"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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COMMON FIXED POINT THEOREMS FOR PROBABILISTIC Φ- CONTRACTION MAPPINGS

R. A. RASHWAN

Abstract. In this paper, we establish some common fixed point theorems for probabilistic Φ- contraction mappings on Menger spaces. our results improve some known results.

1. Introduction

K. Menger [6] introduced the notion of probabilistic metric spaces(or statistical metric spaces), which is a generalization of metric spaces, the study of these spaces was performed extensively, by B. Schweizer and A. Sklar [8]and [9]. Especially, the theory of probabilistic metric spaces is of fundamental importance in probabilistic functional analysis. S. M. Mishra [7] obtained a fixed point theorem for two pairs of compatible mappings on a probabilistic metric spaces. Also, Y. J. Cho, p. p. Murthy and M. Stojakovic [1] obtain a fixed point theorem for pairs of compatible of type (A) on the such space. On the other hand, R. Dedeic and N. Sarapa [2] proved some theorems on common fixed points for a sequence of mappings on complete Menger spaces, while S. L. Singh and B. D. Pant [10] established a fixed point theorem for a family of mappings in Menger spaces.

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In this paper, some common fixed point theorems are proved for a class of Φ- contraction mappings on Menger spaces which improve some resulte in [1]. [2], [7] and [10].

2. Preliminaries

Let R denote the set of reals, R^+ the nonnegative reals and N denote the set of all natural numbers. A mappings $F : R^+ \to R^+$ is called a distribution function if it is nondecreasing and left continuous with $\inf F = 0$ and $\sup F = 1$. We will denote $\triangle$ by the set of all distribution functions.
A probabilistic metric space (briefly, PM-space) is a pair (X, ξ) where X is a nonempty set and ξ is a mapping from $X \times X$ to $\triangle$. For $(u, v) \in X \times X$, the distribution function $F(u, v)$ is denoted by $F_{u,v}$. The function $F_{u,v}$. are assumed to satisfy the following conditions:
(P1) $F_{u,v}(x) = 1$ for every $x > 0$ iff $u = v$,
(P2) $F_{u,v}(0) = 0$ for every $u, v \in X$,
(P3) $F_{u,v}(x) = F_{v,u}(x)$ for every $u, v \in X$,
(P4) if $F_{u,w}(x) = 1$ and $F_{w,v}(y) = 1$ then $F_{u,v}(x + y) = 1$ for every $u, v, w \in X$.
In a metric space (X, d) the metric d induces a mapping $F : X \times X \to \triangle$ such that

$$F(u, v)(x) = F_{u,v}(x) = H(x - d(u, v)),$$

for every $u, v \in X$ and $x \in R$, where H is a specific distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0, \\ \\ 1, & x > 0. \end{cases}$$

The following definitions and lemmas are needed in the sequel.

Definition 2.1. *([9]) A T-norm is a function $t : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies:*
(T1) $t(a, 1) = a$ *and* $t(0, 0) = 0$,
(T2) $t(a, b) = t(b, a)$,
(T3) $t(c, d) \geq t(a, b), \quad c \geq a, d \geq b$,
(T4) $t(t(a, b), c) = t(a, t(b, c))$.

Definition 2.2. *([8]) A Menger space is an order triple (X, ξ, t) where (X, ξ) is a probabilistic metric space and t is T-norm satisfying:*
$(P4)'$ $F_{u,v}(x + y) \geq t(F_{u,w}(x), F_{w,v}(y))$ for all $u, v, w \in X$ and $x, y \geq 0$.
As Schweizer and Sklar [8]pointed out, if $T-$norm t of Menger space (X, ξ, t) is continuous, then there exists a topology τ on X such that X, τ is a Hausdorff topological space in the τ topology induced by the family of neighbourhoods $\{U_x(\epsilon, \lambda) : x \in X, \epsilon > 0, \lambda > 0\}$ where

$$U_x(\epsilon, \lambda) = \{y \in X; F_{x,y}(\epsilon) > 1 - \lambda\}.$$

Definition 2.3. [8] *A sequence $\{x_n\}$ in a Meneger space X is said to be convergent to a point $x \in X$ if for every $\epsilon > 0$ and $\lambda > 0$, there is an integer $N(\epsilon, \lambda)$ such that*

$$F_{x_n,x}(\epsilon) > 1 - \lambda \text{ for all } n \geq N(\epsilon, \lambda)$$

The sequence $\{x_n\}$ is called a Cauchy sequence if for each $\epsilon > 0$ and $\lambda > 0$,there is an integer $N(\epsilon, \lambda)$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \geq N(\epsilon, \lambda)$.

For complete topological preliminaries on Menger spaces see, for example [].

Definition 2.4. *([4])A T-norm t is said to be an h-type T-norm, if the family $\{t^m(u)\}_{m=1}^{\infty}$ is equicontinuous at $u = 1$, where*

$$t^1 = t(u, u)$$

$$t^m(u) = t(u, t^{m-1}(u)), m = 1, 2, ..., u \in [0, 1].$$

$t(a, b) = \min\{a, b\}$ is an h-type T-norm which is the unique T-norm such that

$$t(a, a) \geq a, \;\; \forall a \in [0, 1].$$

Another example of an h-type T-norm was given in [4].
The following two basic lemmas are due to Fang [3]

Lemma 2.1. *Let the function $\phi(u)$ satisfy the following condition:*
(ϕ) $\phi(u) : R^+ \to R^+$ is nondecreasing and $\sum_{n=0}^{\infty} \phi^n(u) < +\infty$ for all $u > 0$,
where $\phi^n(u)$ denote the n-th iterative function of $\phi(u)$. Then $\phi(u) < u$.

Lemma 2.2. *Let (X, ξ, t)be a Menger space with an h-type T-norm t. Suppose that $\{x_n\} \subset X$ such that*

$$F_{x_n,x_{n+1}}(\phi^n(u)) \geq F_{x_0,x_1}(u) \quad \textit{for} \quad u > 0,$$

where the function $\phi(u)$ satisfies the condition (ϕ). Then $\{x_n\}$ is a Cauchy sequence.

Recently, G. Jungck [5] proposed a generalisation of concepts of commuting and weakly commuting mappings in metric space, which is called compatible mappings.
S. N. Mishra [7] introduced this notion in a Menger spaces as follows.

Definition 2.5. *Two self mappings S and T of Menger space (X, ξ, t), where t is continuous will be called compatible if and only $F_{STx_n,TSx_n}(u) \to 1$ for all $u > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to z$ for some $z \in X$.*

The following two lemmas are the analogies of proposition 2.2(2(a)) an (1) of G. Jungck [5].

Lemma 2.3. *If S and T are compatible self mappings of Meneger space (X, ξ, t), where t is continuous and $t(u, u) \geq u$ for all $u \in [0, 1]$ and $Sx_n, Tx_n \to z$ for some z in X ($\{x_n\}$ being a sequence in X), then $TSx_n \to Sx$ provided S is continuous.*

Lemma 2.4. *If S and T are compatible self mappings of Meneger space (X, ξ, t), where t is continuous. Then if $Sz = Tz$ for some z in X, then $STz = TSz$.*

The following definition and lemmas are due to Cho et al. [1]

Definition 2.6. *Let (X, ξ, t)be a Menger space such that T-norm t is continuous and S, T be mappings from X into itself. S and T are said to be compatible of type (A) if*

$$\lim_{n\to\infty} F_{TSx_n,SSx_n}(u) = 1 \quad \textit{and} \quad \lim_{n\to\infty} F_{STx_n,TTx_n}(u) = 1, \quad \textit{for} \quad u > 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Lemma 2.5. *Let (X, ξ, t)be a Menger space such that T-norm t is continuous and*
$t(u, u) \geq u$ for all $u \in [0, 1]$, and let $S, T : X \to X$ be mappings. If S and T are compatible mappings of type (A) and $Sz = Tz$ for some $z \in X$, then $STz = TTz = TSz = SSz$.

Lemma 2.6. *Let (X, ξ, t) be a Menger space such that T-norm t is continuous and*
$t(u, u) \geq u$ for all $u \in [0, 1]$, and $S, T : X \to X$ be mappings. Let S and T be compatible mappings of type (A) and $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Then we have

(1) $\lim_{n\to\infty} TSx_n = Sz$ *if S is continuous at z.*

(2) $STz = TSz$ *and $Sz = Tz$ if S and T are continuous at z.*

3. Common fixed points of compatible mappings

The following lemma is basic in proving of our first main result.

Lemma 3.1. *Let A, B, S and T be self mapping of the Meneger space (X, ξ, t), where t is continuous and $t(u, u) \geq u$ for all $u \in [0, 1]$ such that $A(X) \subset T(X)$ and $B(X) \subset S(X)$ and let $x_0 \in X$. If the condition*

$$\begin{aligned}(F_{Ap,Bq}(\phi(u)) \geq t(F_{Ap,Sp}(u), t(F_{Bq,Tq}(u), t(F_{Sp,Tq}(u),\\ t(F_{Ap,Tq}(\alpha u), F_{Bq,Sp}(2u - \alpha u))))),\end{aligned} \tag{1}$$

is satisfied for $p, q \in X$ and $u > 0$ and $\alpha \in (0, 2)$, where $\phi \in \Phi$, then there is a Cauchy sequence $\{y_n\}$ in X starting at x_0 and defined by

$$\begin{cases} y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}, \\ y_{2n} = Sx_{2n} = Bx_{2n-1}, \qquad n \in N. \end{cases} \tag{2}$$

Proof. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, we may choose x_1 and x_2 in X such that
$Ax_0 = Tx_1 = y_1$ and $Bx_1 = Sx_2 = y_2$. Inductively, one can define a sequence $\{y_n\}$ for which

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2},$$

$$y_{2n} = Sx_{2n} = Bx_{2n-1}.$$

By using (1) and (2) and properties of the T-norm t, for $k_1 \in (0,1)$ we have:

$$\begin{aligned}
F_{y_{2n+1},y_{2n+2}}(\phi(u)) &= F_{Ax_{2n},Bx_{2n+1}}(\phi(u)) \\
&\geq t(F_{Ax_{2n},Sx_{2n}}(u)t(F_{Bx_{2n+1},Tx_{2n+1}}(u), t(F_{Ax_{2n},Tx_{2n+1}}((1-k_1)u), \\
&\quad F_{Bx_{2n+1},Sx_{2n}}((1+k_1)u)))), \\
&= t(F_{y_{2n+1},y_{2n}}(u), t(F_{y_{2n+2},y_{2n+1}}(u), t(F_{y_{2n},y_{2n+1}}(u), \\
&\quad t(F_{y_{2n+1},y_{2n+1}}((1-k_1)u), F_{y_{2n+2},y_{2n}}((1+k_1)u)) \\
&\geq t(F_{y_{2n},y_{2n+1}}(u), t(F_{y_{2n+1},y_{2n+2}}(u), t(F_{y_{2n},y_{2n+1}}(u), \\
&\quad t(F_{y_{2n+1},y_{2n+1}}(u), t(F_{y_{2n+1},y_{2n+2}}((k_1)u)))) \\
&\geq t(F_{y_{2n},y_{2n+1}}(u)), t(F_{y_{2n+1},y_{2n+2}}(u)), F_{y_{2n+1},y_{2n+2}}(k_1 u))) \\
&\geq t(F_{y_{2n},y_{2n+1}}(u)), F_{y_{2n+1},y_{2n+2}}(k_1 u)).
\end{aligned}$$

Since t is continuous and the distribution function is left-continuous, making $k_1 \to 1$, we have

$$F_{y_{2n+1},y_{2n+2}}(\phi(u)) \geq t(F_{y_{2n},y_{2n+1}}(u), F_{y_{2n+1},y_{2n+2}}(u)).$$

Similarly,

$$F_{y_{2n+2},y_{2n+3}}(\phi(u)) \geq t(F_{y_{2n+1},y_{2n+2}}(u), F_{y_{2n+2},y_{2n+3}}(u)).$$

Therefore

$$F_{y_n,y_{n+1}}(\phi(u)) \geq t(F_{y_{n-1},y_n}(u), F_{y_n,y_{n+1}}(u)).$$

If $F_{y_{n-1},y_n}(u) > F_{y_n,y_{n+1}}(u)$, then $F_{y_n,y_{n-1}}(\phi(u)) \geq F_{y_{n-1},y_n}(u)$ which is contradiction, since $\phi(u) < u$. Then we have

$$F_{y_n,y_{n+1}}(\phi(u)) \geq F_{y_{n-1},y_n}(u).$$

Hence, for any $n \in N$ and all $u > 0$, we have

$$F_{y_n,y_{n+1}}(\phi^n(u)) \geq F_{y_0,y_1}(u).$$

By Lemma 2.2, it follows that $\{y_n\}$ is a Cauchy sequence.

Theorem 3.1. *Let A, B, S and T be self mappings of a complete Meneger space (X, ξ, t), where t is continuous and $t(u,u) \geq u$ for all $u \in [0,1]$, suppose that S and T are continuous, the pairs $\{A, S\}$ and $\{B, T\}$ are compatible and $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If there exists a function ϕ satisfying the condition Φ such that for all $u > 0$, condition (1) is satisfied, then A, B, S and T have a unique common fixed point in X.*

Proof. From Lemma 3.1, there is a sequence $\{x_n\}$ in X such that

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2},$$
$$y_{2n} = Sx_{2n} = Bx_{2n-1} \quad n \in N,$$

and that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, then the subsequences
$\{Ax_{2n}\}, \{Sx_{2n}\}, \{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ all converge to a point z in X.
Continuity of S and T implies that $SSx_{2n} \to Sz$ and $TTx_{2n-1} \to Tz$ with the compatibility of $\{A, S\}$ and $\{B, T\}$ and Lemma 2.3 give $ASx_{2n} \to Sz$ and $BTx_{2n-1} \to Tz$. Put $p = Sx_{2n}$ and $q = Tx_{2n-1}$ in (1), we have

$$F_{ASx_{2n},BTx_{2n-1}}(\phi(u)) \geq t(F_{ASx_{2n},SSx_{2n}}(u), t(F_{BTx_{2n-1},TTx_{2n-1}}(u),$$
$$t(F_{SSx_{2n},TTx_{2n-1}}(u), \quad t(F_{ASx_{2n},TTx_{2n-1}}(\alpha u), F_{BTx_{2n-1}}, SSx_{2n}(2u - \alpha u))))).$$

Taking $n \to \infty$ and $\alpha \to 1$, we have

$$\begin{aligned} F_{Sz,Tz}(\phi(u)) &\geq t(F_{Sz,Sz}(u), t(F_{Tz,Tz}(u), t(F_{Sz,Tz}(u), \\ &\quad t(F_{Sz,Tz}(u), F_{Tz,Sz}(u))))). \\ &\geq F_{Sz,Tz}(u), \end{aligned}$$

which means that $Sz = Tz$.
By using condition (1) again, we have

$$\begin{aligned} F_{Az,BTx_{2n-1}}(\phi(u)) &\geq t(F_{Az,Sz}(u), t(F_{BTx_{2n-1},TTx_{2n-1}}(u), t(F_{Sz,TTx_{2n-1}}(u), \\ &\quad t(F_{Az,TTx_{2n-1}}(u), F_{BTx_{2n-1}}, Sz(u))))). \end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned} F_{Sz,Bz}(\phi(u)) &\geq t(F_{Az,Sz}(u), t(F_{Bz,Tz}(u), t(F_{Sz,Tz}(u), \\ &\quad t(F_{Az,Tz}(u), F_{Bz}, Tz(u))))), \end{aligned}$$

which implies that $Az = Bz$. Therefore $Az = Bz = Sz = Tz$.
We will prove that z is a common fixed point of A, B, S and T. Putting $p = x_{2n}$ and $q = z$ in (1), one gets

$$\begin{aligned} F_{Ax_{2n},Bz}(\phi(u)) &\geq t(F_{Ax_{2n},Sx_{2n}}(u), t(F_{Bz,Tz}(u), t(F_{Sx_{2n},Tz}(u), \\ &\quad t(F_{Ax_{2n},Tz(u)}, F_{Bz}, Sx_{2n}(u))))). \end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned} F_{z,Bz}(\phi(u)) &\geq t(F_{z,z}(u), t(F_{Bz,Bz}(u), t(F_{z,Bz}(u), F_{z,Bz}(u), F_{Bz,z}(u))))), \\ (3) \qquad &\geq F_{z,Bz}(u). \end{aligned}$$

Hence, $z = Bz$ and z is a common fixed point of A, B, S and T. For uniqueness, let z' be another common fixed point such that, $z' \neq z$,

$$\begin{aligned} F_{z,z'}(\phi(u)) &= F_{Az,Bz'}(\phi(u)) \\ &\geq t(F_{z,z}(u), t(F_{z',z'}(u), t(F_{z,z'}(u), F_{z,z'}(u), F_{z',z}(u))))), \\ &\geq F_{z,z'}(u). \end{aligned}$$

Thus $z = z'$ and z is a unique common fixed point of A, B, S and T. Taking $\phi(u) = ku,\ (0 < k < 1)$ in Theorem (1) we obtain the following corollary.

Corollary 3.1. [7] *Let A, B, S and T be self mappings of a complete Menger space (X, ξ, t), where t is continuous and $t(u, u) \geq u$ for all $u \in [0, 1]$. Suppose that S and T are continuous, the pairs $\{A, S\}$ and $\{B, T\}$ are compatible and that $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If there exists a constant $k \in (0, 1)$ such that for all $p, q \in X, u > 0$ and $\alpha \in (0, 2)$, we have*

$$\begin{aligned} F_{Ap,Bq}(ku) &\geq t(F_{Ap,Sp}(u), t(F_{Bq,TQ}(u), t(F_{Sp,Tq}(u), \\ &\qquad t(F_{Ap,Tq}(\alpha u), F_{Bq,Sp}(2u - \alpha u))))), \end{aligned}$$

then A, B, S and T have a unique common fixed point in X.

4. Common fixed points of compatible mappings of type (A)

Now, we prove our second main result.

Theorem 4.1. *Let (X, ξ, t) be a complete Menger space with $t(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$ and A, B, S, T be mappings from X into itself such that*

(4.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,
(4.2) the pair $\{A, S\}$ and $\{B, T\}$ are compatible of type (A),
(4.3) one of A, B, S and T is continuous,
(4.4) there exists a function $\phi \in \Phi$ such that

$$(F_{Ap,Bq}(\phi(u)))^2 \geq \min\{(F_{Sp,Tq}(u))^2, F_{Sp,Ap}(u)F_{Tq,Bq}(u), F_{Sp,Bq}(2u)$$
$$F_{Tq,Ap}(u), F_{Sp,Aq}(u)F_{Tq,Ap}(u), F_{Sp,Bq}(2u)F_{Tq,Bq}(u)\}$$

for all $p, q \in X$ *and* $u > 0$. *Then* A, B, S *and* T *have a unique common fixed point in* X.

Proof. Since $A(X) \subset T(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, one can define a sequence $\{y_n\}$ for which

$$y_{2n} = Tx_{2n+1} = Ax_{2n},$$

$$y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}, \quad n \geq 0. \tag{4}$$

We shall prove that for any $n \in N$ and $u > 0$ $F_{y_{2n},y_{2n+1}}(\phi(u)) \geq F_{y_{2n-1},y_{2n}}(u)$

Suppose (4) is not true. Then there exists $n \in N$ and $u > 0$ such that

$$F_{y_{2n},y_{2n+1}}(\phi(u)) < F_{y_{2n-1},y_{2n}}(u) \tag{5}$$

If follows from (4.4) and (5) that

$$\begin{aligned}
(F_{y_{2n},y_{2n+1}}(\phi(u)))^2 &= (F_{Ax_{2n},Bx_{2n+1}}(\phi(u)))^2\\
&\geq \min\{(F_{Sx_{2n},Tx_{2n+1}}(u))^2, F_{Sx_{2n},Ax_{2n}}(u)F_{Tx_{2n+1},Bx_{2n+1}}(u),\\
&F_{Sx_{2n},Bx_{2n+1}}(2u)F_{Tx_{2n+1},Ax_{2n}}(u), F_{Sx_{2n},Ax_{2n}}(u)F_{Tx_{2n+1},Ax_{2n}}(u),\\
&F_{Sx_{2n},Bx_{2n+1}}(2u)F_{Tx_{2n+1},Bx_{2n+1}}(u),\\
&= \min\{(F_{y_{2n-1},y_{2n}}(u))^2, F_{y_{2n-1},y_{2n}}(u)F_{y_{2n},y_{2n+1}}(u),\\
&F_{y_{2n-1},y_{2n+1}}(2u)F_{y_{2n},y_{2n}}(2u), F_{y_{2n-1},y_{2n}}(u)F_{y_{2n},y_{2n}}(u),\\
&F_{y_{2n-1},y_{2n+1}}(2u)F_{y_{2n},y_{2n+1}}(u),\\
&\geq \min\{(F_{y_{2n-1},y_{2n}}(u))^2, F_{y_{2n-1},y_{2n}}(u)F_{y_{2n},y_{2n+1}}(u),\\
&t(F_{y_{2n-1},y_{2n}}(u)F_{y_{2n},y_{2n+1}}(u)), F_{y_{2n-1},y_{2n}}(u)), t(F_{y_{2n-1},y_{2n}}(u),\\
&F_{y_{2n},y_{2n+1}}(u))F_{y_{2n},y_{2n+1}}(u)\}\\
&> \min\{(F_{y_{2n},y_{2n+1}}(u))^2, (F_{y_{2n},y_{2n+1}}(u))^2, F_{y_{2n},y_{2n+1}}(u),\\
&F_{y_{2n},y_{2n+1}}(u), (F_{y_{2n},y_{2n+1}}(u))^2\}\\
&= (F_{y_{2n},y_{2n+1}}(u))^2,
\end{aligned}$$

a contradiction. Therefore (4) holds

$$F_{y_{2n},y_{2n+1}}(\phi(u)) \geq F_{y_{2n-1},y_{2n}}(u).$$

Similarly, we obtain

$$F_{y_{2n+1},y_{2n+2}}(\phi(u)) \geq F_{y_{2n},y_{2n+1}}(u).$$

Therefore

$$F_{y_{2n},y_{2n+1}}(\phi(u)) \geq F_{y_{n-1},y_n}(u).$$

Hence, for all $n \in N$ and $u > 0$, we have

$$F_{y_{2n},y_{2n+1}}(\phi^n(u)) \geq F_{y_{n-1},y_n}(u).$$

By Lemma 1.2, it follows that $\{y_n\}$ is a Cauchy sequence in X. Since the Menger space (X, ξ, t) is complete, $\{y_n\}$ converges to a point z in X and the subsequences $\{Ax_{2n}\}, \{Sx_{2n}\}, \{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converge to z.

Now, suppose that T is continuous. Since B and T are compatible of type (A) by Lemma (2.6),

$$BTx_{2n+1},\ TTx_{2n+1} \to Tz \quad \text{as} \quad n \to \infty.$$

Putting $p = x_{2n}$ and $q = Tx_{2n+1}$ in (4.4), we have
$(F_{Ax_{2n},Bx_{2n+1}}(\phi(u)))^2 \geq \min\{(F_{Sx_{2n},TTx_{2n+1}}(u))^2,$
$F_{Sx_{2n},Ax_{2n}}(u)F_{TTx_{2n+1},BTx_{2n+1}}(u),$
$F_{Sx_{2n},BTx_{2n+1}}(2u)F_{TTx_{2n+1},Ax_{2n}}(u), F_{Sx_{2n},Ax_{2n}}(u)F_{TTx_{2n+1},Ax_{2n}}(u),$
$F_{Sx_{2n},BTx_{2n+1}}(2u)F_{TTx_{2n+1},BTx_{2n+1}}(u).$
Taking limit as $n \to \infty$, we have
$(F_{z,Tz}(\phi(u)))^2 \geq \min\{(F_{z,Tz}(u))^2, F_{z,z}(u)F_{Tz,Tz}(u), F_{z,Tz}(2u), F_{Tz,z}(u)$
$F_{z,z}(u)F_{Tz,z}(u), F_{z,Tz}(2u)F_{Tz,Tz}(u)\}$
$\geq \min\{(F_{z,Tz}(u))^2, 1, (F_{z,Tz}(u))^2, F_{z,Tz}(u), F_{z,Tz}(u)\}$
$= (F_{z,Tz}(u))^2,$
which implies that $Tz = z$. Again, replacing p by x_{2n} and q by z in (4.4), we have

$$\begin{aligned}(F_{Ax_{2n},Bz}(\phi(u)))^2 &\geq \min\{(F_{Sx_{2n},Tz}(u))^2, F_{Sx_{2n},Ax_{2n}}(u)F_{Tz,Bz}(u),\\ &\qquad F_{Sx_{2n},Bz}(2u)F_{Tz,Ax_{2n}}(u), F_{Tz,Ax_{2n}}(u)F_{Sx_{2n},Bz}(2u)\},\end{aligned}$$

Taking limit as $n \to \infty$ and using $Tz = z$, we have

$$\begin{aligned}(F_{z,Bz}(\phi(u)))^2 &\geq \min\{(F_{z,Tz}(u))^2, F_{z,z}(u)F_{Tz,Bz}(u), F_{z,Bz}(2u)F_{Tz,z}(u),\\ &\quad F_{z,z}(u)F_{Tz,z}(u), F_{z,Bz}(2u)F_{Tz,Bz}(u)\}\\ &= \min\{1, F_{z,Bz}(u), F_{z,Bz}(2u), 1, (F_{z,Bz}(u))^2\},\end{aligned}$$

which implies that $Bz = z$. Since $B(X) \subset S(X)$, there exists a point w in X such that $Bz = Sw = z$.
by using (4.4) again, we have

$$\begin{aligned} (F_{Aw,z}(\phi(u)))^2 &= (F_{Aw,Bz}(\phi(u)))^2 \\ &\geq \min\{(F_{Sw,Tz}(u))^2, F_{Sw,Aw}(u)F_{Tz,Bz}(u), F_{Sw,Bz}(2u)F_{Tz,Aw}(u), \\ &\quad F_{Sw,Aw}(u)F_{Tz,Aw}(u), F_{Sw,Bz}(2u)F_{Tz,Bz}(u)\} \\ &= \min\{1, F_{z,Aw}(u), F_{z,Aw}(u), (F_{z,Aw}(u))^2, 1\} \\ &= (F_{z,Aw}(u))^2, \end{aligned} \tag{6}$$

which implies that $Aw = z$. Since A and S are compatible of type (A) and $Aw = Sw = z$ by Lemma(2.5), we have

$$Az = ASw = SSw = Sz.$$

By using (4.4) again, we have $Az = z$. Therefore $Az = Bz = Sz = Tz = z$, that is z is a common fixed point of the given mappings. For uniqueness, let z' be another common fixed point such that $z' \neq z$,

$$\begin{aligned} (F_{z,z'}(\phi(u)))^2 &= (F_{Az,Bz'}(\phi(u)))^2 \\ &\geq \min\{(F_{z,z'}(u))^2, F_{z,z}(u)F_{z',z'}(u), F_{z,z'}(2u)F_{z',z}(u), \\ &\quad F_{z',z}(u)F_{z',z}(u), F_{z,z'}(2u)F_{z',z'}(u)\} \\ &= (F_{z,z'}(u))^2, \end{aligned} \tag{7}$$

which means that $z = z'$. Thus z is a unique common fixed point of A, B, S and T.
Taking $\phi(u) = ku, (0 < k < 1)$ in Theorem (4.1) we obtain the following corollary,

Corollary 4.1. *Let (X, ξ, t) be a complete Menger space with $t(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$ and A, B, S, T be mappings from X into itself such that*
(4.1) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,
(4.2) the pair $\{A, S\}$ *and* $\{B, T\}$ *are compatible of type (A),*
(4.3) one of A, B, S *and* T *is continuous,*
(4.4) there exists a constant $k \in (0, 1)$ *such that*

$$\begin{aligned} (F_{Ap,Bq}(ku))^2 &\geq \min\{(F_{Sp,Tq}(u))^2, F_{Sp,Ap}(u)F_{Tq,Bq}(u), \\ &F_{Sp,Bq}(2u)F_{Tq,Ap}(u), F_{Sp,Aq}(u)F_{Tq,Ap}(u), F_{Sp,Bq}(2u)F_{Tq,Bq}(u)\} \end{aligned}$$

for all $p, q \in X$ *and* $u > 0$. *Then* A, B, S *and* T *have a unique common fixed point in* X.

5. Sequence of Φ - contraction mappings and fixed points

Theorem 5.1. *Let* (X, ξ, t) *be a complete Menger space, where* t *is an* $h-$*type* $T-$*norm and* $T_n : X \to X$ $(n \in N)$ *be a sequence of mappings. Suppose that for any* $x, y \in X, u > 0$ *and any* $n, m \in N, n \neq m$, *the following condition holds*

$$F_{T_n^r x, T_m^r x}(\phi(u)) \geq F_{x,y}(u), \tag{8}$$

for some $r \in N$ *where* ϕ *satisfies the condition* (Φ). *Then the sequence* $T_n,\ n \in N$ *has a unique common fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary and let $\{x_n\},\ n \in N$ be a sequence in X defined by

$$x_n = T_n^r x_{n-1}, \quad n \in N.$$

By using (8), we have

$$\begin{aligned} F_{x_n, x_{n+1}}(\phi(u)) &= F_{T_n^r x_n, T_{n+1}^r x_n}(\phi(u)) \\ &\geq F_{x_{n-1}, x_n}(u), \end{aligned}$$

for all $n \in N$ and $u > 0$. Thus

$$F_{x_n, x_{n+1}}(\phi^n(u)) \geq F_{x_0, x_1}(u).$$

By Lemma (2.2), $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete, $x_n \to x^* \in X$.
First, we prove that x^* is a fixed point of $T_j^r \forall j \in N$. Since t is an $h-$type $T-$ norm, then for any $\lambda \in (0,1)$, there exists $\delta \in (0,1)$ such that $t(1-\delta, 1-\delta) > 1 - \lambda$. From Lemma (2.2), $u - \phi(u) > 0$, for any $u > 0$. Thus

$$F_{x_n, x^*}(u - \phi(u)) \to 1 \text{ and } F_{x_{n-1}, x^*}(u) \to 1 \quad \text{as} \quad n \to \infty.$$

This implies that there exists an $n_0 \in N$ such that

$$F_{x^*, x_{n_0}}(u - \phi(u)) > 1 - \delta \text{ and } F_{x^*, x_{n_0 - 1}}(u) > 1 - \delta.$$

By using (8), for $j \in N$, one gets

$$\begin{aligned} F_{x^*, T_j^r x^*}(u) &\geq t(F_{x^*, x_{n_0}}(u - \phi(u)), F_{x_{n_0}, T_j^r x^*}(\phi(u))) \\ &= t(F_{x^*, x_{n_0}}(u - \phi(u)), F_{T_{n_0}^r x_{n_0 - 1}, T_j^r x^*}(\phi(u))), \\ &\geq t(F_{x^*, x_{n_0}}(u - \phi(u)), F_{x_{n_0 - 1}, x^*}). \end{aligned} \tag{9}$$

Thus $F_{x^*,T_j^r x^*}(u) \geq t(1-\delta, 1-\delta) > 1-\lambda$. Since λ is arbitrary, we get $F_{x^*,T_j^r x^*}(u) = 1$, $\forall u > 0$. Thus $x^* = T_j^r x^*$ for every $j \in N$, and x^* is a fixed point of T_j^r

Let $x\acute{}$ be another fixed point of T_j^r, then

$$\begin{aligned} F_{x^*,x\acute{}}(\phi(u)) &= F_{T_j^r x^*, T_j^r x\acute{}}(\phi(u)) \\ &\geq F_{x^*,x\acute{}}(u), \end{aligned}$$

This is a contradiction, since $\phi(u) < u, \forall u > 0$. Thus $x^* = x\acute{}$ and x is a unique fixed point of T_j^r. Then, we get

$$\begin{aligned} T_j^r(T_j^r x^*) &= T_j^{r+1} x^* \\ &= T_j(T_j^r x^*) \\ &= T_j x^*. \end{aligned}$$

This implies that $T_j x^*$ is another fixed point of T_j^r. Since x^* is the unique fixed point of T_j^r, then $x^* = T_j x$ for all $j \in N$, and $x^*$is a common fixed point sequence $\{T_n\}, n \in N$. for the uniqueness of x^* if possible, let x_1 be another common fixed point of T_n. Then

$$\begin{aligned} F_{x^*,x_1}(\phi(u)) &= F_{T_n x^*, T_m x_1}(\phi(u)) \\ &= F_{T_n^r x^*, T_m^r x_1}(\phi(u)) \\ &\geq F_{x^*,x_1}(u), \end{aligned}$$

which implies that $x^* = x_1$. Thus x^* is the unique common fixed point of $\{T_n\}, n \in N$. The completes the proof.

Corollary 5.1. *([2]) Let (X, ξ, t) be a complete Menger space with continuous $T-$norm t, where $t(a,b) = \min\{a,b\}$ for $a, b \in [0,1]$. Suppose $T_n : X \to X$ be a sequence of mappings such that for some $r \in N$ and some $k \in (0,1)$, we have*

$$F_{T_{n+1}^r p, T_n^r q}(ku) \geq F_{p,q}(u),$$

$$F_{T_n^r p, T_m^r q}(ku) \geq F_{p,q}(u),$$

for all $n, m \in N, p, q \in X$ and every $u > 0$. Then the sequence T_n, $n \in N$ has a unique common fixed point.

Theorem 5.2. *Let (X, ξ, t) be a Menger space with continuous $T-$norm t, satisfying $t(u,u) \geq u$ for all $u \in [0,1]$ and $\{S_i\} : X \to X, i \in N$. If there exists a function ϕ satisfying the condition (Φ) and* *a mapping $T : X \to X$ such that $S_i(X) \subset T(X), i \in N$ and for every $x, y \in X, i, j \in N, i \neq j$:*

$$F_{S_ix,S_iy}(\phi(u)) \geq \min\{F_{Tx,Ty}(u), F_{S_ix,Tx}(u), F_{S_iy,Ty}(u), F_{S_ix,Ty}(2u), F_{S_iy,Tx}(2u)\},$$

for all $u > 0$. If $T(X$ is a complete subspace of X and each S_i is compatible with T, then for each $i \in N, T$ and the family $\{S_i\}$ have a unique common fixed point.

Proof. Let $x_0 \in X$ Since $S_i(X) \subset T(X)$, choose $x_1 \in X$ such that $Tx_i = S_1x_0 = y_0$. Inductively, one can define a sequence $\{y_n\}$ for which $Tx_n = S_nx_{n-1} = y_{n-1}$ for n=1,2,... .By (5.1), we have

$F_{y_{n-1},y_n}(\phi(u)) = F_{S_nx_{n-1},S_{n+1}x_n}(\phi(u)) \geq$
$\geq \min\{F_{Tx_{n-1},Tx_n}(u), F_{S_nx_{n-1},Tx_{n-1}}(u), F_{S_{n+1}x_n,Tx_n}(u),$

$F_{S_nx_{n-1},Tx_n}(2u), F_{S_{n+1}x_n,Tx_{n-1}}(2u)\}$
$= \min\{F_{y_{n-2},y_{n-1}}(u), F_{y_{n-1},y_{n-2}}(u), F_{y_n,y_{n-1}}(u), F_{y_{n-1},y_{n-1}}(2u), F_{y_n,y_{n-1}}(2u)\}.$

Since $F_{y_{n-2},y_n}(2u) \geq \min\{F_{y_{n-2},y_{n-1}}(u), F_{y_{n-1},y_n}(u)\}$, thus

$$F_{y_{n-2},y_n}(\phi(u)) \geq \min\{F_{y_{n-2},y_{n-1}}(u), F_{y_{n-1},y_n}(u)\}.$$

If $\min\{F_{y_{n-2},y_{n-1}}(u), F_{y_{n-1},y_n}(u)\} = F_{y_{n-1},y_n}(u)$, then we get $F_{y_{n-1},y_n}(\phi(u)) \geq F_{y_{n-1},y_n}(u)$ which is a contradiction. Then

$$F_{y_n,y_{n-1}}(\phi(u)) \geq F_{y_{n-2},y_{n-1}}(u).$$

Hence, for all $n \in N$ and $u > 0$, we have

$$F_{y_n,y_{n+1}}(\phi^n(u)) \geq F_{y_0,y_1}(u).$$

By Lemma 2.2, it follows that $\{y_n\}$ is a Cauchy sequence in $T(X)$. Since $T(X)$ is complete, then $y_n \to p \in T(X)$. Thus, there exists a point z in X such that $Tz = p$. For every $n \in N, i \in N$, we have
$F_{y_{n-1},S_iz}(\phi(u)) = F_{S_nx_{n-1},S_iz}(\phi(u)) \geq$
$\geq \min\{F_{T_nx_{n-1},Tz}(u), F_{S_nx_{n-1},Tx_{n-1}}(u), F_{S_iz,Tz}(u), F_{S_nx_{n-1},Tz}(2u), F_{S_iz,Tx_{n-1}}(2u)\}.$
Taking limit as $n \to \infty$, we have

$$\begin{aligned} F_{Tz,S_iz}(\phi(u)) &\geq \min\{1, 1, F_{S_iz,Tz}(u), 1, F_{S_iz,Tz}(2u)\} \\ &= F_{S_iz,Tz}(u). \end{aligned}$$

Thus $S_iz = Tz = p$ for every $i \in N$. Since each S_i is compatible with T, then from Lemma 2.4, $TS_iz = S_iTz$ i.e. $Tp = S_ip$. To prove that p is a common fixed point of $\{S_i\}$ and T.
$F_{y_n,Tp}(\phi(u)) = F_{S_{n+1}x_n,S_ip}(\phi(u)) \geq$

$\geq \min\{F_{y_{n-1},Tp}(u), F_{y_n,y_{n-1}}(u), F_{S_ip,Tp}(u), F_{y_n,Tp}(2u), F_{S_ip,y_{n-1}}(2u)\}$.
Taking limit as $n \to \infty$, we have

$$\begin{aligned} F_{p,Tp}(\phi(u)) &\geq \min\{F_{p,Tp}(u), 1, 1, F_{p,Tp}(2u), F_{p,Tp}(2u)\} \\ &= F_{p,Tp}(u). \end{aligned}$$

Thus $p = Tp = S_ip$ for every $i \in N$ and p is a common fixed point for T and the family $\{S_i\}, i \in N$. For uniqueness of p, if possible, let q be another common fixed point for T and $\{S_i\}$. Then

$$\begin{aligned} F_{p,q}(\phi(u)) &= F_{S_ip,S_jq}(\phi(u)) \\ &\geq \min\{F_{Tp,Tq}(u), F_{S_ip,Tp}(u), F_{S_jq,Tq}(u), F_{S_ip,Tq}(2u), F_{S_jq,Tp}(2u)\} \\ &= F_{p,q}(u). \end{aligned}$$

Thus $p = q$ and p is the unique common fixed point for T and the family $\{S_i\}, i \in N$.
Taking $\phi(u) = ku, (0 < k < 1)$ in Theorem (5.2) we obtain the following corollary.

Corollary 5.2. [10] *Let (X, ξ, t) be a Menger space with continuous $T-$norm t, satisfying $t(u, u) \geq u$ for all $u \in [0, 1]$ and $\{S_i\} : X \to X, i \in N$. If there exists a constant $k \in (0, 1)$ and a mapping $T : X \to X$ such that $S_i(X) \subset T(X), i \in N$ and for every $x, y \in X, i, j \in N, i \neq j$:*

$$F_{S_ix,S_iy}(ku) \geq \min\{F_{Tx,Ty}(u), F_{S_ix,Tx}(u), F_{S_iy,Ty}(u), F_{S_ix,Ty}(2u), F_{S_iy,Tx}(2u)\},$$

for all $u > 0$. If $T(X$ is a complete subspace of X and each S_i is commutes with T, then for each $i \in N, T$ and the family $\{S_i\}$ have a unique common fixed point.

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Department of Mathematics, Faculty of Science,
Assiut University, Egypt.
E-mail: r_$r$ashwan@yahoo.com