"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 19 (2009), No. 1, 217 - 222

CHARACTERISTIC PROPERTIES OF THE INDICATRIX GIVEN BY A RANDERS CHANGE

RYOTA SHIMIZU AND MASASHI KITAYAMA

Abstract. C. Shibata [6] investigated the theory of a change which was called a β-change of Finsler metric.We study the behavior of indicatrices given by special β-changes, in particular by a Randers change.

1. Introduction.

In 1984 C. Shibata [6] investigated the theory of a change which was called a β-change of Finsler metric. Many authors treat indicatrices [2], [4], [5], [6], [7]. In the present paper, we shall study the behavior of indicatrices given by special β-changes. In the first section, we shall give the definitions of some special Finsler spaces. In the second section, we consider indicatrices given by a Randers change ($\bar{L} = L + \beta$).
The terminology and notations are referred to Matsumoto's monograph [5].

§1. The definitions of some special Finsler spaces. Let M be an n-dimensional differentiable manifold and $F^n = (M, L)$ be a Finsler space equipped with a fundamental function $L(x, y)(y^i = \dot{x}^i)$ on M and we shall introduce in F^n the Cartan connection $C\Gamma = (F_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$.

Keywords and phrases: Finsler space, β-change of Finsler metric, Randers change
(2000)Mathematics Subject Classification: 53C60, 53B40

Definition 1. *A Finsler space $F^n(n > 2)$ is called P2-like, if there exists such a covariant vector field P_i that the hv-curvature tensor P_{hijk} of F^n is written in the form*

$$P_{hijk} = P_h C_{ijk} - P_i C_{hjk}.$$

Then we have already known the following.

Theorem A.[3] *Assume that a Finsler space F^n is P2-like. Then the hv-curvature tensor P_{hijk} of F^n vanishes, or the v-curvature tensor S_{hijk} of F^n vanishes.*

Definition 2. *A Finsler space $F^n(n > 3)$ is called R3-like, if the third curvature tensor R_{hijk} of Cartan is expressed in the form*

$$R_{hijk} = g_{hj}L_{ik} + g_{ik}L_{hj} - g_{hk}L_{ij} - g_{ij}L_{hk},$$

where $L_{ik} = (R_{ik} - rg_{ik}/2)/(n-2), R_{hj} = R_h{}^m{}_{jm}$ and $r = R^m_{\ m}/(n-1)$.

For the $(v)hv$-torsion tensor P_{hij} and the $(h)hv$-torsion tensor C_{hij}, putting

$${}^*P_{hij} = P_{hij} - \lambda C_{hij},$$

where the scalar λ is homogeneous of degree one with respect to y^i and is given by P_iC^i/C_jC^j for $C_j \neq 0$.

Definition 3. *A Finsler space $F^n(n > 2)$ is called a${}^*P$-Finsler space, if ${}^*P_{hij} = 0$.*

Definition 4. *A Finsler space F^n is called a Landsberg space, if the $(v)hv$-torsion tensor $P_{hij} = 0$.*

Definition 5. *A non-Riemannian Finsler space $F^n(n > 4)$ is called S4-like, if the v-curvature tensor S_{hijk} is written in the form*

$$L^2 S_{hijk} = h_{hj}M_{ik} + h_{ik}M_{hj} - h_{hk}M_{ij} - h_{ij}M_{hk},$$

where M_{ij} is a symmetric and indicatory tensor.

Then we have already known the following.

Theorem B.[8] *Assume that a Finsler space $F^n(n > 4)$ is an R3-like (non-Landsberg) ${}^*P$-Finsler space. Then F^n is S4-like .*

Theorem C.[8] *An R3-like Landsberg space $F^n(n > 3)$ is a Finsler space satisfying $S_{hijk} = 0$, or a Riemannian space of constant* *curvature.*

Theorem D.[6] *If a Finsler space $F^n(n > 4)$ is $S4$-like, the Finsler space $\bar{F}^n$, obtained from F^n by a Randers change, is also $S4$-like.*

§2. Indicatrices given by a Randers change. The tangent space F_x at each point x of F^n is regarded as an n-dimensional Riemannian space with the fundamental tensor $g_{ij}(x, y)$, where $x = (x^i)$ is fixed. Then in terms of the Cartan connection $C\Gamma$ of F^n, components $C_j{}^i{}_k$ of the (h)hv-torsion tensor are Christoffel symbols of F_x and the v-curvature tensor $S_h{}^i{}_{jk}$ is the Riemannian curvature tensor of F_x. The indicatrix I_x at a point x is a hypersurface of the Riemannian space F_x which is defined by the equation $L(x, y) = 1$, where x is fixed. Consequently I_x is regarded as an (n-1)-dimensional Riemannian space.

Now we consider a special β-change called a Randers one which is defined by $\bar{L} = L + \beta$, where $\beta(= b_i(x)y^i)$ is a non-zero 1-form on M. By the Randers change $L_{ij} = h_{ij}/L$ is invariant, where $h_{ij}$is the angular metric tensor. From now on, we shall call a tensor which is invariant under the Randers change an R-invariant tensor. For the v-curvature tensor S_{hijk}, putting

(2.1)

$$S^*{}_{hijk} = [S_{hijk} + \mathfrak{A}_{(jk)}\{h_{ij}S_{hk} + h_{hk}S_{ij} - Sh_{ij}h_{hk}/(n-2)\}/(n-3)]/L,$$

we get $S^*{}_{hijk}$ is R-invariant [6], where we use the notation $\mathfrak{A}_{(jk)}$ to denote the interchange of indices j, k and subtraction.

For the $S4$-likeness, we have the following theorems.

Theorem E.[7] *If a Finsler space $F^n(n > 4)$ is $S4$-like, then the indicatrix I_x is conformally flat.*

Theorem F.[6] *A non-Riemannian Finsler space $F^n(n > 4)$ is $S4$-like, if and only if the R-invariant tensor $S^*{}_{hijk}$vanishes.*

From Theorem A , (2.1) , Theorem F , Theorem D and Theorem E we immediately get

Theorem 1. *Assume that a Finsler space $F^n(n > 4)$ is $P2$-like. Then the indicatrix $\bar{I}_x$ of $\bar{F}^n$, obtained from F^n by a Randers change,* *is conformally flat provided that* $P_{hijk} \neq 0$.

From Theorem B , Theorem D and Theorem E we immediately get

Theorem 2. *Assume that a Finsler space* $F^n (n > 4)$ *is an R3-like (non-Landsberg)* $^*P$*-Finsler space. Then the indicatrix* $\bar{I}_x$ *of* $\bar{F}^n$*, obtained from* F^n *by a Randers change, is conformally flat .*

From Theorem C , (2.1) , Theorem F , Theorem D and Theorem E we immediately get

Theorem 3. *Assume that a Finsler space* $F^n (n > 4)$ *is an R3-like Landsberg space . If* F^n *is not a Riemannian space of constant curvature , then the indicatrix* $\bar{I}_x$ *of* $\bar{F}^n$*, obtained from* F^n *by a Randers change, is conformally flat .*

Theorem G.[1] *Assume that a Finsler space* $F^n (n > 2)$ *is a* $^*P$*-Finsler space. If the hv-curvature tensor* P_{hijk} *is symmetric in j, k, then* $P_{hijk}=0$ *, or the v-curvature tensor* $S_{hijk} = 0$.

From Theorem G , (2.1) , Theorem F , Theorem D and Theorem E we immediately get

Theorem 4. *Assume that a Finsler space* $F^n (n > 2)$ *is a* $^*P$*-Finsler space. If the hv-curvature tensor* P_{hijk} *is symmetric in j, k, then the indicatrix* $\bar{I}_x$ *of* $\bar{F}^n$*, obtained from* F^n *by a Randers change, is conformally flat provided that* $P_{hijk} \neq 0$.

By a β-change the v-curvature tensor $S_h{}^i{}_{jk}$ changes as follows [6]:

(2.2) $\bar{S}_h{}^i{}_{jk} = S_h{}^i{}_{jk} + \mathfrak{A}_{(jk)}(C_m{}^i{}_k V_h{}^m{}_j - C_h{}^m{}_k V_m{}^i{}_j - V_m{}^i{}_k V_h{}^m{}_j)$,
$V_i{}^h{}_j = C_i{}^h{}_j - \bar{C}_i{}^h{}_j$.

In the case of Randers change, from (2.2), we get

Theorem 5. *Let* $S_h{}^i{}_{jk} = \mathfrak{A}_{(jk)}(C_h{}^m{}_k V_m{}^i{}_j + V_m{}^i{}_k V_h{}^m{}_j - C_m{}^i{}_k V_h{}^m{}_j)$. *Then we have* $\bar{S}_h{}^i{}_{jk} = 0$ *, where* $V_i{}^h{}_j = C_{ijr} b^r l^h/\mu + (2m_i m_j + m^2 h_{ij}) l^h / 2L\mu^2 - (h_i^h m_j + h_j^h m_i + h_{ij} m^h)/2\bar{L}, \mu = \bar{L}/L, m^2 = g^{ij} m_i m_j$ *and* $m_i = b_i - \beta y_i / L^2$.

We have already known the following.
Theorem I.[5] *The v-curvature tensor* S_{hijk} *of a Finsler space* $F^n (n > 2)$ *vanishes at a point* x*, if and only if the indicatrix* I_x *is of constant curvature 1.*

Using Theorem 5 and Theorem I , we have
Theorem 6. *Let* $S_h{}^i{}_{jk} = \mathfrak{A}_{(jk)}(C_h{}^m{}_k V_m{}^i{}_j + V_m{}^i{}_k V_h{}^m{}_j - C_m{}^i{}_k V_h{}^m{}_j)$. *Then the indicatrix* $\bar{I}_x$ *of* $\bar{F}^n$*, obtained from* $F^n (n > 2)$ *by a Randers change, is of constant curvature 1.*

REFERENCES

[1] M. Hashiguchi, **On the hv-cuvature tensors of Finsler spaces**, Rep.Fac.Sci. Kagoshima Univ., (Math. Phys. Chem.) 4(1971), 1-5.
[2] M. Kitayama , **Indicatrices of Randers change** , to appear in Tensor.
[3] M. Matsumoto , **On Finsler spaces with curvature tensors of some special forms**, Tensor, N.S.,22(1971), 201-204.
[4] M. Matsumoto , **On the indicatrices of a Finsler space**, Period. Math. Hungarica 8(1977), 185-191.
[5] M. Matsumoto, **Foundations of Finsler geometry and special Finsler spaces**, Kaiseisha Press, Otsu, Japan, 1986.
[6] C. Shibata, **On invariant tensors of β-changes of Finsler metrics**, J. Math. Kyoto Univ., 24(1984), 163-188.
[7] S. Watanabe and F. Ikeda , **On some properties of Finsler spaces based on the indicatrices**, Publicationes Math. Debrecen, 28(1981), 129-136.
[8] M. Yoshida , **On an $R3$-like Finsler space and its special cases**, Tensor, N.S.,34(1980), 157-166.

Shiranuka High School
Shiranuka, Hokkaido 088-0323
JAPAN

Department of Mathematics
Kushiro Campus, Hokkaido
University of Education
Kushiro, Hokkaido 085-8580, JAPAN