leq x \\ \dfrac{t-b}{b-a}, x \leq t \leq b. \end{cases}$$

The weighted version of the identity (4) given by Pečarić in [8] is given as:

$$f(x) = \int_a^b r(t) f(t)\,dt + \int_a^b P_w(x,t) f'(t)\,dt \tag{5}$$

where $r(t)$ is some probability density function and the weighted Peano kernel is defined as:

$$P_w(x,t) = \begin{cases} R(t), a \leq t \leq x \\ R(t) - 1, x \leq t \leq b, \end{cases}$$

provided $R(t) = \int_a^t r(s)\,ds$ is the cumulative distribution function of $r(t)$. We now give a further generalization to (5) by using a weight function
$w : [a,b] \to [0,+\infty)$ which is not necessarily the probability density function. We define $m(a,b) = \int_a^b w(s)\,ds$ as total area of w and $m(a,x) = \int_a^x w(s)\,ds$, so that $m(a,x) = 0$ for $x < a$. Now using these notations, we define the generalized weighted Peano kernel as:

$$P_{w,\varphi}(x,t) = \begin{cases} \varphi(m(a,t)), a \le t \le x \\ \varphi(m(a,t)) - \varphi(m(a,b)), x \le t \le b \end{cases} \tag{6}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function on $\mathbb{R}_+$ with $\varphi(0) = 0$, $\varphi(m(a,b)) \neq 0$ and φ' is integrable on $\mathbb{R}_+$, and also a generalization of (2) as follows:

$$(7)(w,f,g,\varphi') = \frac{1}{\varphi(m(a,b))}\int_a^b w(x)\varphi'(m(a,x)) f(x) g(x)\,dx - \left(\frac{1}{\varphi(m(a,b))}\int_a^b w(x)\varphi'(m(a,x)) f(x)\,dx\right) \times \left(\frac{1}{\varphi(m(a,b))}\int_a^b w(x)\varphi'(m(a,x)) g(x)\,dx\right).$$

We have given the generalized weighted Montgomery's identity in the following theorem.

Theorem 1.. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous, then*

$$f(x) = \frac{1}{\varphi(m(a,b))}\int_a^b w(t)\varphi'(m(a,t)) f(t)\,dt + \frac{1}{\varphi(m(a,b))}\int_a^b P_{w,\varphi}(x,t) f'(t)\,dt, \tag{8}$$

for all $x \in [a,b]$, where $P_{w,\varphi}(x,t)$ is defined in (6).

Proof. Consider the kernel defined in (6). Using the hypothesis of φ and (6), we have:

$$\begin{aligned}\int_a^b P_{w,\varphi}(x,t) f'(t)\,dt &= \int_a^x \varphi(m(a,t)) f'(t)\,dt \\ &+ \int_x^b (\varphi(m(a,t)) - \varphi(m(a,b))) f'(t)\,dt \\ &= \int_a^b \varphi(m(a,t)) f'(t)\,dt - \varphi(m(a,b)) \int_x^b f'(t)\,dt.\end{aligned}$$

Integrating by parts and simplifying we obtain:

$$\int_a^b P_{w,\varphi}(x,t) f'(t)\,dt = \varphi(m(a,b)) f(x) - \int_a^b w(t)\varphi'(m(a,t)) f(t)\,dt.$$

We now give an generalization of the Čebyšev inequality in the following theorem:

Theorem 2.. *Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions on $[a,b]$. Also let the functions w and φ satisfy the conditions given in Theorem 1. Suppose*

$$f', g', \varphi' \in L_\infty[a,b],$$

and

$$\int_a^b w(x)\varphi'(m(a,x))\,dx = \varphi(m(a,b)) \text{ for all } x \in [a,b], \tag{9}$$

then

(10)

$$|T(w,f,g,\varphi')| \le \frac{1}{\varphi^3(m(a,b))} \|f'\|_\infty \|g'\|_\infty \|\varphi'\|_\infty \int_a^b w(x) H^2(x)dx,$$

for all $x \in [a,b]$, where $H(x) = \int_a^b |P_{w,\varphi}(x,t)|\,dt.$

Proof. Since the functions f and g are absolutely continuous, we have:

$$\begin{aligned}&f(x) - \frac{1}{\varphi(m(a,b))}\int_a^b w(t)\varphi'(m(a,t)) f(t)\,dt \\ &= \frac{1}{\varphi(m(a,b))}\int_a^b P_{w,\varphi}(x,t) f'(t)\,dt,\end{aligned} \tag{11}$$

and

$$g(x) - \frac{1}{\varphi(m(a,b))}\int_a^b w(t)\,\varphi'(m(a,t))\,g(t)\,dt = \frac{1}{\varphi(m(a,b))}\int_a^b P_{w,\varphi}(x,t)\,g'(t)\,dt. \tag{12}$$

Using (11) and (12) we have:

$$\begin{aligned}
&\frac{1}{\varphi^2(m(a,b))}\left(\int_a^b P_{w,\varphi}(x,t)\,f'(t)\,dt\right)\left(\int_a^b P_{w,\varphi}(x,t)\,g'(t)\,dt\right) \\
&= \left(f(x) - \frac{1}{\varphi(m(a,b))}\int_a^b w(t)\,\varphi'(m(a,t))\,f(t)\,dt\right)\times \\
&\left(g(x) - \frac{1}{\varphi(m(a,b))}\int_a^b w(t)\,\varphi'(m(a,t))\,g(t)\,dt\right) \\
&= f(x)\,g(x) - \frac{f(x)}{\varphi(m(a,b))}\int_a^b w(t)\,\varphi'(m(a,t))\,g(t)\,dt \\
&-\frac{g(x)}{\varphi(m(a,b))}\int_a^b w(t)\,\varphi'(m(a,t))\,f(t)\,dt \\
&+\frac{1}{\varphi^2(m(a,b))}\left(\int_a^b w(t)\varphi'(m(a,t))f(t)dt\right)\left(\int_a^b w(t)\varphi'(m(a,t))\,g(t)\,dt\right)
\end{aligned}$$

Simplifying by using the condition (9) and (7), we get

$$\begin{aligned}
T(w,f,g,\varphi') \;&=\; \frac{1}{\varphi^3(m(a,b))}\int_a^b w(x)\,\varphi'(m(a,t))\times \\
&\left(\int_a^b P_{w,\varphi}(x,t)\,f'(t)\,dt\right)\left(\int_a^b P_{w,\varphi}(x,t)\,g'(t)\,dt\right)dx
\end{aligned}$$

which implies that

$$|T(w,f,g,\varphi')| \le \frac{1}{\varphi^3(m(a,b))}\|f'\|_\infty\,\|g'\|_\infty\,\|\varphi'\|_\infty\int_a^b w(x)\,H^2(x)\,dx.$$

Remark 1.. *If in (8) and (10), w is the probability density function, then we recapture the corrected version of the results obtained in* [1].

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