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MORE ON SLIGHTLY-β-CONTINUOUS FUNCTIONS

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Abstract. β-continuity was introduced by Monsef et al. [1] and then the weak and strong forms of β-continuity are studied. In this paper, we obtain several new properties of slightly β-continuous function which is defined by Noiri [8].

1. Introduction

The concepts of β-open sets and β-continuity were introduced and studied by Monsef et al. [1]. Further Popa and Noiri [11] introduced weak β-continuity and subsequently Noiri and Popa [7] introduced the concept of almost β-continuity. Recently Noiri [8] introduced the concept of slightly β-continuous functions which is a generalization of above mentioned variations of β-continuity. It is also the generalization of slightly semi- continuous functions introduced by Nour [10] and slightly precontinuity introduced by Baker [2]. The purpose of this paper is to study some new properties of slightly β-continuous functions and to relate it with some new variations of β-continuity. Also the concept of set β-connected function is defined and it is established that with in the class of surjective functions, the class of slightly β-continuous functions and set β-connected functions coincide.

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2. Preliminaries

Throughout the present paper, X and Y denote topological spaces. Let A be a subset of X. We denote the interior and closure of A by $Int(A)$ and $Cl(A)$ respectively. A subset A of a space X is called *β-open* [1] (resp. semi-open [5], *preopen* [6]) if $A \subseteq Cl(Int(Cl(A)))$ (resp. $A \subseteq Cl(Int(A)), A \subseteq Int(Cl(A))$). The complement of β-open set is *β-closed* [1]. The intersection of all β-closed sets containing A is called *β-closure* of A and is denoted by $\beta Cl(A)$. A subset A of a space X is said to be *β-clopen* [8] (resp. *semi-clopen* [10], *preclopen* [2]) if it is β- open (resp. semi-open, preopen) and β-closed (resp. semi-closed, preclosed). A subset A of a space X is said to be *regular open* if $A = Int(Cl(A))$ and itfs complement is said to be *regular closed* if $A = Cl(Int(A))$. A space X is said to be *mildly compact* [12] if every clopen cover of a space X has a finite subcover and a space X is said to be *β-closed* [13] if every β-clopen cover of a space X has a finite subcover.

Definition 2.1. A function $f : X \to Y$ is called *β-continuous* [1] (resp. *almost β-continuous* [7], *weakly β-continuous* [11] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a β-open set U such that $f(U) \subseteq V$(resp. $f(U) \subseteq Int(Cl(V)), f(U) \subseteq Cl(V)$).

Definition 2.2. A function $f : X \to Y$ is called *slightly β-continuous* [8] (resp. *slightly semi-continuous* [10], *slightly precontinuous* [2]) if for each $x \in X$ and each clopen set V of Y containing $f(x)$,there exists a β-open (resp. semi-open, preopen) set U such that $f(U) \subseteq V$.

Definition 2.3. A function $f : X \to Y$ is said to be *strongly θ-β-continuous* [9] (resp. *almost strongly θ-β-continuous* [13]) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \subseteq \beta O(X, x)$ such that $f(\beta Cl(U)) \subseteq V$ (resp. $f(\beta Cl(U)) \subseteq \beta Cl(V)$).

From the definitions stated above, we have the following diagram:
strongly θ-β-continuity $\rightarrow$ almost strongly θ-β-continuity $\rightarrow$ almost β continuity $\rightarrow$ weak β-continuity $\rightarrow$ slight β-continuity $\leftarrow$ slight semi-continuity
$\uparrow$ slight precontinuity

None of the above implications are reversible as can be seen from the

following examples:
The reverse implication of first horizontal line may not hold as can be seen from [13, Examples 2.5,2.6]. Moreover weak β-continuity does not necessarily imply almost β-continuity as can be seen in example of [11].

Example 2.1. Let $X = Y = \{a, b, c\}$. Let $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}, \sigma = \{\emptyset, X, \{a\}, \{b, c\}\}$ be the topologies on X and Y respectively. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is slightly β-continuous but not slightly precontinuous since $\{a\}$ and $\{b, c\}$ are clopen sets in (Y, σ) which are β clopen in (X, τ) but not preclopen in (X, τ).

Example 2.2. Let $X = Y = \{a, b, c\}$. Let τ =indiscrete topology, $\sigma = \{\emptyset, X, \{a\}, \{b, c\}\}$ be the topologies on X and Y respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is slightly β-continuous but not slightly semicontinuous since in an indiscrete topological space, every subset is β-open but not semi-open.

Example 2.3. Let $X = Y = \{a, b, c, d\}$. Let $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}, \sigma = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$ be the topologies on X and Y respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is slightly β-continuous but not weakly β-continuous.

3. Slightly β-continuous functions

In this section, we study some more properties on slightly β-continuous functions. A space X is called *extremally disconnected* if the closure of each open set of X is open. Also X is said to be *ultra regular* [3] if for each open set U and for each $x \in U$,there exists a clopen set V such that $x \in V \subseteq U$.

Theorem 3.1. *If $f : X \to Y$ is a slightly β-continuous function of a space X into an extremally disconnected space Y, then f is almost strongly θ-β-continuous.*

Proof: Let $x \in X$ and let V be an open subset of Y containing $f(x)$. Now, $\beta Cl(V) = Int(Cl(V))$ is regular open in Y. Since Y

is extremally disconnected, $\beta Cl(V)$ is clopen. Since f is slightly β-continuous, by [8, Theorem 3.2], there exists a β-open set U such that $f(\beta Cl(U)) \subseteq \beta Cl(V)$. Thus f is almost strongly θ-β-continuous.
Corollary 3.1. ([8, Theorem 4.13]). *If $f : X \to Y$ is a slightly β-continuous function of a space X into an extremally disconnected space Y, then f is weakly β-continuous..*

Theorem 3.2. *If $f : X \to Y$ is slightly β-continuous and Y is ultra regular, then f is strongly θ-β-continuous.*

Proof: Let $x \in X$ and let V be an open in Y containing $f(x)$. Since Y is ultra regular, there exists a clopen set W such that $f(x) \subseteq W \subseteq V$. Since f is slightly β- continuous, there is β-open set U such that $f(\beta Cl(U)) \subseteq W$ [8, Theorem 3.2] and so $f(\beta Cl(U)) \subseteq V$. Thus f is strongly θ-β-continuous.
Corollary 3.2. ([8, Theorem 4.13]). *If $f : X \to Y$ is slightly β-continuous and Y is ultra regular, then f is β-continuous.*

As we see in the following example,the restriction of a slightly β-continuous function,even to β-open set may not be slightly β-continuous as can be seen from the following example:

Example 3.1. Let $X = Y = \{a, b, c, d\}$ and let $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}, \tau = \{\emptyset, X, \{a\}, \{b, c, d\}\}$ be the topologies on X and Y respectively. Then the identity mapping $f : (X, \sigma) \to (Y, \tau)$ is slightly β-continuous but if $A = \{a, c, d\}$ and σ_A is the relative topology on A induced by s, then $f|_A : (A, \sigma_A) \to (X, \tau)$ is not slightly β-continuous. Note that $\{b, c, d\}$ is clopen in (X, τ), but $(f|_A)^{-1}(\{b, c, d\}) = \{c, d\}$ is not β-open in (A, σ_A).

Next we show that the restriction of a slightly β-continuous function even to α-open set is slightly β-continuous. The following lemma will be useful:

Lemma 3.1. ([1, Theorem 2.5]) *If A is α-open and B is β-open, then $A \cap B$ is β-open in the relative topology of A.*

Theorem 3.3. *If $f : X \to Y$ is slightly β-continuous and A*

is α-open subset of X, then $f|_A : A \to Y$ is slightly β-continuous.
Proof: Let V be a clopen subset of Y. Then $(f|_A)^{-1}(V) = f^{-1}(V) \cap A$. Since $f^{-1}(V)$ is β-open and A is α-open, it follows from Lemma 3.1 that $(f|_A)^{-1}(V)$ is β-open in the relative topology of A.

Definition 3.1. (i) A space X is said to be *β-connected* [11] if X cannot be expressed as the union of two disjoint non empty β-open sets of X. (ii) A function $f : X \to Y$ is called *β-connected* if the image of every β-connected subset of X is a connected subset of Y.

The following example shows that a slightly β-continuous function is not necessarily β-connected:

Example 3.2. Let X be a set containing three distinct elements p, q, r. For each $x \in X$, let $\sigma_x = \{U \subseteq X : U = \emptyset \text{ or } x \in U\}$ be the corresponding particular point topology. Let $f : (X, \sigma_p) \to (X, \sigma_q)$ be the identity map. Since (X, σ_q) is connected, f is slightly β-continuous. The set $\{p, r\}$ is β-connected in (X, σ_p) as the β-open sets of (X, σ_x) are precisely the open sets. However $f(\{p, r\}) = \{p, r\}$ is not connected in (X, σ_q). It follows that f is not β-connected.

Next we show by the example that a β-connected function need not be slightly β-continuous:

Example 3.3. Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and let s be the usual relative topology on X. Let $Y = \{0, 1\}$ and let t be the discrete topology on Y. Define $f : (X, s) \to (Y, t)$ as $f(1/n) = 0$ for every $n \in \mathbb{N}$ and $f(0) = 1$. It can be seen that the β-open sets in (X, s) are the precisely the open sets. It then follows that f is β-connected but not slightly β-continuous.

Thus we established that slight β-continuity and β-connectedness are independent. Next,we use the concept of a β-open set to define an analogue of a notion of a set connected function introduced by Kwak [4] and set preconnected function introduced by Baker [2].

Definition 3.2. A space X is said to be *β-connected between the subsets A and B of X* provided there is no β-clopen set F for which $A \subseteq F$ and $F \subseteq B = \emptyset$.

Definition 3.3. A function $f : X \to Y$ is said to be *set β-connected* if whenever X is β-connected between subsets A and B of X, then $f(X)$ is connected between $f(A)$ and $f(B)$ with respect to the relative topology on $f(X)$.

Theorem 3.4. *A function $f : X \to Y$ is set β-connected if and only if $f^{-1}(F)$ is β-clopen in X for every clopen set F of $f(X)$ (with respect to the relative topology on $f(X)$).*

Proof: Assume that F is a clopen subset of $f(X)$ with respect to the relative topology on $f(X)$. Suppose that $f^{-1}(F)$ is not β-closed in X. Then there exists $x \in X \setminus f^{-1}(F)$ such that for every β-open set U with $x \in U, U \cap f^{-1}(F) \neq \emptyset$. We claim that X is set β-connected between x and $f^{-1}(F)$. Suppose that there exists a β-clopen set A such that $f^{-1}(F) \subseteq A$ and $x \in A$. Thus $x \in X \setminus A \subseteq X \setminus f^{-1}(F)$ and evidently $X \setminus A$ is a β- open set containing x and disjoint from $f^{-1}(F)$. This contradiction implies that X is set β -connected between x and $f^{-1}(F)$. Since f is set β-connected, $f(X)$ is connected between $f(x)$ and $f(f^{-1}(F))$. But $f(f^{-1}(F)) \subset F$ which is clopen in $f(X)$ and $f(X) \subseteq F$,which is a contradiction. Therefore $f^{-1}(F)$ is β-closed in X and an argument using complements will show that $f^{-1}(F)$ is also β-open.
Conversely suppose that there exists subsets A and B of X for which $f(X)$ is not connected between $f(A)$ and $f(B)$ (in the relative topology on $f(X)$). Thus there is a set $F \subseteq f(X)$ that is clopen in the relative topology on $f(X)$ such that $f(A) \subseteq F$ and $F \cap f(B) = \emptyset$. Then $A \subseteq f^{-1}(F), B \cap f^{-1}(F) = \emptyset$ and $f^{-1}(F)$ is β-clopen which implies that X is not β-connected between A and B. It follows that X is set β-connected.

Theorem 3.5. *Eery set β-connected function is slightly β-continuous.*
Proof: Assume that $f : X \to Y$ is set β-connected. Let F be a clopen subset of Y. Then $F \cap f(X)$ is clopen in the relative topology on $f(X)$. Since f is set β connected, by Theorem 3.4, $f^{-1}(F) = f^{-1}(F \cap f(X))$ is β-clopen in X.

The following result is immediate consequence of Theorem 3.4.

Theorem 3.6. *Every slightly β-continuous surjection is set β-connected.*

Combining Theorems 3.5 and 3.6, we see that within the class of surjective functions, the classes of set β-connected functions and slightly β-continuous functions coincide.

In general,slightly β-continuous function is not necessarily set β-connected as can be seen from the following example:

Example 3.4. Let $X = \{0, 1\}$ and $t = \{\emptyset, X, \{1\}\}$. Let $Y = \{a, b, c\}$ and $s = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, t) \to (Y, s)$ be a function defined as $f(\emptyset) = a$ and $f(1) = b$. Then f is slightly β-continuous by Definition 2.2 but not set β- connected as $\{a\}$ is clopen in the relative topology on $f(X)$ but $f^{-1}(a) = \{0\}$ which is not β-open in (X, t).

Theorem 3.7. *If $f : X \to Y$ is slightly β-continuous surjection and X is β-closed, then Y is mildly compact.*
Proof: Let $\{V_a : a \in \Delta\}$ be a clopen cover of Y. Since f is slightly β- continuous, so $\{f^{-1}(V_a) : a \in \Delta\}$ is a β-clopen cover of X [8, Theorem 3.2] and so there is a finite subset Δ_0 of Δ such that $X = \cup\{f^{-1}(V_a) : a \in \Delta_0\}$. Therefore $Y = \bigcup\{V_a : a \in \Delta_0\}$, since f is surjective. Thus Y is mildly compact.

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