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SMOOTH DEPENDENCE ON RIEMANNIAN METRIC OF EIGENVALUES OF HODGE-DE RHAM OPERATORS

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Abstract. Let M be an oriented, closed, smooth ($= C^{\infty}$) manifold of dimension $n \geq 2$, $A^k(M)$ the space of smooth differential k-forms on M, and $\mathfrak{M}(M)$ the space of all Riemannian metrics on M endowed with the canonical structure of smooth Fréchet manifold (for details, see M. Golubitsky and V.G. Guillemin [8], pp.74-78). Using an idea of J.Wenzelburger [10], [11], we prove that the eigenvalues of the Hodge de-Rham operator $\Delta^{(p)} : A^p(M) \to A^p(M)$ depend smoothly ($= C^{\infty}$) on Riemannian metric $g \in \mathfrak{M}(M)$ for each $k \in \{0, \ldots, n\}$ if on the space $\mathfrak{M}(M)$ of all Riemannian metrics on such manifold is considered the canonical structure of Fréchet smooth manifold. In Corollary 14 it is shown that operators $\delta_g^{(k)} : A^k(M) \to A^{k-1}(M)$ and $\Delta_g^{(k)} : H^2 A^k(M) \to H^0 A^k(M)$ [see Definition 1 (vi) and (vii)] depend smoothly by $g \in \mathfrak{M}(M)$ for each $k \in \{0, \ldots, n\}$. Minimax principle (see Theorem 2.2 of M. Craioveanu, M. Puta, Th.M. Rassias [5], p. 286) and Theorem 6 imply the smoothly dependence on Riemannian metric of eigenvalues of Hodge-de Rham operators and of restrictions of these operators on spaces of differential exact forms, respectively coexact forms (see Corollary 15).

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Finally, another interesting consequence of Theorem 11 is presented in Corollary 16, namely that the same assumptions as those stated above M, Hodge-de Rham decomposition

$$H^0A^k(M) = d^{(k-1)}\left(H^1A^{k-1}(M)\right) \oplus \delta_g^{(k+1)}\left(H^1A^{k+1}(M)\right) \oplus \ker\left(\Delta_g^{(k)}\right)$$

smoothly depends on $g \in \mathfrak{M}(M)$ for each $k \in \{0, \ldots, n\}$, the meaning of the Remark 12.

1. Introduction

Let M be a smooth differential n-dimensional manifold and $A(M) = \sum_{p=0}^{n} A^p(M)$ the exterior algebra of smooth differential forms on M.

Definition 1. *Let M be a manifold with boundary. For assertions (ii), (iii), (vi) and (vii) we assume in addition that M is equipped with a Riemannian metric g.*

(i) *the outer product (or $\wedge$-product) of differential forms is defined by*

$$\wedge : A^k(M) \times A^l(M) \to A^{k+l}(M),$$

$$(\omega \wedge \eta)(X_1, \ldots, X_{k+l}) = \sum_{\sigma \in S(k,k+l)} (\mathrm{sgn}\sigma)\, \omega\left(X_{\sigma(1)}, \ldots, X_{\sigma(k)}\right) \cdot \eta\left(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}\right),$$

where $X_1, \ldots, X_{k+l}$ are arbitrary vector fields on M.

(ii) *Let $(E_1, \ldots, E_n)$ be a local g-orthonormal frame on $U \subset M$. It defines a product on $A^k(M)$, defined locally by*

$$(\ |\ )_{\Lambda^k} : A^k(M) \times A^k(M) \to C^\infty(M),$$

$$(\omega|\eta)_{\Lambda^k(U)} = \sum_{\sigma \in S(k,n)} \omega\left(E_{\sigma(1)}, \ldots, E_{\sigma(k)}\right) \cdot \eta\left(E_{\sigma(1)}, \ldots, E_{\sigma(k)}\right).$$

(iii) *The Hodge star operator $S^{(k)} : A^k(M) \to A^{n-k}(M)$ is defined by equality*

$$\eta \wedge S^{(k)}\omega = (\eta|\omega)_{\Lambda^k(M)}\, \mu,\ \forall \eta \in A^k(M).$$

Here $\mu \in A^n(M)$ is the Riemannian volume form on M.

(iv) *The inner product (or contraction) with a vector field* $Y \in \Gamma(TM)$ *is defined by*

$$i(Y) : A^k(M) \to A^{k-1}(M)$$

$$(i(Y)\omega)(X_1, \dots, X_{k-1}) = \omega(Y, X_1, \dots, X_{k-1}), \ \forall X_1, \dots, X_{k-1} \in \Gamma(TM).$$

(v) *The exterior differential* $d : A^k(M) \to A^{k+1}(M)$*, sometimes denoted* $d^{(k)}$*, is defined (for* $k < n$*) by*

$$d\omega(X_0, X_1, \dots, X_k) = \sum_{0 \le j \le k} (-1)^j D\left[\omega\left(X_0, \dots, \hat{X}_j, \dots, X_k\right)\right](X_j) + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega\left([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k\right).$$

Here $X_0, \dots, X_k$ *are arbitrary vector fields on* M*, and the symbol* $\hat{X}_j$ *expresses that argument* X_j *is missing. For* $\omega \in A^n(M)$*, by definition* $d\omega = 0$*.*

(vi) *The codifferential operator is defined as the application* $\delta : A^k(M) \to A^{k-1}(M)$*, sometimes noted* $\delta^{(k)}$*,*

$$\delta\omega = (-1)^{nk+n+1} S^{(n-k-1)} d\left(S^{(k)}\omega\right), \omega \in A^k(M).$$

(vii) *The Hodge-de Rham operator is defined as the application* $\Delta^{(k)} : A^k(M) \to A^k(M)$*,*

$$\Delta^{(k)}\omega = (d\delta + \delta d)\omega, \ \omega \in A^k(M).$$

From Definition 1. it follows that:

$$S^{(0)}(1) = v_g, S^{(n)}(v_g) = 1 \tag{1}$$

where v_g denotes the canonical volume form of (M, g), 1 is the real constant function on M having value 1 and

$$S^{(n-p)} \circ S^{(p)}(\omega) = (-1)^{p(n-p)} \omega \text{ for any } \omega \in A^p(M). \tag{2}$$

Remark 2. *Note that* $\Delta^{(0)}$ *is just the Laplace-Beltrami operator* $\Delta : C^\infty(M) \to C^\infty(M)$*.*

Proposition 3. *The operator* $\Delta^{(p)}$*, for each* $0 \le p \le n$*, has the following properties:*

(i) $\Delta^{(p)}$ *is formally self adjoint;*
(ii) $\Delta^{(p)}$ *is formally positive, i.e.* $\left\langle \Delta^{(p)}\alpha, \alpha \right\rangle \ge 0$ *for any* $\alpha \in A^p(M)$*;*
(iii) $\Delta^{(p)}\alpha = 0$ *if and only if* $d^{(p)}\alpha = 0$ *and* $\delta^{(p)}\alpha = 0$*;*

(iv) $\Delta^{(n-p)} S^{(p)} = S^{(p)} \Delta^{(p)}$.

Proof. (i)

$$\begin{aligned}\left\langle \Delta^{(p)}\alpha, \beta \right\rangle &= \left\langle d^{(p-1)} \circ \delta^{(p)}\alpha + \delta^{(p+1)} \circ d^{(p)}\alpha, \beta \right\rangle = \\ &= \left\langle \delta^{(p)}\alpha, \delta^{(p)}\beta \right\rangle + \left\langle d^{(p)}\alpha, d^{(p)}\beta \right\rangle = \\ &= \left\langle \alpha, d^{(p-1)} \circ \delta^{(p)}\beta \right\rangle + \left\langle \alpha, \delta^{(p+1)} \circ d^{(p)}\beta \right\rangle \\ &= \left\langle \alpha, \Delta^{(p)}\beta \right\rangle\end{aligned}$$

for each $\alpha, \beta \in A^p(M)$.

(ii) $\left\langle \Delta^{(p)}\alpha, \alpha \right\rangle = \left\langle d^{(p)}\alpha, d^{(p)}\alpha \right\rangle + \left\langle \delta^{(p)}\alpha, \delta^{(p)}\alpha \right\rangle = \left\| d^{(p)}\alpha \right\|^2 + \left\| \delta^{(p)}\alpha \right\|^2 \geq 0$ for each $\alpha \in A^p(M)$.

(iii) If $\Delta^{(p)}\alpha = 0$, the equality $\left\langle \Delta^{(p)}\alpha, \alpha \right\rangle = \left\langle d^{(p)}\alpha, d^{(p)}\alpha \right\rangle + \left\langle \delta^{(p)}\alpha, \delta^{(p)}\alpha \right\rangle$ implies that $\left\langle d^{(p)}\alpha, d^{(p)}\alpha \right\rangle + \left\langle \delta^{(p)}\alpha, \delta^{(p)}\alpha \right\rangle = 0$, or equivalently $d^{(p)}\alpha = 0$ and $\delta^{(p)}\alpha = 0$. Conversely, if $d^{(p)}\alpha = 0$ and $\delta^{(p)}\alpha = 0$, then by the definition of $\Delta^{(p)}$, $\Delta^{(p)}\alpha = 0$.

(iv) We shall consider two cases:

Case 1: n is even. Then we can write

$$\begin{aligned}\Delta^{(n-p)} = &- \big(d^{(n-p-1)} \circ S^{(p+1)} \circ d^{(p)} \circ S^{(n-p)} \\ &+ S^{(p)} \circ d^{(p-1)} \circ S^{(n-p+1)} \circ d^{(n-p)}\big)\end{aligned}$$

and therefore using the relation (2),

$$\begin{aligned}S^{(p)} \circ \Delta^{(p)} &= - \big(S^{(p)} \circ d^{(p+1)} \circ S^{(n-p+1)} \circ d^{(n-p)} \circ S^{(p)} \\ &+ S^{(p)} \circ S^{(n-p)} \circ d^{(n-p-1)} \circ S^{(p+1)} \circ d^{(p)}\big) \\ &= - \big[-\delta^{(n-p+1)} \circ d^{(n-p+1)} \circ S^{(p)} \\ &+ (-1)^{2p(n-p)} (-1)^{(n-p-1)n+1} d^{(n-p-1)} \circ \delta^{(n-p)} \circ S^{(p)}\big] \\ &= \Delta^{(n-p)} \circ S^{(p)}.\end{aligned}$$

Case 2: n is odd. Then

$$\Delta^{(n-p)} \circ S^{(p)} = d^{(n-p-1)} \circ \delta^{(n-p)} \circ S^{(p)} + \delta^{(n-p+1)} \circ d^{(n-p)} \circ S^{(p)}$$

and therefore using the Definition 1 and relation (2),

$$\begin{aligned}
\Delta^{(n-p)} \circ S^{(p)} &= (-1)^{n-p} d^{(n-p-1)} \circ S^{(p+1)} \circ d^{(p)} \\
&+ (-1)^{n-p+1} S^{(p)} \circ d^{(p+1)} \circ S^{(n-p+1)} \circ d^{(n-p)} \circ S^{(p)} \\
&= (-1)^{n-p} (-1)^{2p^2+p(n-p)+1} S^{(p)} \circ \delta^{(p+1)} \circ d^{(p)} \\
&+ (-1)^{n-p+1} (-1)^{p} S^{(p)} \circ d^{(p+1)} \circ \delta^{(p)} = \\
&= S^{(n-p)} \circ \left(\delta^{(n-p+1)} \circ d^{(n-p)} + d^{(n-p+1)} \circ \delta^{(n-p)}\right) \\
&= S^{(p)} \circ \Delta^{(p)}.
\end{aligned}$$

as desired. Q.E.D.

□

Remark 4. *The normalization of $\wedge$-product is chosen using the convention used by Abraham, Marsden and Ratiu [1].*

Proposition 5. (i) *The outer differential operator and codifferential operator are nilpotents, ie*

$$d\,(d\omega) = 0 \text{ and } \delta\,(\delta\omega) = 0,\ \forall\omega \in A^k\,(M)\,; \tag{3}$$

(ii) *The Hodge star operator is idempotent, ie*

$$S^{(n-k)}\left(S^{(k)}\omega\right) = (-1)^{k(n-k)}\,\omega,\ \forall\omega \in A^k\,(M)\,; \tag{4}$$

(iii) *The operators d and δ are in Hodge sense, adjoint each other, ie*

$$S^{(k-1)}\delta\omega = (-1)^k\, dS^{(k)}\omega,\ \text{ and } S^{(k+1)}d\omega = (-1)^{k+1}\,\delta S^{(k)}\omega,\ \forall\omega \in A^k\,(M)\,; \tag{5}$$

(iv) *If $(E_1,\ldots,E_n)$ is a local g-orthonormal frame on $U \subset M$ and $\sigma \in S\,(k,n)$ the Hodge star operator is calculated as*

$$\left(S^{(k)}\omega\right)\left(E_{\sigma(k+1)},\ldots,E_{\sigma(n)}\right) = (\mathrm{sgn}\sigma)\,\omega\left(E_{\sigma(1)},\ldots,E_{\sigma(k)}\right),\ \forall\omega \in A^k\,(M)\,. \tag{6}$$

Property (iv) can be deduced from the definition of Hodge star operator noting that

$$\sum_{\sigma\in S(k,n)} \eta\left(E_{\sigma(1)},\ldots,E_{\sigma(k)}\right)\cdot\omega\left(E_{\sigma(1)},\ldots,E_{\sigma(k)}\right) = \left(\eta\wedge S^{(k)}\omega\right)(E_1,\ldots,E_n)$$

$$= \sum_{\sigma\in S(k,n)} (\mathrm{sgn}\sigma)\,\eta\left(E_{\sigma(1)},\ldots,E_{\sigma(k)}\right)\cdot\left(S^{(k)}\omega\right)\left(E_{\sigma(k+1)},\ldots,E_{\sigma(n)}\right).$$

Since $\forall\omega \in A^k(M)$ it is arbitrary follows that (6) is true for any $\sigma \in S(k,n)$.

The following theorem gives a precise characterization of eigenvalues, $\lambda'_{k,p}(M,g)$, $k \in \mathbb{N}$, which not involves derivatives of Riemannian metric g.

Theorem 6. *(Dodziuk [7]) Let $g \in \mathfrak{M}(M)$ and $p \in \{1,\dots,n\}$ fixed. Let*

$$0 < \lambda'_{1,p}(M,g) \le \lambda'_{2,p}(M,g) \le \dots$$

be the eigenvalues of the restriction

$$\Delta^{(p)}_{g|_{d^{(p-1)}(A^{p-1}(M))}} : d^{(p-1)}\left(A^{p-1}(M)\right) \to d^{(p-1)}\left(A^{p-1}(M)\right)$$

of $\Delta^{(p)}_g$ to the real vector space of exact differential p-forms $d^{(p-1)}(A^{p-1}(M))$, counted with their multiplicity. Then

$$\lambda'_{k,p} = \inf_{V_k}\sup\left\{\frac{\left\|d^{(p-1)}\theta\right\|_g^2}{\|\theta\|_g^2}\,\middle|\,d^{(p-1)}\theta \in V_k\setminus\{0\}\right\} \tag{7}$$

where V_k through the family of all k-dimensional real vector subspace of it.

Proof. Let us note first that taking supremum in (7) can be done in two stages. For each exact differential p-form let choose $\theta \in A^{p-1}(M)\setminus\{0\}$ to maximize quotient $\dfrac{\left\|d^{(p-1)}\theta\right\|_g}{\|\theta\|_g}$. Let choose $\theta \in A^{p-1}(M)\setminus\{0\}$ arbitrarily, with Hodge-de Rham decomposition

$$\theta = H^{(p-1)}(\theta) + d^{(p-2)}\omega_1 + \delta^{(p)}_g\omega_2,\ \omega_1 \in A^{p-2}(M),\ \omega_2 \in A^p(M),$$

where $H^{(p-1)}$ denotes the harmonic projector, and be $\theta_0 := \delta^{(p)}_g\omega_2 \in \delta^{(p)}_g(A^p(M))$. Therefore,

$$\inf_{V_k}\sup\left\{\frac{\left\|d^{(p-1)}\theta\right\|_g^2}{\|\theta\|_g^2}\,\middle|\,d^{(p-1)}\theta \in V_k\setminus\{0\}\right\}$$

$$= \inf_{V_k}\sup\left\{\frac{\left\|d^{(p-1)}\theta\right\|_g^2}{\|\theta_0\|_g^2}\,\middle|\,d^{(p-1)}\theta \in V_k\setminus\{0\}\right\}$$

$$= \inf_{W_k} \sup \left\{ \frac{\left\| d^{(p-1)}\theta_0 \right\|_g}{\left\| \theta_0 \right\|_g} \left| d^{(p-1)}\theta \in W_k \backslash \{0\} \right. \right\} \tag{8}$$

where W_k through the set of all k-dimensional linear subspace of $\delta_g^{(p)} A^p (M)$.

Because

$$\delta_g^{(p)}\theta_0 = \left(\delta_g^{(p-1)} \circ \delta_g^{(p)}\right)(\omega_2) = 0$$

and

$$d^{(p-1)}\theta = d^{(p-1)}\left(H^{(p-1)}(\theta) + d^{(p-2)}\omega_1 + \delta_g^{(p)}\omega_2\right) = d^{(p-1)}\theta_0,$$

it follows that

$$\left\| d^{(p-1)}\theta \right\|_g^2 = \left\| d^{(p-1)}\theta_0 \right\|_g^2 = \left\| d^{(p-1)}\theta_0 \right\|_g^2 + \left\| \delta_g^{(p-1)}\theta_0 \right\|_g^2.$$

Therefore, the right side of (8) gives the mini-max characterization of the k eigenvalues $\lambda''_{k,p-1}(M,g)$ of the restriction of $\Delta_g^{(p-1)}$ to the vector space of differential coexact $(p-1)$-forms, which coinciding with the k eigenvalue $\lambda'_{k,p}(M,g)$ of the restriction of $\Delta_g^{(p)}$ to the p-forms exact differential space. Q.E.D. □

Let $\mathbb{H}$ be a real Hilbert space with the inner product $\langle , \rangle_0 : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ and $\| \cdot \|_0$ the induced norm. Let $\mathcal{G}(\mathbb{H})$ be the family of all closed vector subspaces of $\mathbb{H}$.

If $\mathbb{E}, \mathbb{F} \in \mathcal{G}(\mathbb{H})$ are fixed, let $L(\mathbb{E}, \mathbb{F})$ be the real vector space of all the bounded linear operators from $\mathbb{E}$ into $\mathbb{F}$. With respect to the canonical norm of a bounded linear operator from the Hilbert space $\mathbb{E}$ into the Hilbert space $\mathbb{F}$, still denoted with $\| \cdot \|_0$, $L(\mathbb{E}, \mathbb{F})$ is a Banach space. For each $\mathbb{E} \in \mathcal{G}(\mathbb{H})$ fixed, let

$$\mathcal{G}_{\mathbb{E}} := \{\mathbb{F} \in \mathcal{G}(\mathbb{H}) | \quad \mathbb{H} = \mathbb{E} \oplus \mathbb{F}\}$$

be the set of all closed complements of $\mathbb{E}$ in $\mathbb{H}$ and let us notice that $\mathbb{E}^{\perp}$(= the orthogonal complement of $\mathbb{E}$ in $\mathbb{H}$ with respect to the inner product $\langle , \rangle_0$) $\in \mathcal{G}_{\mathbb{E}}$. Let

$$\mathcal{P}(\mathbb{H}, \mathbb{E}) := \{\pi \in L(\mathbb{H}) := L(\mathbb{H}, \mathbb{H}) | \quad \pi \circ \pi = \pi \quad \text{and} \quad Im(\pi) = \mathbb{E}\}$$

be the space of all continuous projections of $\mathbb{H}$ onto $\mathbb{E} \in \mathcal{G}(\mathbb{H})$ endowed with the relative topology induced by the canonical topology on $L(\mathbb{H})$.

Lemma 7. *(see [11]). If the subspace $\mathbb{E} \in \mathcal{G}(\mathbb{H})$ is fixed, then the map*

$$Ker : \mathcal{P}(\mathbb{H}, \mathbb{E}) \to \mathcal{G}_{\mathbb{E}}, \pi \mapsto Ker(\pi),$$

is a bijection.

For $\mathbb{F}_0, \mathbb{F} \in \mathcal{G}_{\mathbb{E}}$, let $\pi_0, \pi \in \mathcal{P}(\mathbb{H}, \mathbb{E})$ so that $Ker(\pi_0) = \mathbb{F}_0$ and $Ker(\pi) = \mathbb{F}$ (see Lemma 7). If the closed vector subspace $\mathbb{F}_0$ of $\mathbb{H}$ is fixed, the map $\varphi_{\mathbb{F}_0,\mathbb{E}} : \mathcal{G}_{\mathbb{E}} \to L(\mathbb{F}_0, \mathbb{E})$, which associates to each vector subspace $\mathbb{F} \in \mathcal{G}_{\mathbb{E}}$ the map from $\mathbb{F}_0$ into $\mathbb{E}$ having as graph the subspace $\mathbb{F}$ of $\mathbb{H} = \mathbb{F}_0 \oplus \mathbb{E}$, is a bijection. Moreover, the set of all charts of the type $\{(\mathcal{G}_{\mathbb{E}}, \varphi_{\mathbb{F}_0,\mathbb{E}}, L(\mathbb{F}_0, \mathbb{E}))| \quad \mathbb{F}_0, \mathbb{E} \in \mathcal{G}(\mathbb{H}), \mathbb{H} = \mathbb{F}_0 \oplus \mathbb{E}\}$ is a smooth atlas for $\mathcal{G}(\mathbb{H})$. Endowed with the Banach smooth manifold structure defined by this atlas, $\mathcal{G}(\mathbb{H})$ is called the Grassmann manifold associated to the Hilbert space $\mathbb{H}$ (see N. Bourbaki [3], p. 38). In addition, one can show that the topological space $\mathcal{G}(\mathbb{H})$ is metrisable.

Lemma 8. *Let $\mathbb{E} \in \mathcal{G}(\mathbb{H})$ be fixed and $\mathcal{G}_{\mathbb{E}}$ with the C^∞-manifold structure induced by the one previously defined on $\mathcal{G}(\mathbb{H})$. Then there is a unique C^∞-manifold structure on $\mathcal{P}(\mathbb{H}, \mathbb{E})$, whose subjacent topology coincides with the one induced on $\mathcal{P}(\mathbb{H}, \mathbb{E})$ by the real Banach space structure of $L(\mathbb{H})$, so that the bijection*

$$Ker : \mathcal{P}(\mathbb{H}, \mathbb{E}) \to \mathcal{G}_{\mathbb{E}}, \pi \mapsto Ker(\pi),$$

(see Lemma 7) is a C^∞-diffeomorphism.

For the proof, see E.Binz, J.Śniatycki and H.Fischer [2].

Let us denote by $L^2_{sim}(\mathbb{H}; \mathbb{R})$ the real vector space of all symmetric and continuous $\mathbb{R}$-bilinear forms $\beta : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$. With respect to the supremum norm

$$\|\beta\| := sup\{\frac{|\beta(u,v)|}{\|u\|_0 \|v\|_0}| \quad u, v \in \mathbb{H} \setminus \{0\}\},$$

$L^2_{sim}(\mathbb{H}; \mathbb{R})$ is a real Banach space. Let $\mathfrak{M}(\mathbb{H}) \subset L^2_{sim}(\mathbb{H}; \mathbb{R})$ be the set of all inner products on $\mathbb{H}$ which are continuous with respect to the topology induced by $g_0 := \langle , \rangle_0$ on $\mathbb{H}$ and let us notice that $\mathfrak{M}(\mathbb{H})$ is a non-empty open subset [since $g_0 \in \mathfrak{M}(\mathbb{H})$] of $L^2_{sim}(\mathbb{H}; \mathbb{R})$.

Lemma 9. *(see [11]). Let $\mathbb{H}$ be a real Hilbert space with the inner product g_0. Then, for each $g \in \mathfrak{M}(\mathbb{H})$, the topologies induced on $\mathbb{H}$ by g and g_0, respectively, coincide. In particular, $\mathbb{H}$ is a complete metric space with respect to each inner product $g \in \mathfrak{M}(\mathbb{H})$.*

If the subspace $\mathbb{E} \in \mathcal{G}(\mathbb{H})$ is fixed, then one agrees to denote the subspace $\mathbb{F} \in \mathcal{G}_{\mathbb{E}}$ which is orthogonal on $\mathbb{E}$ with respect to the inner product $g \in \mathfrak{M}(\mathbb{H})$ with $\mathbb{F}_g$ and also to call $\mathbb{F}_g$ the g-orthogonal complement of $\mathbb{E}$ in $\mathbb{H}$: $\mathbb{H} = \mathbb{F}_g \oplus \mathbb{E}$ and $g(u,v) = 0$ for any $u \in \mathbb{F}_g$ and any $v \in \mathbb{E}$.

Definition 10. *If the subspace $\mathbb{E} \in \mathcal{G}(\mathbb{H})$ and the inner product $g \in \mathfrak{M}(\mathbb{H})$ are fixed, then the orthogonal projection $\pi \in \mathcal{P}(\mathbb{H}, \mathbb{E})$ of $\mathbb{H} = \mathbb{F}_g \oplus \mathbb{E}$ onto $\mathbb{E}$, denoted with π_g, i.e. $Ker(\pi_g) = \mathbb{F}_g$ (see Lemma 7), is called the g-orthogonal projection of $\mathbb{H}$ onto $\mathbb{E}$.*

Using the Lemmas 8 and 9 it follows the next theorem.

Theorem 11. *Let $\mathbb{H}$ be a real Hilbert space, $\mathbb{E} \in \mathcal{G}(\mathbb{H})$ a fixed subspace and $\mathbb{F}_g$ the g-orthogonal complement of $\mathbb{E}$ in $\mathbb{H}$, where $g \in \mathfrak{M}(\mathbb{H})$. Then the g-orthogonal projection π_g of $\mathbb{H}$ onto $\mathbb{E}$ depends smoothly on the inner product $g \in \mathfrak{M}(\mathbb{H})$, that is the map $\mathfrak{M}(\mathbb{H}) \ni g \mapsto \pi_g \in \mathcal{P}(\mathbb{H}, \mathbb{E}) \subset L(\mathbb{H})$ is of class C^∞ (meaning Fréchet differentiability).*

Proof. Lemma 9 shows that all the topologies induced on $\mathbb{H}$ by inner products $g \in \mathfrak{M}(\mathbb{H})$ coincide. Let $g_0 \in \mathfrak{M}(\mathbb{H})$ fixed. As we have already shown during the demonstration of Lemma 9, for each $g \in \mathfrak{M}(\mathbb{H})$ there is an unique $\mathbb{R}$-linear autoadjunct operator so that

$$g\left(u, v\right) = g_0\left(u, A_{g_0 g}\left(v\right)\right) \tag{9}$$

for any $u, v \in \mathbb{H}$. Moreover, $A_{g_0 g} : \mathbb{H} \to \mathbb{H}$ is a omeomorfism for each $g \in \mathfrak{M}(\mathbb{H})$. Therefore $A_{g_0 g}(\mathbb{E})$ is a closed vector subspace of $\mathbb{H}$ and - on the basis of (9) - applications

$$_{\mathbb{F}_{g_0}}\left|A_{g_0 g}\right|_{\mathbf{F}_g} : \mathbb{F}_g \to \mathbb{F}_{g_0} \text{ and } _{A_{g_0 g}(\mathbb{E})}\left|A_{g_0 g}\right|_{\mathbb{E}} : \mathbb{E} \to A_{g_0 g}\left(\mathbb{E}\right) \tag{10}$$

are $\mathbb{R}$-linear isomorphisms and homeomorphisms. If $\pi_{g_0} \in \mathcal{P}(\mathbb{H}, \mathbb{E})$ notes the g_0-orthogonal projection of $\mathbb{H}$ on $\mathbb{E}$ (see Definition 10), so that $\mathrm{Ker}\left(\pi_{g_0}\right) = \mathbb{F}_{g_0}$, then

$$\mathbb{B}_g := A_{g_0 g}^{-1} \circ \pi_{g_0} \circ A_{g_0 g} \in \mathcal{P}\left(\mathbb{H}, A_{g_0 g}\left(\mathbb{E}\right)\right) \subset L\left(\mathbb{H}\right)$$

and $\mathrm{Ker}\left(\mathbb{B}_g\right) = \mathbb{F}_g$. Since applications

$$\mathfrak{M}(\mathbb{H}) \ni g \mapsto A_{g_0 g} \in L\left(\mathbb{H}\right) \text{ and } \mathfrak{M}(\mathbb{H}) \ni g \mapsto A_{g_0 g}^{-1} \in L\left(\mathbb{H}\right)$$

are of class C^∞ (in the Fréchet sense) [$\mathfrak{M}(\mathbb{H}) \neq \Phi$ is an open subset of Banach space $L_{sim}^2\left(\mathbb{H}; \mathbb{R}\right)$ and $L\left(\mathbb{H}\right)$ is a Banach space] and the composition of C^∞-applications between Banach spaces is all the C^∞-class (see for example M. Craioveanu, T.S. Ratiu [6]) results that the application $\mathfrak{M}(\mathbb{H}) \ni g \mapsto \mathbb{B}_g \in L\left(\mathbb{H}\right)$ is the C^∞-class. Because applications (10) are $\mathbb{R}$-linear isomorphisms and homeomorphisms, $\mathbb{F}_g \in \mathbb{G}_{\mathbb{E}} \cap \mathbb{G}_{A_{g_0 g}^{-1}(\mathbf{E})}$ for any $g \in \mathfrak{M}(\mathbb{H})$. On the other hand, in the basis of Lemma 8, the application

$$\mathcal{P}\left(\mathbb{H}, A_{g_0 g}^{-1}\left(\mathbb{E}\right)\right) \ni \pi \xrightarrow{\mathrm{Ker}} \mathrm{Ker}\left(\pi\right) \in \mathbb{G}_{A_{g_0 g}^{-1}(\mathbb{E})}$$

is C^∞-difeomorfism for any $g \in \mathfrak{M}(\mathbb{H})$. Since $\mathrm{Ker}\,(\mathbb{B}_g) = \mathbb{F}_g = \mathrm{Ker}\,(\pi_g)$ for any $g \in \mathfrak{M}(\mathbb{H})$, the application

$$\mathfrak{M}(\mathbb{H}) \ni g \mapsto \pi_g \in \mathcal{P}\,(\mathbb{H}, \mathbb{E}) \subset L\,(\mathbb{H})$$

is C^∞- class. Q.E.D. □

Remark 12. *Since the map $\mathfrak{M}(\mathbb{H}) \ni g \mapsto \pi_g \in \mathcal{P}(\mathbb{H}, \mathbb{E}) \subset L(\mathbb{H})$ is smooth (see Theorem 11), in this case one may also say that the orthogonal decomposition $\mathbb{H} = \mathbb{F}_g \oplus \mathbb{E}$ depends smoothly on $g \in \mathfrak{M}(\mathbb{H})$.*

Let M be again an oriented, closed n-dimensional ($n \geq 2$), C^∞-manifold, $A^k(M)$ the space of smooth differential k-forms on M, $k \in \{0, 1, \ldots, n\}$, and $\mathfrak{M}(M)$ the set of all smooth Riemannian metrics on M, endowed with the smooth Fréchet manifold structure. Using Riesz's representation theorem, for any $g_o, g \in \mathfrak{M}(M)$, it follows that there is a smooth automorphism of vector bundles $\Phi_{g_o g} : TM \to TM$ such that

$$g(X, Y) = g_o(\Phi_{g_o g} \circ X, \Phi_{g_o g} \circ Y), \tag{11}$$

for any $X, Y \in \mathfrak{X}(M)$. The automorphism $\Phi_{g_o g}$ is uniquely determined modulo an isometry of (M, g_o) and the maps

$$\mathfrak{M}(M) \ni g \mapsto \Phi^*_{g_o g} \in L(L^2(A^k(M))), \quad \mathfrak{M}(M) \ni g \mapsto (\Phi^*_{g_o g})^{-1} \in L(L^2(A^k(M))),$$

induced by $\Phi_{g_o g}$, are smooth for any $k \in \{0, 1, \ldots, n\}$ (for further details, see E.Binz, J.Śniatycki and H.Fischer [2]).

Lemma 13. *Let $g_o, g \in \mathfrak{M}(M)$ be arbitrary, but fixed, Riemannian metrics and $S^{(k)}_{g_o}$ (respectively $S^{(k)}_g$) $: A^k(M) \to A^{n-k}(M)$ the star Hodge operator associated to g_o (respectively g), $k \in \{0, 1, \ldots, n\}$. Under the previous assumptions, the following equality*

$$\Phi^*_{g_o g} \circ S^{(k)}_{g_o} = S^{(k)}_g \circ \Phi^*_{g_o g} : L^2(A^k(M)) \to L^2(A^{n-k}(M))$$

is valid for any $k \in \{0, 1, \ldots, n\}$. In particular, the map

$$\mathfrak{M}(M) \ni g \mapsto S^{(k)}_g \in L(L^2(A^k(M)), L^2(A^{n-k}(M))),$$

is smooth for any $k \in \{0, 1, \ldots, n\}$.

Proof. If $(E_1, \ldots, E_n)$ is a local g-orthonormal arbitrary frame on M, then - on the basis of (11) -$\Phi_{g_0 g} \circ E_1, \ldots, \Phi_{g_0 g} \circ E_n$ is a local g_0-orthonormal frame on M. In the basis of Proposition 5 (iv) true and for Riemannian manifold without boundary, result that

$$\begin{aligned}
&\left(\left(\Phi^*_{g_0 g} \circ S^{(k)}_{g_0}\right)(\omega)\right)\left(E_{\sigma(k+1)}, \ldots, E_{\sigma(n)}\right) \\
&= \left(\Phi^*_{g_0 g}\left(S^{(k)}_{g_0}(\omega)\right)\right)\left(E_{\sigma(k+1)}, \ldots, E_{\sigma(n)}\right) \\
&= \left(S^{(k)}_{g_0}(\omega)\right)\left(\Phi_{g_0 g} \circ E_{\sigma(k+1)}, \ldots, \Phi_{g_0 g} \circ E_{\sigma(n)}\right) \\
&= \operatorname{sgn}(\sigma)\, \omega\left(\Phi_{g_0 g} \circ E_{\sigma(1)}, \ldots, \Phi_{g_0 g} \circ E_{\sigma(k)}\right) \\
&= \operatorname{sgn}(\sigma)\left(\Phi^*_{g_0 g}(\omega)\right)\left(E_{\sigma(1)}, \ldots, E_{\sigma(k)}\right) \\
&= \left(S^{(k)}_{g}\left(\Phi^*_{g_0 g}(\omega)\right)\right)\left(E_{\sigma(k+1)}, \ldots, E_{\sigma(n)}\right)
\end{aligned}$$

for any $\omega \in A^k(M)$ and every $\sigma \in S(k,n)$ note the set of all permutations σ of the set $\{1, \ldots, n\}$ so that $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(1) < \ldots < \sigma(k)$. Therefore, in the basis of the considerations set out preceding this lemma, the application

$$\mathfrak{M}(M) \ni g \mapsto S^{(k)}_{g} = \Phi^*_{g_0 g} \circ S^{(k)}_{g_0} \circ \left(\Phi^*_{g_0 g}\right)^{-1} \in L\left(A^k(M), A^{n-k}(M)\right)$$

is smooth for all $k \in \{0, \ldots, n\}$. Q.E.D.

□

The Hodge star operator $S^{(k)}_{g_o}$(respectively $S^{(k)}_{g}$) associated to the Riemannian metric g_o (respectively g) $\in \mathfrak{M}(M)$ induces the inner product $\langle , \rangle_{g_o}$ (respectively $\langle , \rangle_{g}$) on the Hilbert space $L^2(A^k(M)) = H^0(A^k(M))$, hence:

$$\langle \omega_1, \omega_2 \rangle_{g_o} := \int_M \omega_1 \wedge S^{(k)}_{g_o}(\omega_2) = (-1)^{k(n-k)} \int_M \omega_1 \wedge S^{(k)}_{g}(S^{(n-k)}_{g} \circ S^{(k)}_{g_o})(\omega_2)$$

$$=: \langle \omega_1, A^{(k)}_{g g_o}(\omega_2) \rangle_g \tag{12}$$

for any $\omega_1, \omega_2 \in A^k(M)$, where

$$A^{(k)}_{g g_o} : A^k(M) \to A^k(M), \quad A^{(k)}_{g g_o} := (-1)^{k(n-k)} S^{(n-k)}_{g} \circ S^{(k)}_{g_o}, \tag{13}$$

$k \in \{0, 1, \ldots, n\}$.

$A^{(k)}_{g g_o}$ is a continuous and formally self-adjoint (symmetric) $\mathbb{R}$-linear operator, which can be extended to the Hilbert space $L^2(A^k(M)) =: H^0(A^k(M))$. Therefore, (12) shows that all Riemannian metrics $g \in \mathfrak{M}(M)$ induce the same topology on $L^2(A^k(M)) =: H^0(A^k(M))$, for each $k \in \{0, 1, \ldots, n\}$. The same property is also true for the Sobolev spaces $H^1(A^k(M))$ and $H^2(A^k(M))$ for each $k \in \{0, 1, \ldots, n\}$.

Lemma 13 and the definitions of the codifferential $\delta_g^{(k)} : A^k(M) \to A^{k-1}(M)$ and of the Hodge-de Rham operator $\Delta_g^{(k)} : H^2(A^k(M)) \to H^0(A^k(M))$ therefore lead to the following results.

Corollary 14. *Let M be a closed, n-dimensional smooth manifold and $H^s(A^k(M))$ the Sobolev space of class H^s, $s \in \{0,1,2\}$, associated to the pre-Hilbertian vector space $A^k(M)$, $k \in \{0,1,\dots,n\}$. Then, the next two statements are true:*

(i) The map

$$\mathfrak{M}(M) \ni g \mapsto \delta_g^{(k)} \in L(H^1(A^k(M)), H^0(A^k(M)))$$

is smooth for each $k \in \{0,1,\dots,n\}$, i.e. the codifferential $\delta_g^{(k)}$ smoothly depends on $g \in \mathfrak{M}(M)$ for each $k \in \{0,1,\dots,n\}$;

(ii) The map

$$\mathfrak{M}(M) \ni g \mapsto \Delta_g^{(k)} := d^{(k-1)} \circ \delta_g^{(k)} + \delta_g^{(k+1)} \circ d^{(k)} \in L(H^2(A^k(M)), H^0(A^k(M)))$$

is smooth for each $k \in \{0,1,\dots,n\}$, that is the Hodge-de Rham operator $\Delta_g^{(k)}$ smoothly depends on $g \in \mathfrak{M}(M)$ for each $k \in \{0,1,\dots,n\}$.

Let us also remark that still Lemma 13 shows that the map

$$\mathfrak{M}(M) \ni g \mapsto A_{gg_o}^{(k)} \in L(H^0(A^k(M))),$$

where $A_{gg_o}^{(k)}$ is the $\mathbb{R}$-linear operator defined by the equality (13), smoothly depends on $g \in \mathfrak{M}(M)$, for each $k \in \{0,1,\dots,n\}$ so that Corollary 14 (i), the minimax principle (see Theorem 2.2 of M.Craioveanu, M.Puta, Th.M.Rassias [5], p.286) and Theorem 6 imply the following result regarding the smooth dependence on the Riemannian metric of the eigenvalues of Hodge-de Rham operators and of the eigenvalues of their restrictions to the spaces of exact and co-exact, smooth differential forms on M respectively.

Corollary 15. *If M is a closed, n-dimensional smooth manifold, $\lambda_{j,k}(M,\cdot)$,
$\lambda'_{j,k}(M,\cdot)$, $\lambda''_{j,k}(M,\cdot) : \mathfrak{M}(M) \to \mathbb{R}$, are the real functions given by the eigenvalues of Hodge-de Rham operator $\Delta^{(k)}$ and the eigenvalues of the restriction of $\Delta^{(k)}$ to the space of exact (resp. co-exact) smooth differential k-forms on M, $j \in \mathbb{N}$ and $k \in \{0,1,\dots,n\}$, then those functions are smooth with respect to the canonical Fréchet manifold structure considered on $\mathfrak{M}(M)$.*

Finally, in this context, we mention another interesting consequence of Theorem 11.

Corollary 16. *Under the same assumptions as those stated in corollary 15, the Hodge-de Rham decomposition*

$$H^0A^k(M) = d^{(k-1)}\left(H^1A^{k-1}(M)\right) \oplus \delta_g^{(k+1)}\left(H^1A^{k+1}(M)\right) \oplus \ker\left(\Delta_g^{(k)}\right)$$

(see Theorem 1.3.4 [9]) smoothly depends of $g \in \mathfrak{M}(M)$ for each $k \in \{0, \dots, n\}$, the meaning of the Remark 12.

Proof. In fact, to note that Hilbert space $\mathbb{E} := d^{(k-1)}\left(H^1A^{k-1}(M)\right)$ does not depend on the choice of Riemannian metric on M, so our assertion is an immediate consequence of Theorem 11, where we considered

$$\mathbb{F}_g := \delta_g^{(k+1)}\left(H^1A^{k+1}(M)\right) \oplus \ker\left(\Delta_g^{(k)}\right).$$

Q.E.D. □

Remark 17. *The Fréchet manifold topology of $\mathfrak{M}(M)$is just the C^∞-topology on $\mathfrak{M}(M)$, so that Corollary 15 includes in particular the continuity property of the real functions $\lambda_{j,k}(M,\cdot)$, $\lambda'_{j,k}(M,\cdot)$, $\lambda''_{j,k}(M,\cdot) : \mathfrak{M}(M) \to \mathbb{R}$ with respect to this topology for each $j \in \mathbb{N}$ and $k \in \{0, 1, \dots, n\}$ (see M.Craioveanu and M.Puta* [4]*, M.Craioveanu, M.Puta and Th.M.Rassias* [5]*).*

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