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SOME OPEN PROBLEMS IN THE THEORY OF ALMOST PERIODIC FUNCTIONS

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Abstract. The theory of almost periodic functions is a well established branch of modern mathematics, with roots in the work of mathematicians as H. Poincaré, P. Bohl and E. Esclangon. The essentials of the theory have been published by H. Bohr (1922–1926), whose work immediately attracted the interest of many distinguished mathematicians like V. Stepanov, S. Bochner, H. Weyl, N. Wiener, A.S. Besicovitch, J. von Neumann, N.N. Bogolinbov, with significant contributions to the field.

An outline of this initial period of the theory of almost periodic functions, with pertinent references, can be found in our book [5], or in A.M. Fink [8], B.M. Levitan [9], B.M. Levitan, V.V. Zhikov [10]. In our recent book [6], the concept of function space is exploited in building up the classical theory (basically, the case of almost periodic functions of a real variable with complex or real values).

The hierarchy of classes/spaces of almost periodic functions has been recently investigated, with thorough discussion and many examples, by J. Andres et al. [2].

To remain in the framework of an abstract, we will mention here several inclusions of various spaces of almost periodic functions. The notations used below have the following meaning.

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1) AP_1 will mean the space of almost periodic functions whose Fourier series converges absolutely (also, uniformly).
2) AP stands for the space of almost periodic functions in the sense of Bohr (i.e., uniformly almost periodic functions).
3) S will denote the Stepanov's almost periodic space. Actually, $S = S^1$ is the richest space among all S^p-spaces, $p \geq 1$.
4) APW stands for the Weyl's space of almost periodic functions (not a Banach space!).
5) B^p, $p \geq 1$, denotes the Besicovitch space of almost periodic functions (with $B^1 = B$ the richest, and $B^2 = AP_2$ the one commonly used).
6) PAP will designate the space of pseudo almost periodic functions, as defined and studied by C. Zhang [12]. These functions appear as "small" perturbations of the almost periodic functions (Bohr), and certainly present interest in applications.

The hierarchy is given by various inclusions. See [2], [6] for more details.

$$AP_1 \subset AP \subset S \subset APW \subset B, \tag{1}$$

$$AP \subset S^2 \subset B^2 \subset B, \tag{2}$$

$$AP \subset PAP \subset BC \cap B. \tag{3}$$

It is appropriate to mention that Besicovitch spaces B^p and W (as well as W^p, $p > 1$) have as elements classes of equivalent functions, corresponding to a certain relation of equivalence, defined in each case.

The fact that a function belongs to a space of almost periodic functions (classical or generalized) is conferring to it a certain set of properties. The *oscillation* is the most important feature of such functions.

We are briefly describing now a scale of spaces, containing almost periodic functions in different sense than these mentioned above. These functions spaces are summarily defined in our book [6], and in other publications mentioned therein. This scale of spaces depends on a parameter r, $1 \leq r \leq 2$, say $AP_r(R, \mathcal{C})$, with $AP_1(R, \mathcal{C})$ the space of almost periodic functions with absolutely convergent series, and $AP_2 = B^2 =$ the Besicovitch space (with index 2).

More precisely, let $r \in [1, 2]$, and consider the (generic) trigonometric polynomial

$$T(t) = a_1 e^{i\lambda_1 t} + a_2 e^{i\lambda_2 t} + \cdots + a_n e^{i\lambda_n t}, \tag{4}$$

with $a_k \in \mathcal{C},\ \lambda_k \in R,\ k = 1, 2, ..., n,\ n \in N$.

The following is a functional on the linear space $\mathcal{T}$ of all trigonometric polynomials of the form (4), when $n \in N$ is arbitrary. The reals λ_k are always assumed distinct in the representation of each polynomial (4):

$$\|T\|_r = (|a_1|^r + |a_2|^r + \cdots + |a_n|^r)^{1/r}. \tag{5}$$

It is true that $(\mathcal{T}, \|\cdot\|_r)$ is a linear normed space (obviously, incomplete). By the completion procedure of metric/normed spaces, one obtains the space $AP_r(R, \mathcal{C})$. One has $AP_1(R, \mathcal{C})$ as described above, and $AP_2(R, \mathcal{C}) = B^2(R, \mathcal{C})$. The elements of AP_r could be usual functions, as it is for $r = 1$, or classes of equivalent functions, as it appears for $r = 2$.

Based on properties of the norm (5), one finds out that the following inclusions hold for the spaces AP_r:

$$AP_1 \subset AP_r \subset AP_s \subset AP_2, \tag{6}$$

for any $1 < r < s < 2$.

A long list of new problems is now emerging from the discussion carried out above. We shall list just a few of them.

a) What are the relationships between the spaces AP_r and the classical spaces mentioned above?

We know that AP_1 is the space of almost periodic functions whose Fourier series converges absolutely, while AP_2 is the Besicovitch space. Hence, the question a) makes sense for $r \in (1, 2)$.

For instance, we know that AP contains AP_1, and is contained in AP_2. How does AP stand when we compare it with AP_r, $r \in (0, 1)$?

b) What can we say about the nature of elements of the spaces AP_r, $r \in (1; 2)$?

For $r = 1$ they are functions, but for $r = 2$ they are classes of equivalent functions.

c) Since $AP_1 \subset AP_2 = B^2$, for $r \in (1, 2)$, it implies that the Fourier series is defined for each element of AP_r. From the definition of the norm $\|\cdot\|_r$, we easily find that for any function (or class), the Fourier series has the property that

$$\sum_{k=1}^{\infty} |a_k|^r < \infty. \tag{7}$$

In other words, the sequence of Fourier coefficients is in the space ℓ^r (real or complex).

Does this property possess a converse?

We know that this is true for $r = 1$ and $r = 2$, the latter case being clarified by Besicovitch [3], with his Fischer–Riesz type theorem.

Further, the fact that AP_r appears to be modeled on ℓ^r, because the Fourier series is completely determining the generating function/class, what more can be said about this relationship between AP_r and ℓ^r?

d) Investigate properties of the derivative or the integral of an element in AP_r.

e) Investigate properties of the spaces AP_r, $r \in (1, 2)$, as compactness, approximation of their elements by trigonometric polynomials (for example, of Féjer–Bochner type).

f) Build up the theory of AP_r spaces in case of functions with values in Banach spaces. (See [2], [7], [10] and [11] for background.)

Many other problems could be formulated, starting from the theory on almost periodicity for classical types of functions.

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