"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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CONTROLLING CHAOTIC DYNAMICAL SYSTEMS THROUGH FIXED POINT ITERATIVE TECHNIQUES

VASILE BERINDE

Abstract. An extremely simple and efficient controlling mechanism has been developed to stabilize discrete dynamical systems. The new technique is essentially based on considering controllers taken from typical fixed point iterative methods. Theoretical analysis as well as computer simulations have been provided to show the simplicity, great power, effectiveness and efficiency of this new method in practice.

1. Introduction

In recent years, deterministic chaos has been observed when applying simple models to various phenomena in nature and science: population dynamics, chemical reactions, electronic circuits, cardiology, laser technology etc.

One of the topics related to chaotic dynamical systems has been the development of techniques for the control of chaotic phenomena. Some of the basic methods of controlling chaos are summarized in Lynch [15], where a selection of various applications of chaos control in the real world are listed, see also Ditto et al. [7], Chan [6]; Ott et al. [17] etc. Stabilizing unstable dynamical systems through feedback adjustment methods have dominated the recent research in the field of chaos control, see Huang [13] and references therein.

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A nonlinear feedback mechanism through controlling the growth rate has been developed by Huang [13]. This method has been shown theoretically and by numerical simulations to be effective in stabilizing unstable periodic points of chaotic discrete systems. In this article, a simple growth-rate type mechanism for controlling chaos in discrete systems is developed. We show in theory and by numerical simulations that our technique of stabilizing unstable periodic points of chaotic discrete systems in effective and, moreover, compared to other stabilizing methods, has a high speed.

It is worth noting that idea of the methods used here are inspired from recent and classical methods in the iterative approximation of fixed points, see Berinde [3], although they were independently developed in economics as "adaptive" methods, see [11]-[13].

2. Fixed point type controlling mechanisms

Consider a one-dimensional discrete system defined by a first order difference equation:

$$x_i = \theta(x_{i-1}), \tag{2.1}$$

where $\theta : [a, b] \to [a, b]$, $a, b \in \mathbb{R}$, is a continuous function.

We know by the Brower's fixed point theorem, that θ has at least one fixed point in the interval $[a, b]$, that is, there exists at least one $\overline{x}$ such that $\theta(\overline{x}) = \overline{x}$. Suppose further that $\theta'(\overline{x})$ exists. By a fixed point type (FPT) mechanism, we mean the following modification to the original system (2.1):

$$x_i = \widetilde{\theta}(x_{i-1}) \equiv (1 - \gamma)x_{i-1} + \gamma\theta(x_{i-1}), \tag{2.2}$$

where γ is a control parameter that can take any value in the interval $[0, 1]$.

Note that for $\gamma = 1$ we find the original system (2.1), while for $\gamma = 0$ we get the trivial stable system $x_i = x_{i-1}$. Therefore, we will consider in the following only control parameters in the open interval $(0, 1)$.

Inspired by the fixed point iterative methods in [3] we also consider the fixed point type (FPT) mechanism associated to the system (2.1) and given by:

$$x_i = \widehat{\theta}(x_{i-1}) \equiv (1 - \gamma_i)x_{i-1} + \gamma_i\theta(x_{i-1}), \tag{2.3}$$

where γ_i is a control parameter sequence that can take any values in the interval $[0, 1]$.

Remark 1. *1) The concept of growth rate, which is used extensively in many fields that involves discrete dynamics, has been used in Huang* [13] *to name his controlling mechanism. Note that the controlling mechanism defined by Eq. (2.2) is of growth-rate type, too.*

Indeed, for an one-dimensional discrete dynamics given by Eq. (2.1) the growth rate is defined as

$$g_i \equiv \frac{x_i - x_{i-1}}{x_{i-1}} = \frac{\theta(x_i)}{x_{i-1}} - 1. \tag{2.4}$$

The FPT controlling mechanism defined by Eq. (2.2) is mathematically equivalent to the following growth-rate type equation

$$\frac{x_i - x_{i-1}}{x_{i-1}} = \gamma \left[\frac{\theta(x_i)}{x_{i-1}} - 1 \right], \tag{2.5}$$

which in fact controls the growth rates of the original system Eq. (2.1) adaptively.

2) In order to ensure that the growth rate is well defined, we may assume in the following that the origin is excluded from the interval $[a, b]$.

The following theorem essentially ensures two desired properties of the FPT controlling mechanism.

Theorem 1. *The controlled system defined by Eq. (2.2) possesses the following mathematical characteristics:*

i) (Generic property) The process θ and $\widetilde{\theta}$ share exactly the same set of fixed points, that is, if $\theta(\overline{x}) = \overline{x}$ for some $\overline{x} \in [a, b]$, then $\widetilde{\theta}(\overline{x}) = \overline{x}$, and vice versa.

ii) (Necessary and sufficient condition) For an unstable fixed point $\overline{x}$, with $\theta'(\overline{x}) < -1$, there always exists an effective regime for the control parameter, such that $\left|\overline{\theta}'(\overline{x})\right| < 1$ for $\gamma \in \Gamma$.

Proof. . If $\overline{x} \in [a, b]$ is a fixed point of θ, that is $\theta(\overline{x}) = \overline{x}$, then

$$\widetilde{\theta}(\overline{x}) = (1 - \gamma)\overline{x} + \gamma\theta(\overline{x}) = (1 - \gamma)\overline{x} + \gamma\overline{x} = \overline{x},$$

that is, $\overline{x}$ is a fixed point of the adjusted system $\widetilde{\theta}$, too.

Conversely, if $\widetilde{\theta}(\overline{x}) = \overline{x}$ then by Eq. (2.2), it results $\theta(\overline{x}) = \overline{x}$.

As the derivative of $\widetilde{\theta}$ is given by $\widetilde{\theta}'(\overline{x}) = 1 - \gamma(1 - \theta'(\overline{x})$, if $\theta'(\overline{x}) < -1$, then we have

$$\widetilde{\theta}'(\overline{x}) = 1 - \gamma(1 - \theta'(\overline{x})) < 1.$$

Moreover, in order to have $\widetilde{\theta}'(\overline{x}) > -1$, γ must satisfy

$$\gamma < \frac{2}{1 - \theta'(\overline{x})}.$$

As $\theta'(\overline{x}) < -1$, it results that $1 - \theta'(\overline{x}) > 2$ and therefore

$$\gamma_{max} < \frac{2}{1 - \theta'(\overline{x})} < 1.$$

This shows that indeed there exists an effective regime for the control parameter: $\Gamma = (0, \gamma_{max})$, provided $\overline{x}$ is an unstable fixed point for the original system satisfying $\theta'(\overline{x}) < -1$. $\square$

Remark 2. *We illustrate the conclusion of Theorem 1 by some numerical simulations. To simultaneously offer a comparison to the results in Huang* [13]*, we shall consider first the examples there.*

Note that all numerical tests presented in this paper have been obtained by means of the software package FIXPOINT.

Example 1. ([13])

Consider the cubic process defined by $g : [1, 2] \to [1, 2]$, $g(x) = 1 + (x - 1)(4x - 7)^2$, *that is*

$$x_i = g(x_{i-1}) \equiv 1 + (x_{i-1} - 1)(4x_{i-1} - 7)^2, \tag{2.6}$$

which has three unstable fixed points $\overline{x}_1 = 1$, $\overline{x}_2 = 1.5$ *and* $\overline{x}_3 = 2$, *because* $g'(\overline{x}_1) = 9 > 1$, $g'(\overline{x}_2) = -3 < -1$ *and* $g'(\overline{x}_1) = 9$.

In view of Theorem 1, $\overline{x}_2 = 1.5$ *will become a stable fixed point for the controlled system associated to the original system given Eq. (2.6):*

$$x_i = \widetilde{g}(x_{i-1}) \equiv (1 - \gamma)x_{i-1} + \gamma\left[1 + (x_{i-1} - 1)(4x_{i-1} - 7)^2\right] =$$

$$= 16\gamma x^3 - 72\gamma x^2 + (104\gamma + 1)x - 48\gamma. \tag{2.7}$$

It follows by Theorem 1 that the unstable fixed point $\overline{x}_2$ *of* g *will be a stable fixed point of* $\widetilde{g}$.

The range for the control parameter γ *will be* $\Gamma = \left(0, \frac{1}{2}\right)$, *since*

$$\gamma_{max} = \frac{2}{1 - g'(\overline{x}_2)} = \frac{1}{2}.$$

Table 1 shows the first 12 *iterations for the systems defined by* $\theta = g$, $\widetilde{\theta}$ *and* $\widehat{\theta}$, *respectively, starting with the same initial point* $x_0 = 1.3333$. *While* θ *shows a totally unstable trajectory,* $\widetilde{\theta}$ *presents a stabilized trajectory after the* 12*nd iteration and* $\widehat{\theta}$ *presents a stabilized trajectory after the* 10*th iteration, both at the unstable fixed point* $\overline{x}_2 = 1.5$.

It is easy to see that the process defined by $\widehat{\theta}$ *(corresponding to Mann fixed point iteration procedure, see* [3]*) converges faster than the process defined by* $\widetilde{\theta}$ *(corresponding to Krasnoselskij fixed point iteration procedure, see* [14] *and* [3]*).*

Example 2. ([13])

Consider now the well known Logistic system defined by $f : [0,1] \to [0,1]$, $f(x) = 4x(1-x)$, *that is*

$$x_i = f(x_{i-1}) \equiv 4x_{i-1}(1 - x_{i-1}), \tag{2.8}$$

which has two unstable fixed points $\overline{x}_1 = 0$ *and* $\overline{x}_2 = 0.75$, *because* $f'(\overline{x}_1) = 4 > 1$ *and* $f'(\overline{x}_2) = -2 < -1$.

In view of Theorem 1, $\overline{x}_2 = 0.75$ *will become a stable fixed point for the controlled system associated to the original system given Eq. (2.6):*

$$x_i = \widetilde{f}(x_{i-1}) \equiv (1-\gamma)x_{i-1} + \gamma(4x_{i-1}(1 - x_{i-1}) = -4\gamma x^2 + (3\gamma + 1)x. \tag{2.9}$$

It follows by Theorem 1 that the unstable fixed point $\overline{x}_2$ *of* f *will be a stable fixed point of* $\widetilde{f}$.

The range for the control parameter γ *will be* $\Gamma = \left(0, \frac{2}{3}\right)$, *since*

$$\gamma_{max} = \frac{2}{1 - f'(\overline{x}_2)} = \frac{2}{3}.$$

Table 2 shows the first 12 *iterations for the systems defined by* $\theta = f$, $\widetilde{\theta}$ *and* $\widehat{\theta}$, *respectively, starting with the same initial point* $x_0 = 0.8333$. *While* θ *shows a totally unstable trajectory,* $\widetilde{\theta}$ *presents a stabilized*

TABLE 1. The first 12 iterations for the dynamical system defined by $\theta = g$ and $x_0 = 1.3333$; here $\gamma = \frac{1}{6}$, $\gamma_i = \frac{1}{i+1}$

n	θ	$\widetilde{\theta}$	$\widehat{\theta}$
0	1.3333	1.333	1.3333
1	1.925981	1.432080	1.925981
2	1.458835	1.476525	1.692408
3	1.622380	1.492140	1.473854
4	1.162186	1.497379	1.499929
5	1.896630	1.499126	1.499986
6	1.308444	1.499709	1.499995
7	1.962206	1.499903	1.499998
8	1.693273	1.499968	1.499999
9	1.035695	1.499989	1.499999
10	1.291405	1.499996	1.500000
11	1.980563	1.499999	1.500000
12	1.834016	1.500000	1.500000

trajectory after the 8th iteration and $\widehat{\theta}$ presents a stabilized trajectory after the 21th iteration, both at the unstable fixed point $\overline{x}_2 = 0.75$.

It is easy to see that in this case the process defined by $\widetilde{\theta}$ (corresponding to Krasnoselskij fixed point iteration procedure, see [3] *and* [14]*) converges faster than the process defined by $\widehat{\theta}$ (corresponding to Mann fixed point iteration procedure).*

Following the same pattern we can stabilize unstable periodic fixed point or multi-dimensional systems, like in [13].

3. The case when Theorem 1 does not apply

Note that an unstable fixed point $\overline{x}$ of θ, with $\theta'(\overline{x}) > 1$ cannot be stabilized by applying Theorem 1.

In this section we consider an example of chaotic dynamical system for which we can give not only empirical results like in the case of the previous two examples but also analytical results. For this dynamical system, Theorem 1 does not apply but, as it will be seen, the chaotic dynamical can be, however, stabilized by the fixed point iterative schemes (2.2), in view of the theoretical results in [5].

TABLE 2. The first 12 iterations for the dynamical system defined by $\theta = f$ and $x_0 = 0.8333$; here $\gamma = \frac{2}{5}$, $\gamma_i = \frac{1}{i+1}$

n	θ	$\widetilde{\theta}$	$\widehat{\theta}$
0	0.833300	0.833300	0.833300
1	0.555644	0.722238	0.555644
2	0.987615	0.754319	0.771630
3	0.048927	0.749106	0.749376
4	0.186134	0.750177	0.749844
5	0.605952	0.749964	0.749937
6	0.955097	0.750007	0.749969
7	0.171547	0.749999	0.749982
8	0.568474	0.750000	0.749989
9	0.981245	0.750000	0.749993
10	0.073612	0.750000	0.749995
11	0.272772	0.750000	0.749996
12	0.793469	0.750000	0.749997

Example 3. ([1])

Consider the nonlinear dynamical system defined by $h : \left[-\frac{3}{2}, \frac{1}{2}\right] \to \left[-\frac{3}{2}, \frac{1}{2}\right]$, $h(x) = 2x^2 + 2x - 1$, *that is*

$$x_{n+1} = 2x_n^2 + 2x_n - 1,\ n = 0, 1, 2, ..., \tag{3.1}$$

which has two unstable fixed points: $\overline{x}_1 = -1$ *and* $\overline{x}_2 = 0.5$, *because* $h'(\overline{x}_1) = 11 > 1$ *and* $f'(\overline{x}_2) = 3 > 1$.

Hence, Theorem 1 cannot be applied here in order to establish if one or both fixed points will become stable fixed points for the controlled systems associated to the original system.

It is a simple task to show that for all $x_0 \in \left[-\frac{3}{2}, \frac{1}{2}\right]$ *we have*

$$x_n \in \left[-\frac{3}{2}, \frac{1}{2}\right],\ n \geq 1. \tag{3.2}$$

By (3.1) we have

$$x_{n+1} + \frac{1}{2} = 2\left(x_n + \frac{1}{2}\right)^2 - 1,\ n \geq 0$$

and using the fact that, by (3.2)

$$x_n + \frac{1}{2} \in [-1, 1],\ n \geq 0,$$

we can denote $x_0 + \frac{1}{2} = \cos\alpha$ *and then by the formula*

$$\cos 2\alpha = 2\cos\alpha^2 - 1,$$

we inductively obtain that

$$x_n + \frac{1}{2} = \cos(2^n\alpha),\ n \geq 0$$

that is,

$$x_n = -\frac{1}{2} + \cos(2^n\alpha),\ n \geq 0. \tag{3.3}$$

For $x_0 = \frac{1}{2}$, $\lim_{n\to\infty} x_n = \frac{1}{2}$, *while for* $x_0 = -1$, $\lim_{n\to\infty} x_n = -1$. *There are also other values of* x_0 *for which* $\{x_n\}$ *converges to* -1, *e.g.* $x_0 = 0$ *and, similarly, there are other values of* x_0, *e.g.* $-\frac{3}{2}, -\frac{1}{2}$ *etc., for which* (x_n) *converges to* $\frac{1}{2}$.

By (3.3) and the previous assertion, it follows that for a certain x_0, $\{x_n\}$ *is convergent if and only if it is "embeded" in a fixed point of* h, *that is, if there exists a rank* n *such that* $x_n \in \left\{-1, \frac{1}{2}\right\}$.

We thus deduce that for x_0 *in the set*

$$C = \left\{-\frac{1}{2} + \sin\frac{k\pi}{2^n} \mid k \in Z,\ n \in N\right\}, \tag{3.4}$$

$\{x_n\}$ *gets stabilized, while for* x_0 *in the set*

$$P = \left\{-\frac{1}{2} + \cos\frac{2k\pi}{2^p \pm 1} \mid k \in Z,\ n \in N\right\}, \tag{3.5}$$

$\{x_n\}$ *is periodic, of period* p, *hence divergent.*

For example, when starting from

$$x_0 = -\frac{1}{2} + \cos\frac{2\pi}{5} = -\frac{3}{4} + \frac{\sqrt{5}}{4},$$

$\{x_n\}$ *is periodic of period* $p = 2$.

The sets (3.4) and (3.5) are infinite and it is not very difficult to show they are both dense in the interval $\left[-\frac{3}{2}, \frac{1}{2}\right]$.

This dynamical system allows us to better illustrates chaotic behaviour: for any $\epsilon > 0$, *there exists* $\omega \in C$ *and* $\omega' \in P$, *satisfying* $|\omega - \omega'| < \epsilon$ *and such that starting from* $x_0 = \omega$, $\{x_n\}$ *converges, while, starting from* $x_0 = \omega'$, $\{x_n\}$ *diverges (being periodic).*

Table 3 shows some of the first 30 *iterations for the systems defined by* $\theta = h$, $\widetilde{\theta}$ *and* $\widehat{\theta}$, *respectively, starting with the same initial point* $x_0 = 0.4999$. *While* θ *shows a totally unstable trajectory,* $\widetilde{\theta}$ *presents a stabilized trajectory after the* 28*th iteration at the unstable fixed point* $\overline{x}_1 = -1$ *but the Mann type controlled system* $\widehat{\theta}$ *gets not stabilized after* 30*th iterations. Note that* $\overline{x}_2 = 0.5$ *remains an unstable fixed point for the controlled system* $\widetilde{\theta}$, *too.*

4. Conclusions and bibliographical comments

Theorem 1 and Examples 1-3 in this paper illustrate how a chaotic discrete dynamical system can be stabilized by means of fixed point iterative techniques. Theorem 1 ensures the framework of stabilizing only fixed points $\overline{x}$ of *differentiable* dynamical systems for which, in addition, the condition $\theta'(\overline{x})$ is satisfied, like in Examples 1-2.

This condition is not satisfied by the two fixed points of the dynamical system in Example 3. A more general result that do not involve any assumption on the first order derivative, established in [5], is needed to guarantee the stabilization of an unstable fixed point of the dynamical system in Example 3. The only property that is required to the function that defines such a dynamical system is to be Lipschitzian.

Note that in order to prove the theorems in [5], some classical results in [14], [8], [2] and [9] have been used and extended.

TABLE 3. The first 30 iterations for the dynamical system defined by $\theta = h$ and $x_0 = 0.4999$; here $\gamma = \frac{1}{2}$, $\gamma_i = \frac{1}{i+1}$

n	θ	$\widetilde{\theta}$	$\widehat{\theta}$
0	0.4999	0.4999	0.4999
1	0.499600	0.499750	0.499600
2	0.498400	0.499375	0.499000
3	0.493607	0.498438	0.498001
4	0.474509	0.496098	0.496504
5	0.399334	0.490260	0.494411
...	...	...	...
25	-1.157044	-0.999992	0.224155
26	-0.636587	-1.000004	0.198180
27	-1.462688	-0.999998	0.171392
28	0.353536	-1.000001	0.143898
29	-0.042953	-1.000000	0.115805
30	-1.082217	-1.000000	0.087226

Note also that the techniques on which the controlling techniques of chaotic dynamical systems are based, appear to have been developed rather in parallel in mathematics and economics, see [12] and references therein.

It is interesting to note that the well known Krasnoselskij-Mann fixed point iterative method [3] have been used in economics as early as 1958, under the name of "adaptive adjustment", see [16], that is, three years later than the Krasnoselskij-Mann fixed point iterative method has been developed by Krasnoselskij [14] in a mathematical context.

REFERENCES

[1] Andrica, D., Berinde, V., **Concursul interjudeţean de matematică "Gr. C. Moisil". Ediţiile I-XX (1986-2005)**, Editura Cub Press 22, Baia Mare, 2006

[2] Bailey, D.F., **Krasnoselski's theorem on the real line**, Amer. Math. Monthly **81** (1971), no. 5, 506–507

[3] Berinde, V., **Iterative Approximation of Fixed Points**, 2nd Ed., Springer Verlag, Berlin Heidelberg New York, 2007

[4] Berinde, V., and Pacurar, M., **Controlling chaotic discrete dynamical systems through fixed point iterative methods**, in PAMM, Special Issue: 79th Annual Meeting of the International Association of Applied Mathematics and Mechanics (GAMM), Bremen 2008, Vol. 8, Issue 1, Pages 10877 – 10878; DOI: 10.1002/pamm.200810877

[5] Berinde, V. and Kovács, G., **Stabilizing discrete dynamical systems by monotone Krasnoselskij type iterative schemes**, Creative Math. Inform. **17** (2008), No. 3, 298-307

[6] Chan, N. P., **Controlling chaos by proportional pulses**, Phys. Lett. A., **234** (1997), 193–197

[7] Ditto, W. L., Raussco, S. N., Spano, M. L., **Experimental control of chaos**, Phys. Rev. Lett., **65** (1990), 3211–3214

[8] Franks, R. L., Mrazec, R. P., **A theorem on mean-value iterations**, Proc. Amer. Math. Soc., **30** (1971), 324–326

[9] Hillam, B. P., **A generalization of Krasnoseski's theorem on the real line**, Math. Magazine, **48** (1975), 167–168

[10] Holmgren, R. A., **First course in discrete dynamical systems**, Second edition, Springer Verlag, 2000

[11] Huang, W., **Stabilizing nonlinear dynamical systems by an adaptive adjustment mechanism**, Phys. Rev. E, **61** (2000), No. 2, 1012–1015

[12] Huang, W., **Theory of adaptive adjustment**, Discrete Dyn. Nat. Soc., **5** (2001), 247–263

[13] Huang, W., **Controlling chaos through growth rate adjustment**, Discrete Dynamics in Nature and Society, **7** (2002), 191–199

[14] Krasnoselskij, M.A., **Two remarks on the method of successive approximations** (in Russian), Uspehi Math. Nauk (N.S.), **10** (1955), no. 1, 123–127

[15] Lynch, S., **Dynamical Systems with Applications using MATLAB**, Birkhäuser, Boston, Basel, Berlin, 2004

[16] Nerlove, M., **Adaptive Expectations and Cobweb Phenomena**, Quarterly Journal of Economics, **72** (1958), 227–240

[17] Ott, E., Grebogi, C., Yorke, J. A., **Controlling chaos**, Phys. Rev. Lett., **64** (1990), 1196–1199

Department of Mathematics and Computer Science
Faculty of Sciences
Northern University of Baia Mare
Victoriei 76, 430122 Baia Mare ROMANIA
vberinde@ubm.ro; vasile_berinde@yahoo.com