"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 19 (2009), No. 2, 71 - 80

INFINITESIMAL MOTIONS OF THE $2-\pi$ STRUCTURES ON THE TANGENT BUNDLE

VICTOR BLĂNUȚĂ, MANUELA GÎRȚU and VALENTIN GÎRȚU

Abstract. We define the motion of an almost $2-\pi$ structure on the tangent bundle and study the main properties of these. We study here the existence and arbitrariness of d-connection $D\Gamma(N)$ determining all these connections. Finally we determine the infinitesimal motions of this structure and we study the properties of these motions.

INTRODUCTION

M. Yawata studied in [7] the infinitesimal transformations and motions on a vector bundle establishing the main properties of these. We have defined an almost 2π structure on TM as a d-tensor field $\varphi^i_j(x, y)$ with the property $\varphi^i_j\varphi^j_k = \lambda^2\delta^i_k$, λ being nonvanish complex number $[1], [2], [3]$. For $\lambda = \pm i$, we have an almost complex *d*-structure on TM and for $\lambda = \pm 1$, we get an almost product *d*-structure on TM. We study the existence and arbitrariness of d-connection $D\Gamma(N)$ for which $\varphi^i_{j|k} = 0, \varphi^i_j |_k = 0$ and determine all d-connections $D\Gamma(N)$ with these properties [1], [2]. These connections and the composition of the mapping give us a group $G_{2\pi}$.

Finally we determine the infinitesimal motions of φ^j_i establishing the equations (3.10) and the consequences (3.14). There are also studied the main properties of these motions.

Some notations and fundamental result are taken from the book [6].

Keywords: almost 2π structure, almost 2π d-connection, infinitesimal transformation, infinitesimal motion
(2000) Mathematics Subjects Classification: 53C60, 53C80.

1. PRELIMINARIES

Let M be an n-dimensional (n=2p) differentiable manifold of class C^∞ and (TM, π, M) it is tangent bundle. The canonical coordinates of a point $(x, y) \in TM$ are (x^i, y^i), where indices i, j, k,… take the values $1, 2, \dots, n$.

A local coordinate transformation on TM is given by

(1.1) $\widetilde{x}^i = \widetilde{x}^i(x^1, x^2, \dots, x^n)$, $\det\left\|\dfrac{\partial \widetilde{x}^i}{\partial x^j}\right\| \neq 0$, $\widetilde{y}^i = \dfrac{\partial \widetilde{x}^i}{\partial x^j} y^j$.

Let V be the vertical distribution on TM, i.e. $V : u = (x, y) \in TM \to V_u \in T_u TM$, for any point $u \in TM$. Then V is an integrable distribution having $\left\{\dfrac{\partial}{\partial y^i}\right\}$ as adapted basis.

A nonlinear connection N is a C^∞-distribution on TM with the property

$$(1.2)\ T_u TM = N_u \oplus V_u$$

for any point $u \in TM$. N will be called a horizontal distribution on TM.

Locally it is characterized by an adapted basis

$$(1.3)\ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^i}$$

where the function $N_i^j(x, y)$ are the coefficients of the nonlinear connection N.

Then $\left(\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^i}\right)$ are the local basis of F(TM)-module of vector fields $X(TM)$, adapted to the supplementary distribution N and V. We denote these basic shortly by $(\delta_i, \dot{\partial}_i)$ and call it adapted basis. The dual basis are given by

$$(1.4)\ dx^i,\ \delta y^i = dy^i + N_j^i dx^j .$$

A vector field $X \in X(TM)$ is uniquely expressed in the form

$$(1.5)\ X = X^H + X^V,\ X^H \in N,\ X^V \in V$$

and for 1-form

$$(1.6)\ \omega = \omega^H + \omega^V$$

where $\omega^H(X^V) = 0$ and $\omega^V(X^H) = 0$.

A tensor field $t \in T_s^r(TM)$ is called *distinguished tensor field* (shortly *d-field*) if it has the properties:

$$(1.7)\ t(\omega_1, \ldots, \omega_r, X_1, \ldots X_s) = 0$$

for $\omega_i = \omega^H$ or $\omega_i = \omega^V$ $(i = 1, \ldots, r)$ and $X_j = X_j^H$ or $X_j = X_j^V$, $(j = 1, 2, \ldots, s)$.

For example X^H, X^V are *d*-tensor fields and ω^H, ω^V are *d*-fields 1-form.

The components of a *d*-tensor field on *TM* determine *d*-tensor fields on *M* and with respect to (1.1), they are transformed by classical rule.

We remark that *F*(*TM*)-linear mapping $J : X(TM) \to X(TM)$ defined by

$$(1.8)\ J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i},\ J\left(\frac{\partial}{\partial y^i}\right) = 0$$

is globally defined on *TM*. It has the property $J^2 = 0$ and is an integrable structure called *tangent structure on TM*.

A linear connection *D* on *TM* is called *distinguished connection* if *D* preserve by parallelism the horizontal distribution *N* and the tangent structure is absolute parallel with respect to *D*.

We write

$$(1.9)\ D_X^H = D_{X^H},\ D_X^V = D_{X^V}$$

where D^H and D^V are the *h*-and-*v*- covariant derivatives in the algebra of *d*-tensor fields $T_d(TM) \subset T(TM)$, respectively.

When the nonlinear connection *N* is given, *d*-connection *D* on *TM* is completely determined by its coefficients:

$$(1.10)\ D_{\delta_k}\delta_j = F_{jk}^i\delta_i; D_{\delta_k}\dot{\partial}_j = F_{jk}^i\dot{\partial}_i; \quad D_{\dot{\partial}_k}\delta_j = C_{jk}^i\delta_i; D_{\dot{\partial}_k}\dot{\partial}_j = C_{jk}^i\dot{\partial}_i.$$

We shall denote the system of the coefficients of *D* by $D\Gamma(N) = (F_{jk}^i, C_{jk}^i)$. Whether *N* is fixed it is denoted by $D\Gamma = (N_j^i, F_{jk}^i, C_{jk}^i)$.

If we write the torsion tensor field *T* of $D\Gamma(N)$ in the adapted basis $(\delta_i, \dot{\partial}_i)$ as follows:

$$(1.11)\ \begin{cases} T(\delta_k, \delta_j) = T_{jk}^i\delta_i + \widetilde{T}_{jk}^i\dot{\partial}_i \\ T(\dot{\partial}_k, \delta_j) = P_{jk}^i\delta_i + \widetilde{P}_{jk}^i\dot{\partial}_i \\ T(\dot{\partial}_k, \dot{\partial}_j) = S_{jk}^i\dot{\partial}_i \end{cases}$$

then the coefficients $T^i_{jk}, \widetilde{T}^i_{jk}, P^i_{jk}, \widetilde{P}^i_{jk}, S^i_{jk}$ are uniquely determined by

$$(1.12)\quad \begin{aligned} &T^i_{jk} = F^i_{jk} - F^i_{kj}, \widetilde{T}^i_{jk} = R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k, \\ &P^i_{jk} = C^i_{jk}, \widetilde{P}^i_{jk} = \dot{\partial}_k N^i_j - F^i_{kj}, S^i_{jk} = C^i_{jk} - C^i_{kj} \end{aligned}$$

2. ALMOST $2-\pi$ STRUCTURES ON TM

Definition 2.1 A *d-tensor field* $\varphi^i_j(x, y)$ of type (1.1) on TM is called an *almost* $2-\pi$ *structure* if it has the property

$$(2.1)\quad \varphi^i_j \varphi^i_k = +\lambda^2 \delta^i_k$$

where λ is a complex number.

For $\lambda = \pm i$, φ^i_j is an almost complex structure on *TM* and for $\lambda = \pm 1$, φ^i_j is an almost product structure on *TM*.

On *TM*, there exist almost $2-\pi$ structures; one of them is given by

$$(2.2)\quad \varphi^i_j = \lambda \delta^i_j .$$

This is a nontrivial tensor field because its lift to *TM* is not generally integrable.

Evidently rank $\left\| \varphi^i_j \right\| = n$ and the mapping $\varphi : X \in X_d(M) \to \varphi X \in X_d(M)$ is an isomorphism, where $X_d(M)$ is the module of *d*-vector fields on *TM*.

Obata's operators of the structure φ^i_j are given by

$$(2.3)\quad O^{ij}_{hk} = \frac{1}{2}\left(\delta^i_h \delta^j_k + \frac{1}{\lambda^2} \varphi^i_h \varphi^j_k \right),\ {}^*O^{ij}_{hk} = \frac{1}{2}\left(\delta^i_h \delta^j_k - \frac{1}{\lambda^2} \varphi^i_h \varphi^j_k \right).$$

Let $\overset{\circ}{D}\Gamma = (\overset{\circ}{N}{}^i_j, \overset{\circ}{F}{}^i_{jk}, \overset{\circ}{C}{}^i_{jk})$ be a fixed *d*-connection on *TM* and Φ the *d*-tensor field, given by

$$(2.4)\quad \Phi = \varphi^i_j(x, y)\delta_i \otimes dx^j .$$

Definition 2.2 A *d-connection D* is called an *almost* $2-\pi$ *d-connection* if it satisfies

$$(2.5)\quad D_X \Phi = 0 ,$$

for any vector field on *TM*.

This condition is equivalent to

$$(2.5')\quad D^H_X \Phi = 0, D^V_X \Phi = 0 .$$

When φ^i_j and N^i_j are given by direct calculation, we have:

Theorem 2.1. *Let* $\overset{\circ}{D}\Gamma = (\overset{\circ}{N}{}^i_j, \overset{\circ}{F}{}^i_{jk}, \overset{\circ}{C}{}^i_{jk})$ *be a fixed d-connection on TM. Then there exist an almost* 2π *d-connection* $\hat{D}\Gamma$ *whose coefficients are given by*

$$\text{(2.6.)}\quad N^i_j = \overset{\circ}{N}{}^i_j,\ F^i_{jk} = \overset{\circ}{F}{}^i_{jk} - \frac{1}{2\lambda^2}\varphi^l_j \varphi^i_{l\overset{\circ}{|}k},\ C^i_{jk} = \overset{\circ}{C}{}^i_{jk} - \frac{1}{2\lambda^2}\varphi^l_j \varphi^i_l \overset{\circ}{|}_k$$

The connection $\hat{D}\Gamma$ given by (2.6) is called *almost* 2π *d-connection* on *TM* with respect to the 2π *d*-structure φ^i_j.

Using (2.5) we can prove

Theorem 2.2. *The set of all almost* 2π *connection is given by*

$$\text{(2.7)}\quad \begin{aligned} N^i_j &= \overset{\circ}{N}{}^i_j - X^i_j,\ F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + \overset{\circ}{C}{}^i_{jm} X^m_k - \\ &- \frac{1}{2\lambda^2}\varphi^m_j \left(\varphi^i_{m\overset{\circ}{|}k} + \varphi^i_m \overset{\circ}{|}_l X^l_k\right) + O^{mi}_{jl} Y^l_{mk}, \\ C^i_{jk} &= \overset{\circ}{C}{}^i_{jk} - \frac{1}{2\lambda^2}\varphi^m_j \left(\varphi^i_m \overset{\circ}{|}_k\right) + O^{mi}_{jl} Z^l_{mk} \end{aligned}$$

where $\overset{\circ}{D}\Gamma$ is a fixed *d*-connection and $\overset{\circ}{|}, \overset{\circ}{|}$ we denote *h* and v-covariant derivatives with respect to $\overset{\circ}{D}\Gamma$, and $X^i_j, Y^i_{jk}, Z^i_{jk}$ are arbitrary *d*-tensor field respectively.

From (2.6), the transformations $\overline{D}\Gamma(N) \to D\Gamma(N)$ between two almost 2π *d*-connections are given by

$$\text{(2.8)}\quad \overline{F}{}^i_{jk} = F^i_{jk} + O^{lj}_{mk} X^m_{lk},\ \overline{C}{}^i_{jk} = C^i_{jk} + O^{li}_{mj} Y^m_{lk}$$

where X^m_{lk}, Y^m_{lk} are arbitrary –tensor fields. Therefore, we have

Theorem 2.3. *The set of all transformation (2.7) together with the mapping product form are Abelian group* $G_{2\pi}$ *which is isomorphic with the additive group of the pair of d-tensor fields:* $\left(O^{li}_{mj} X^m_{lk}, O^{li}_{bj} Y^m_{lk}\right)$.

3. INFINITESIMAL MOTIONS OF AN ALMOST $2-\pi$ d – STRUCTURE

Let us consider an infinitesimal transformation on *TM*, determined by a vector field $v^i(x)$ on *M* [7].

$$(3.1)\ \ `x^i = x^i + v^i(x)dt,\ `y^i = y^i + y^m \partial_m v^i dt\,.$$

Thus the Lie derivation of a *d*-tensor field $\varphi^i_j(x, y)$ is given by

$$(3.2)\ \ \underset{v}{L}\,\varphi^i_j(x, y) = \theta_v \varphi^i_j - \varphi^m_j \partial_m v^i + \varphi^i_m \partial_j v^m$$

where θ_v is the operator defined by

$$(3.3)\ \ \theta_v = v^m \partial_m + y^m \partial_m v^s \dot{\partial}_s$$

and we denote $\partial_m = \dfrac{\partial}{\partial x^m},\ \dot{\partial}_m = \dfrac{\partial}{\partial y^m}$.

We can give

Definition 3.1. An *infinitesimal transformation* (3.1) is called an *infinitesimal motion* for an almost 2π *d*-structure $\varphi^i_j(x, y)$ if the Lie derivative $\underset{v}{L}\,\varphi^i_j(x, y)$ vanishes:

$$(3.4)\ \ \underset{v}{L}\,\varphi^i_j(x, y) = 0\,.$$

Taking account of (3.2) we can state the following:

Theorem 3.1. *An infinitesimal transformation (3.1) is a motion for an almost 2π d-structure φ^i_j, if and only if the following equations are satisfied*

$$(3.5)\ \ \theta_v \varphi^i_j - \varphi^m_j \partial_m v^i + \varphi^i_m \partial_j v^m = 0\,.$$

Using (2.1) we have

$$(3.6)\ \ \varphi^i_j\, \underset{v}{L}\,\varphi^j_k + \left(\underset{v}{L}\,\varphi^i_j\right)\varphi^i_k = 0\,.$$

Consequently, we have:

Theorem 3.2. *If (3.4) holds, then we have*

$$(3.7)\ \ \underset{v}{L}\,O^{ij}_{hk}(x, y) = 0,\ \underset{v}{L^*}\,O^{ij}_{hk}(x, y) = 0\,.$$

Now, let us consider an almost 2π, *d*-connection $D\Gamma(N) = \left(F^i_{jk}, C^i_{jk}\right)$ with respect to φ^i_j. Of course $D\Gamma(N)$ is given by (2.7), with $X^i_j = 0$. Hence it can be expressed by

$$(3.8)\ \ F^i_{jk} = \overset{\circ}{F}{}^i_{jk} - \frac{1}{2\lambda^2}\varphi^m_j \varphi^i_{\,m\overset{\circ}{|}k} + O^{mi}_{jl} Y^l_{mk}\,,$$

$$C^i_{jk} = \overset{\circ}{C}{}^i_{jk} - \frac{1}{2\lambda^2}\varphi^m_j \varphi^i_m \overset{\circ}{|}_k + O^{mi}_{jl} Z^i_{mk}$$

where $D\overset{\circ}{\Gamma}(N)=\left(\overset{\circ}{F}{}^{i}_{jk}, \overset{\circ}{C}{}^{i}_{jk}\right)$ is a fixed d-connection, and Y^i_{jk}, Z^i_{jk} are arbitrary d-tensor fields.

Any d-connection $D\Gamma(N)=\left(F^i_{jk}, C^i_{jk}\right)$ defined by (3.8) has the property:

$$\text{(3.9)}\ \varphi^i_{j|k}=0, \varphi^i_j\,|_k=0\,.$$

We consider one of these d-connections.

Now, by virtue of (3.8) we can express the Lie derivative $\underset{v}{L}\varphi^i_j$ in new form [6] and can formulate Theorem 3.1 in the following manner

Theorem 3.3. *The infinitesimal transformation (3.1) is a motion of an almost* 2π *d-structure* $\varphi^i_j(x,y)$ *if and only if the following equation is satisfied*

$$\text{(3.10)}\ -\varphi^k_j v^i_{|k}+\varphi^i_k v^k_{|j}+\left(y^k v^m_{|k}-v^k y^m_{|k}\right)\dot{\partial}_m\varphi^i_j=0$$

where the covariant derivatives are taken with respect to an almost 2π *d-connection* $D\Gamma(N)$.

The d-tensor $D^m_K=y^m_{|k}$ is called the *h-deflection tensor field* of $D\Gamma(N)$.

We remark that for a d-connection $D\Gamma(N)=\left(F^i_{jk}, C^i_{jk}\right)$ the Lie derivatives $\underset{v}{L}F^i_{jk}, \underset{v}{L}C^i_{jk}$, with respect to (3.1)are given by

$$\text{(3.11)}\ \begin{cases}\underset{v}{L}F^i_{jk}(x,y)=\theta_v F^i_{jk}-F^m_{jk}\partial_m v^i-F^i_{mk}\partial_j v^m+F^i_{jm}\partial_k v^m+\partial_j\partial_k v^i\\ \underset{v}{L}C^i_{jk}(x,y)=\theta_v C^i_{jk}-C^m_{jk}\partial_m v^i+C^i_{mk}\partial_j v^m+C^i_{jm}\partial_k v^m.\end{cases}$$

If we denote by "||" the covariant derivative with respect to Berwald connection $B\Gamma(N)=\left(B^i_{jk},0\right), B^i_{jk}=\partial_j N^i_k$, then we have the following Lie brackets:

$$\text{(3.12)}\ \begin{cases}\left[\delta_k,\theta_v\right]=\left(\partial_k v^m\right)\delta_m+\left\{v^m R^r_{km}+\left(y^m v^r_{\mathrm{Im}}-v^m y^r_{|m}\right)_{||k}\right\}\dot{\partial}_r\\ \left[\dot{\partial}_k,\theta_v\right]=\left\{v^m B^r_{km}+\dot{\partial}_k\left(y^m v^r_{|m}-v^m y^r_{|m}\right)\right\}\dot{\partial}_r\end{cases}$$

We can prove

Theorem 3.4. *If* $t^i_j(x,y)$ *is a d-tensor field and* $D\Gamma(N)$ *is a d-connection, then we have the following formulas:*

$$(3.13)\quad \begin{aligned} &\left(\underset{v}{L}\, t^i_j\right)_{k} - \underset{v}{L}\, t^i_{j|k} = -t^r_j \underset{v}{L} F^i_{rk} + t^i_r \underset{v}{L} F^r_{jk} + \\ &\qquad + \left\{v^m R^r_{km} + \left(y^m v^r_{|m} - v^m y^r_{|m}\right)_{\|k}\right\}\dot{\partial}_r t^i_j \\ &\left(\underset{v}{L}\, t^i_j\right)|_k - \underset{v}{L}\, t^i_j\,|_k = -t^r_j \underset{v}{L} C^i_{rk} + t^i_r \underset{v}{L} C^r_{jk} + \\ &\qquad + \left\{v^m_{\|} + \dot{\partial}_k\left(y^m v^r_{|m} - v^m y^r_{|m}\right)\right\}\dot{\partial}_r t^i_j \end{aligned}$$

Proof. By a straightforward calculation we get

$$\left(\underset{v}{L} t^i_j\right)_{Ik} = \delta_k\left(\theta_v t^i_j - t^r_j \partial_r v^r + t^i_r \partial_j v^r\right) + F^i_{mk} \underset{v}{L} t^m_j - F^m_{jk} \underset{v}{L} t^i_m$$

$$\underset{v}{L} t^i_{j|k} = \theta_v\left(\delta_k t^i_j + F^i_{mk} t^m_j - F^m_{jk} t^i_m\right) - t^m_{j|k}\partial_m v^i + t^i_{m|k}\partial_j v^m + t^i_{j|m}\partial_k v^m$$

Therefore, taking account of (3.10) and (3.11), we have the formulas (3.12). *q.e.d.*

Applying these formulas we have:

Theorem 3.5. *If (3.1) is an infinitesimal motion of an almost* 2π *d-structure* φ^i_j *and* $D\Gamma(N)$ is an almost 2π *d-connection with respect to* φ^i_j, *then we have*

$$(3.14)\quad \begin{cases} -\varphi^r_j \underset{v}{L} F^i_{rk} + \varphi^i_r \underset{v}{L} F^r_{jk} + \left\{v^m R^r_{mk} + \left(y^m v^r_{|m} - v^m y^r_{|m}\right)_{\|k}\right\}\dot{\partial}_r \varphi^i_j = 0 \\ -\varphi^r_j \underset{v}{L} C^i_{rk} + \varphi^i_r \underset{v}{L} C^r_{jk} + \left\{v^r_{\|k} + \dot{\partial}_k\left(y^m v^r_{|m} - v^m y^r_{|m}\right)\right\}\dot{\partial}_r \varphi^i_j = 0 \end{cases}$$

Proof. Indeed, if (3.1) is an infinitesimal motion for φ^i_j, we have $\underset{v}{L}\varphi^i_j = 0$. $D\Gamma(N)$ being an almost 2π *d*-connection, we get $\varphi^i_{j|k} = 0, \varphi^i_j\,|_k = 0$.

Applying the formulas (3.12) for $t^i_j = \varphi^i_j$ we have the equations (3.14).

Remark. All this theory can be particularized for the almost complex *d*-structure φ^i_j on *TM* taking $\lambda = \pm i$ and for the almost product *d*-structure φ^i_j on *M*, taking $\lambda = \pm 1$.

References

1. Blănuță, V., **Finsler connections compatible with almost $2-\pi$ Finsler structures**, The Proc. of the third national seminar on Finsler spaces. Univ. Braşov (1984), 77-82

2. Blănuţă, V., Gîrţu M., **On some properties of Finsler connection compatible with almost $2-\pi$ Finsler structures**, Studii şi Cercetări Ştiinţifice. Seria: Matematică, Universitatea din Bacău 3 (1993), 27-34
3. Blănuţă, V., **Riemannian almost $2-\pi$ structure on the tangent bundle**, Studii şi Cercetări Ştiinţifice. Seria: Matematică, Universitatea din Bacău 8 (1998), 1-8
4. Hsu, C, J., **On some properties of π structures on differentiable manifold**, Tohaku Mathematical Journal 12 (1960), 429-454
5. Miron, R., Anastasiei, M., **Vector bundle. Lagrange spaces. Applications in relativity**, Editura Academiei R.S.R., 1987.
6. Miron, R., **Introduction to the theory of Finsler spaces**, Proc. Nat. Sem. of Finsler spaces, Braşov; 1980, 131-183
7. Yawata, M., **Infinitesimal transformations on a vector bundle**, PhD Thesis, Univ "Al. I. Cuza" Iaşi (1993)

"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Department of Mathematics and Informatics
Spiru Haret 8, Bacău 600114, ROMANIA
vblanuta@ub.ro
girtum@yahoo.com

National College "Gh.Vrănceanu" Bacău,
Lucreţiu Pătrăşcanu 30, Bacău 600204
ROMANIA