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THETA FUNCTION IDENTITIES ASSOCIATED WITH MODULAR EQUATION

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Abstract. Ramanujan gave simple theta function identities for different bases. We have considered the continued fraction given in section 1, eq. (1.7) which equals quotient of theta functions on base four. This continued fraction is also of Ramanujan and is analogous to his famous continued fraction $R(q)$. In this paper we have given simple theta function identities on base four. These identities will be helpful in deducing modular equations.

1. INTRODUCTION

During the years 1903-1914, Ramanujan recorded most of his mathematical discoveries, without proofs, in notebooks. Although many of his results were already in the literature, more were not. A photostat edition, with no editing, was published by the Tata Institute of Fundamental Research, Bombay in 1957. The formidable task of editing the notebooks was taken up in right earnest by B.C. Berndt. The dedicated work of Berndt, published by Springer-Verlag, is now available in five parts.

In chapter 19 of his second notebook, Ramanujan studied modular equations primary of degrees 3, 5 and 7. For each degree Ramanujan derived series of theta functions identities of appropriate arguments. Each modular equation is equivalent to certain theta-function identity, but a theta-function identity may not have an equivalent modular equation. These theta-function identities are then used to establish astonishing series of modular equations of that degree. Ramanujan published but one paper [9] in which modular equations are discussed, but modular equations were not the main reason for this paper. Ramanujan recorded several hundred modular equations in his three notebooks [10]. Complete proofs for all the modular equations can be found in Berndt's book [4], [5], [6].

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The continued fraction $C(q)$ given in (1.7) is a special case of the continued fraction of Ramanujan [Entry 9 and Entry 13 in Chapter 16 of Ramanujan's Second Note Book],[4],which is analogous to Ramanujan's famous continued fraction $R(q)$. We have given some identities and expansions for this continued fraction $C(q)$ [11,12]. This continued fraction can be expressed as a quotient of theta functions on base four. Naturally I have given simple theta function identities on base 4. These theta function identities derived by us will be analogous to Ramanujan's Entries for base 5. These identities may be helpful in finding modular equations.

Ramanujan defined general theta function by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1 .$$

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty , \ |ab| < 1. \qquad (1.1)$$

which is Jacobi's Triple product identity.

The most important special cases of $f(a,b)$ are in Ramanujan's notation, $|q| < 1$

$$\varphi(q) := f(q,q), \qquad (1.2)$$

By [4, p. 35, Entry 19]

$$\psi(q) := f(q,q^3), \qquad (1.3)$$

and Euler's function

$$f(-q) := f(-q,-q^2) \qquad (1.4)$$

Lastly define

$$\chi(-q) := (q,q^2)_\infty , \qquad (1.5)$$

and Euler's identity [2, p. 27, eq.(1.6.8)]

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}. \qquad (1.6)$$

In section 3 we have given a differential identity of a quotient of theta functions using continued fraction $C(q)$. There are four more identities involving Ramanujan's $\varphi(q),\psi(q)$ functions and the function $\chi(-q)$

In section 4 we have given simple theta function identities.

The author considered in [12] the continued fraction.

$$C(q) = \frac{q^{\frac{1}{8}}}{1+}\frac{1+q}{1+}\frac{q^2}{1+}\frac{q+q^3}{1+}\frac{q^4}{1+.....} \qquad (1.7)$$

and proved that

$$C(q)=q^{\frac{1}{8}}\frac{(q;q^4)_\infty(q^3;q^4)_\infty}{(q^2;q^4)_\infty^2} \tag{1.8}$$

which implies by (1.1)

$$C(q)=q^{\frac{1}{8}}\frac{f(-q,-q^3)}{f(-q^2,q^2)}. \tag{1.9}$$

2. NOTATIONS

We shall be using the customary q-product notation also known as q-Pochhammer symbol. Thus set

$$(a)_0 := (a;q)_0 = 1, \text{ and for } n \geq 1,$$

let

$$(a)_n := (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k).$$

Further set

$$(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \ |q| < 1.$$

$$(a_1,a_2,\ldots\ldots,a_m;q)_\infty = \prod_{i=1}^{m}(a_i;q)_\infty.$$

If the base q is understood, we use $(a)_n$ and $(a)_\infty$ instead $(a;q)_n$ and $(a;q)_\infty$, respectively.

We shall be using the following results frequently in the sequel:

We mention the following straightforward consequences of the series expansion of $f(a,b)$:

$$\varphi(q)=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2}=(-q;q^2)_\infty^2(q;^2q^2)_\infty=\frac{(-q;-q)_\infty}{(q;-q)_\infty}, \ |q|<1 \tag{2.1}$$

$$\psi(q)=f(q,q^3)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \ |q|<1 \tag{2.2}$$

$$f(-q)=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}}=(q;q)_\infty, \ |q|<1 \tag{2.3}$$

where the latter equality is Eulers' pentagonal number theorem. The product representation in (2.1)—(2.3) follows from Jacobi's triple product identity (1.1).

Special cases of this general identity [8, p 825] will be used in deriving identities for $f(a,b)$.:

$$(z, zq, \ldots\ldots, zq^{m-1}; q^m)_\infty = (z;q)_\infty . \tag{2.4}$$

for any integer $m > 0$.

3. SOME THETA FUNCTION IDENTITIES

$$8q\frac{d}{dq}\left[\log q^{\frac{1}{8}}\frac{f(-q,q^3)}{f(-q^2,q^2)}\right] = \frac{(q,q)_\infty^4}{(-q,q)_\infty^4}, \tag{3.1}$$

$$f(q,q^3)\,f(q^2,q^2) = \frac{\varphi(-q^4)\,f(-q^4)}{\chi(-q)}\chi(q^2), \tag{3.2}$$

$$f(-q,-q^3)\,f(-q^2,-q^2) = f(-q)\,f(-q^4)\,\chi(-q^2), \tag{3.3}$$

$$f(q,q^7)\,f(q^3,q^5) = \chi(q)\,f^2(-q^8), \tag{3.4}$$

$$q\psi^2(q^4) = \sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{8n+2}} - \frac{q^{4n+3}}{1-q^{8n+6}}\right). \tag{3.5}$$

Proofs of the identities

Proof of (3.1)

Taking logarithmic differentiation with respect to q of both sides of (1.8), w have

Now

$$\frac{d}{dq}\left[\log C(q)\right] = \frac{1}{8q} - \sum_{n=0}^{\infty}\left[\frac{(4n+1)q^{4n}}{1-q^{4n+1}} + \frac{(4n+3)q^{4n+2}}{1-q^{4n+3}} - \frac{2(4n+2)q^{4n+1}}{1-q^{4n+2}}\right] \tag{3.6}$$

By (1.9) the above identity can be written as

$$8q\frac{d}{dq}\left[\log q^{\frac{1}{8}}\frac{f(-q,q^3)}{f(-q^2,q^2)}\right]$$

$$= 1 - 8\sum_{n=0}^{\infty}\left[\frac{(4n+1)q^{4n+1}}{1-q^{4n+1}} + \frac{(4n+3)q^{4n+3}}{1-q^{4n+3}} - \frac{2(4n+2)q^{4n+2}}{1-q^{4n+2}}\right] \tag{3.7}$$

Using the identity in [8],

$$1 - 8\sum_{n=0}^{\infty}\left[\frac{(4n+1)q^{4n+1}}{1-q^{4n+1}} - \frac{2(4n+2)q^{4n+2}}{1-q^{4n+2}} + \frac{(4n+3)q^{4n+3}}{1-q^{4n+3}}\right] = \frac{(q;q)_\infty^4}{(-q;q)_\infty^4}, \tag{3.8}$$

(3.7) can be written as

$$8q\frac{d}{dq}\left[\log q^{\frac{1}{8}}\frac{f(-q,q^3)}{f(-q^2,q^2)}\right]=\frac{(q;q)_\infty^4}{(-q;q)_\infty^4},$$

which is (3.1).

A note on the identity (3.8). The author gave two proofs of this identity [12]. The first proof is based on the ${}_6\psi_6$ summation formula of Bailey and the second by invoking the residue theorem applied to elliptic functions. Moreover this identity is a well known identity of Jacobi [2, p. 396].

Proof of (3.2)

Taking $z=-q$ and $m=4$ in (2.4), we have

$$(-q;q^4)_\infty(-q^2;q^4)_\infty(-q^3;q^4)_\infty(-q^4;q^4)_\infty=(-q;q)_\infty.$$

Employing identities (1.1) and (1.6) in the above identity, we have

$$\begin{aligned}f(q,q^3)f(q^2,q^2)&=(-q;q^4)_\infty(-q^3;q^4)_\infty(-q^2;q^4)_\infty^2(q^4;q^4)_\infty^2\\&=\frac{(-q;q)_\infty(-q^2;q^4)_\infty(q^4;q^4)_\infty^2}{(-q^4;q^4)_\infty}\\&=\frac{(q^4;q^8)_\infty(-q^2;q^4)_\infty(q^4;q^4)_\infty^2}{(q;q^2)_\infty}.\end{aligned}\tag{3.9}$$

By the definition of $\chi(q)$, (3.9) can be written as

$$\begin{aligned}f(q,q^3)f(q^2,q^2)&=\frac{\chi(-q^4)f^2(-q^4)}{\chi(-q)}(-q^2;q^4)_\infty\\&=\frac{\chi(-q^4)f^2(-q^4)}{\chi(-q)}\chi(q^2)=\frac{\varphi(-q^4)f(-q^4)}{\chi(-q)}\chi(q^2),\end{aligned}$$

which is (3.2).

Proof of (3.3)

Taking $z=q$ and $m=4$ in (2.4), we have by employing (1.1)

$$(q;q^4)_\infty(q^2;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty=(q;q)_\infty$$

Again by the definition of $f(a,b)$, as given in (1.1) and the definition of $\chi(q)$, the above identity simplifies to

$$\begin{aligned}f(-q,-q^3)f(-q^2,-q^2)&=(q;q^4)_\infty(q^3;q^4)_\infty(q^2;q^4)_\infty^2(q^4;q^4)_\infty^2\\&=(q;q)_\infty(q^2;q^4)_\infty(q^4;q^4)_\infty\end{aligned}$$

$$= f(-q)f(-q^4)\chi(-q^2),$$

which is (3.3).

Proof of (3.4)

Making $q \to q^2$, taking $z = -q$ and $m = 8$ in (2.4), we have

$$(-q;q^8)_\infty(-q^3;q^8)_\infty(-q^5;q^8)_\infty(-q^7;q^8)_\infty = (-q;q^2)_\infty,$$

and by the definition of $f(a,b)$, we have

$$\begin{aligned} f(q,q^7)f(q^3,q^5) &= (-q;q^8)_\infty(-q^3;q^8)_\infty(-q^5;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty^2, \\ &= (-q;q^2)_\infty(q^8;q^8)_\infty^2 \\ &= \chi(q)(q^8;q^8)_\infty^2 \\ &= \chi(q)f^2(-q^8), \end{aligned}$$

which is (3.4).

Proof of (3.5)

Equation (8.5) in Chapter 17 [4, p. 116] :

$$\frac{af(-b/a,-a^3b)}{f(-a^2,-b^2)}\varphi(ab)\psi(a^2b^2) = \sum_{k=1}^{\infty}\left(\frac{a^k b^{k-1}}{1-a^{2k}b^{2k-2}} - \frac{a^{k-1}b^k}{1-a^{2k-2}b^{2k}}\right).$$

Taking $a = q$ and $b = q^3$, we have,

$$q\varphi(q^4)\psi(q^8) = \sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{8n+2}} - \frac{q^{4n+3}}{1-q^{8n+6}}\right).$$

Using the relation in Entry 25(iv), Chapter 16 [4, p. 40]:

$$\varphi(q)\psi(q^2) = \psi^2(q).$$

We finally have

$$q\psi^2(q^4) = \sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{8n+2}} - \frac{q^{4n+3}}{1-q^{8n+6}}\right),$$

which is (3.5).

4. IDENTITIES WHICH ARE ANALOGOUS TO THE ENTRIES 2 OF CHAPTER 19 OF RAMANUJAN [4, P. 222]

We shall now prove the following identities:

(i) $f(-q,-q^3)f^3(-q^{12}) = f(-q^4)f(-q,-q^{11})f(-q^3,-q^9)f(-q^5,-q^7)$, (4.1)

(ii) $f(-q^2,-q^2)f^3(-q^{12}) = f(-q^4)f^2(-q^2,-q^{10})f(-q^6,-q^6)$, (4.2)

(iii) $f(-q,-q^5)f^3(-q^{18}) = f(-q^6)f(-q,-q^{17})f(-q^5,-q^{13})f(-q^7,-q^{11})$.(4.3)

Proof of (4.1)

Expanding (4.1) by the definition of $f(a,b)$, as given in (1.1)

$$=(q;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty(q^{12};q^{12})^3_\infty, \tag{4.4}$$

and the right side

$$=(q^4;q^4)_\infty(q;q^{12})_\infty(q^{11};q^{12})_\infty(q^3;q^{12})_\infty(q^9;q^{12})_\infty(q^5;q^{12})_\infty(q^7;q^{12})_\infty(q^{12};q^{12})^3_\infty, \tag{4.5}$$

Making $q \to q^4$ and taking $z = q$, $m = 12$ in (2.4), we have

$$(q;q^{12})_\infty(q^5;q^{12})_\infty(q^9;q^{12})_\infty = (q;q^4)_\infty$$

and making $q \to q^4$ and taking $z = q^3$, $m = 12$ in (2.4), we have

$$(q^3;q^{12})_\infty(q^7;q^{12})_\infty(q^{11};q^{12})_\infty = (q^3;q^4)_\infty .$$

Hence (4.5) is

$$=(q^4;q^4)_\infty(q;q^4)_\infty(q^3;q^4)_\infty(q^{12};q^{12})^3_\infty . \tag{4.6}$$

By (4.4) and (4.6) we have (4.1).

The proofs of (4.2) and (4.3) are similar.

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