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PSEUDO-ATOMS OF FUZZY MULTISUBMEASURES

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Abstract. In this paper we establish some decomposition results using pseudo-atoms of fuzzy multisubmeasures with respect to the Hausdorff topology. We also point out several properties of fuzzy set multifunctions.

1. Introduction

Since Sugeno [11] has introduced in 1974 the concept of *fuzzy measure*, the non-additive case has become very attractive due to its applications in economics, statistics, theory of games, social sciences, engineering domains. Recently, interesting applications have been obtained in human decision making (Liginlal and Ow [9]) and prediction of osteoporotic fractures (Pham, Brandl, N. D. Nguyen and T. V. Nguyen [10]). Non-atomic additive (or non-additive) measures have been studied thanks to their important applications (e.g. in non-atomic games theory). O-continuity is also an important point in measure theory due to its relationships with different types of additivity.

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In this paper we present some properties of exhaustive and o-continuous set multifunctions and establish decomposition results with respect to pseudo-atoms of fuzzy multisubmeasures in Hausdorff topology.

2. Preliminaries

Let X be a real normed space, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of $X, \mathcal{P}_f(X)$ the family of all nonvoid closed subsets of X and $\mathcal{P}_{bfc}(X)$ the family of all nonvoid convex closed bounded subsets of X. On $\mathcal{P}_0(X)$ we consider the Minkowski addition " $\stackrel{\bullet}{+}$ ", defined by:

$$M \stackrel{\bullet}{+} N = \overline{M + N}, \text{ for every } M, N \in \mathcal{P}_0(X),$$

where $M + N = \{x + y | x \in M, y \in N\}$ and $\overline{M + N}$ is the closure of $M + N$ with respect to the topology induced by the norm of X.

For every $M, N \in \mathcal{P}_0(X)$, we denote

$$h(M, N) = \max\{e(M, N), e(N, M)\},$$

where $e(M, N) = \sup_{x \in M} d(x, N)$ is the excess of M over N and $d(x, N)$ is the distance from x to N with respect to the metric d induced by the norm of X. It is well-known that h becomes an extended metric on $\mathcal{P}_f(X)$ (i.e. is a metric which can also take the value $+\infty$) (see Hu and Papageorgiou [8]).

We denote $|M| = h(M, \{0\})$, for every $M \in \mathcal{P}_0(X)$, where 0 is the origin of X.

It is easy to see that $\mu(A) = \{0\}$ if and only if $|\mu(A)| = 0$.

By $i = \overline{1, n}$ we mean $i \in \{1, 2, \ldots, n\}$, for $n \in \mathbb{N}^*$, where $\mathbb{N}^* = \mathbb{N} \backslash \{0\}$. We also denote $\mathbb{R}_+ = [0, +\infty)$ and $\overline{\mathbb{R}}_+ = [0, +\infty]$.

Let T be an abstract nonvoid set, $\mathcal{P}(T)$ the family of all subsets of T and $\mathcal{C}$ a nonempty family of subsets of T. For every set multifunction $\mu : \mathcal{C} \to \mathcal{P}_0(X)$, we denote by $|\mu| : \mathcal{C} \to \overline{\mathbb{R}}_+$, the set function defined by $|\mu|(A) = |\mu(A)|$, for every $A \in \mathcal{C}$.

In the sequel, $\mathcal{C}$ is a ring of subsets of T and X is a real normed space.

3. Fuzzy set multifunctions

In this section we present some properties of exhaustivity and o-continuity (with respect to the Hausdorff topology) of set multifunctions. For the beginning we recall some classical definitions.

Definition 3.1. Let $m : \mathcal{C} \to \mathbb{R}_+$ be a set function, with $m(\emptyset) = 0$. m is said to be:

I) *exhaustive* if $\lim\limits_{n\to\infty} m(A_n) = 0$, for every sequence of mutually disjoint sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$.

II) *increasing convergent* if $\lim\limits_{n\to\infty} m(A_n) = m(A)$, for every increasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ (i.e. $A_n \subset A_{n+1}$, for every $n \in \mathbb{N}^*$), with $A = \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$ (denoted by $A_n \nearrow A$).

III) *decreasing convergent* if $\lim\limits_{n\to\infty} m(A_n) = m(A)$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ (i.e. $A_{n+1} \subset A_n$, for every $n \in \mathbb{N}^*$), with $A = \bigcap\limits_{n=1}^{\infty} A_n \in \mathcal{C}$ (denoted by $A_n \searrow A$).

IV) *monotone* if $m(A) \le m(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

V) *fuzzy* if m is monotone, increasing convergent and decreasing convergent.

VI) a *submeasure* (in the sense of Drewnowski [1]) if m is monotone and *subadditive*, that is, $m(A\cup B) \le m(A)+m(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

Definition 3.2. A set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(X)$, with $\mu(\emptyset) = \{0\}$ is said to be:

I) *a multisubmeasure* (Gavriluţ [2,3]) if μ is monotone (ie. $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$) and $\mu(A \cup B) \subseteq \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$ (or, equivalently, for every $A, B \in \mathcal{C}$);

II) *a multimeasure* if $\mu(A \cup B) = \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

III) Suppose μ is a multisubmeasure (multimeasure respectively). A set $A \in \mathcal{C}$ is said to be a $\{0\}$-*multisubmeasure* ($\{0\}$-*multimeasure* respectively) *set* if $\mu(A) = \{0\}$.

Definition 3.3. Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$.

μ is said to be:

I) *increasing convergent* (with respect to h) if $\lim\limits_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every increasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \nearrow A \in \mathcal{C}$.

II) *decreasing convergent* (with respect to h) if $\lim\limits_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \searrow A \in \mathcal{C}$.

III) *fuzzy* if it is monotone, increasing convergent and decreasing convergent.

IV) *exhaustive* (with respect to h) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every mutually disjoint sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$.

V) *o-continuous* (with respect to h) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \searrow \emptyset$.

VI) *of finite variation* if $\overline{\mu}(A) < \infty$, for every $A \in \mathcal{C}$, where $\overline{\mu} : \mathcal{C} \to \overline{\mathbb{R}}_+$, called *the variation of* μ, is defined, for every $A \in \mathcal{C}$, by:

$$\overline{\mu}(A) = \sup\{\sum_{i=1}^{n} |\mu(A_i)|; A_i \subseteq A,\ A_i \in \mathcal{C},$$
$$\forall i = \overline{1,n}, A_i \cap A_j = \emptyset,\ \text{for } i \neq j\}.$$

Examples 3.4. I. Let $\nu : \mathcal{C} \to \mathbb{R}_+$ be a set function and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ the set multfuncton defined by $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$. If ν is a submeasure (finitely additive, respectively), then μ is a multisubmeasure (monotone multimeasure, respectively).

μ is called *the multisubmeasure* (*multimeasure* respectively) *induced by* ν.

One can easily check that μ is fuzzy if and only if ν is fuzzy.

II. If $m_1, m_2 : \mathcal{C} \to \mathbb{R}_+$ are set functions such that m_1 is an o-continuous finitely additive measure and m_2 is an o-continuous submeasure (finitely additive measure, respectively), then the multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, defined by

$$\mu(A) = [-m_1(A), m_2(A)],\ \text{for every } A \in \mathcal{C},$$

is a fuzzy multisubmeasure (multimeasure respectively).

Examples 3.5.

I. If $\nu : \mathcal{C} \to \mathbb{R}_+$ is an exhaustive submeasure, then the multisubmeasure induced by ν is exhaustive.

II. If $\nu_1, \ldots, \nu_p : \mathcal{C} \to \mathbb{R}_+$, are p exhaustive finitely additive set functions, where $p \in \mathbb{N}^*$, we consider the multivalued set function $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, defined by

$$\mu(A) = \{\nu_1(A), \nu_2(A), \ldots, \nu_p(A)\}, \text{ for every } A \in \mathcal{C}.$$

μ is not a multisubmeasure, but the set multifunction $\widetilde{\mu} : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, defined by

$$\widetilde{\mu}(A) = \overline{\bigcup_{\substack{B \subset A \\ B \in \mathcal{C}}} \mu(B)}, \text{ for every } A \in \mathcal{C},$$

is an exhaustive multisubmeasure.

In what follows we establish several results related to exhaustive fuzzy multifunctions.

Theorem 3.6. *A set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is exhaustive if and only if every monotone sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ is Cauchy with respect to μ, that is,* $\lim\limits_{\substack{n\to\infty \\ m\to\infty}} |\mu(A_n \Delta A_m)| = 0$.

Proof. For the *if part*, suppose without any loss of generality that $(A_n)_{n\in\mathbb{N}}$ is increasing. Let us suppose, by the contrary, that it is not a Cauchy one. Then there exist $\varepsilon_0 > 0$ and an increasing sequence $(n_k)_k \subset \mathbb{N}^*$ so that $|\mu(A_{n_k} \Delta A_{n_{k+1}})| \geq \varepsilon_0$, for every $k \geq 1$.

Let $B_{n_k} = A_{n_{k+1}} \setminus A_{n_k}$, for every $k \geq 1$. Then

$$|\mu(B_{n_k})| = |\mu(A_{n_k} \Delta A_{n_{k+1}})| \geq \varepsilon_0, \text{ for every } k \geq 1,$$

which is false because B_{n_k} are all mutually disjoint and μ is exhaustive.

For the *only if part*, let $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ be a mutually disjoint sequence and consider $B_n = \bigcup\limits_{i=1}^{n} A_i$, for every $n \in \mathbb{N}^*$. Then $(B_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ is increasing, so

$$\lim_{n\to\infty} |\mu(A_n)| = \lim_{n\to\infty} |\mu(B_{n+1} \setminus B_n)| =$$
$$= \lim_{n\to\infty} |\mu(B_{n+1} \Delta B_n)| = 0,$$

as claimed. □

Theorem 3.7. *A monotone set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is exhaustive if and only if for every sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ and every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}^*$ so that $|\mu(A_n \setminus \bigcup_{k=1}^{n_0} A_k)| < \varepsilon$, for every $n > n_0$.*

Proof. For the *only if part,* let $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ be a mutually disjoint sequence. Then for every $\varepsilon > 0$, there exists $n_0 \geq 1$ so that $|\mu(A_n \setminus \bigcup_{k=1}^{n_0} A_k)| < \varepsilon$, for every $n \geq n_0$. Consequently, $|\mu(A_n)| = |\mu(A_n \setminus \bigcup_{k=1}^{n_0} A_k)| < \varepsilon$, for every $n > n_0$, so μ is exhaustive.

The *if part* follows immediately by Theorem 3.6, taking into account that $\mu(A_n \setminus \bigcup_{k=1}^{n_0} A_k) \subset \mu(A_n \setminus A_{n_0})$, so $|\mu(A_n \setminus \bigcup_{k=1}^{n_0} A_k)| \leq |\mu(A_n \setminus A_{n_0})|$, for every $n > n_0$. □

Remarks 3.8. (Gavriluţ [2,3]) Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a multisubmeasure. Then:

I. μ is decreasing convergent if and only if it is o-continuous.

II. If μ is o-continuous , then μ is increasing convergent. So, any decreasing convergent multisubmeasure is increasing convergent.

III. If $\mathcal{C}$ is a σ-ring and μ is o-continuous, then μ is exhaustive.

IV. If μ is increasing convergent and exhaustive, then μ is o-continuous.

Theorem 3.9. *Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a set multifunction.*

I. If μ is exhaustive and fuzzy, then μ is o-continuous.

II. If μ is a multisubmeasure, then μ is fuzzy if and only if it is o-continuous.

Proof. I. Suppose, by the contrary, that there exist $\varepsilon_0 > 0$ and $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \searrow \emptyset$ and $|\mu(A_n)| > \varepsilon_0$. Since for every $k \geq 1$ arbitrary, but fixed, $(A_k \setminus A_n) \nearrow A_k$ and μ is increasing convergent, then $\lim_{n\to\infty} h(\mu(A_1 \setminus A_n), \mu(A_1)) = 0$. Also, $|\mu(A_1)| > \varepsilon_0$.

Since $h(M, N) \geq ||M| - |N||$, for every $M, N \in \mathcal{P}_f(X)$, then

$$h(\mu(A_1 \setminus A_n), \mu(A_1)) \geq ||\mu(A_1)| - |\mu(A_1 \setminus A_n)|,$$

for every $n \in \mathbb{N}^*$.

It results

$$\lim_{n\to\infty} |\mu(A_1 \setminus A_n)| \geq |\mu(A_1)| > \varepsilon_0.$$

Then there is $n_1 \in \mathbb{N}^*$ so that $|\mu(A_1 \backslash A_{n_1})| > \varepsilon_0$.

The same as before, there exists $n_2 > n_1$ such that $|\mu(A_{n_1} \backslash A_{n_2})| > \varepsilon_0$. Continuing this way, we find an increasing sequence $(n_k)_k \subset \mathbb{N}^*$ so that
$|\mu(A_{n_k} \backslash A_{n_{k+1}})| > \varepsilon_0$, which is false because μ is exhaustive.

II. According to Remarks 3.8, any o-continuous multisubmeasure is increasing convergent and decreasing convergent, so it is a fuzzy multifunction. □

4. Atoms and pseudo-atoms of set multifunctions

Non-atomicity is an important topic in measure theory, having interesting applications in mathematical economics, statistics or theory of games. In Gavriluţ [4], Gavriluţ and Croitoru [5,6,7] we introduced and studied the concept of pseudo-atom of a set multifunction. If the notions of atom and pseudo-atom coincide for a finitely additive set function, this is not true for a set multifunction (as we shall see in Example 4.3). So, to define the concept of pseudo-atom in the set-valued case is quite justified).

In this section we present some properties concerning pseudo-atoms for set multifunctions and establish some decomposition results by pseudo-atoms of fuzzy multisubmeasures in Hausdorff topology.

To begin we recall some properties of pseudo-atoms in the set-valued case.

Definition 4.1. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction. A set $A \in \mathcal{C}$ is called:

I) *an atom* of μ if $\mu(A) \supsetneq \{0\}$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = \{0\}$ or $\mu(A \backslash B) = \{0\}$.

II) *a pseudo-atom* of μ if $\mu(A) \supsetneq \{0\}$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = \{0\}$ or $\mu(B) = \mu(A)$.

Remark 4.2. I. Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a multisubmeasure and $A, B \in \mathcal{C}$, with $B \subseteq A$. Then $\mu(A \backslash B) = \{0\}$ implies $\mu(A) = \mu(B)$. It follows that every atom of μ is a pseudo-atom of μ. As we shall see in the following example, the converse is not valid.

II. Let $\mu : \mathcal{C} \to \mathcal{P}_{bfc}(X)$ be a multimeasure and let $A, B \in \mathcal{C}$, with $B \subseteq A$. Then $\mu(A) = \mu(B)$ implies $\mu(A \backslash B) = \{0\}$. It follows that every pseudo-atom of μ is an atom of μ.

Consequently, $A \in \mathcal{C}$ is an atom of a monotone multimeasure $\mu : \mathcal{C} \to \mathcal{P}_{bfc}(X)$ if and only if A is a pseudo-atom of μ.

Example 4.3. Let $T = \{x, y, z\}$, $\mathcal{C} = \mathcal{P}(T)$ and for every $A \in \mathcal{C}$, let

$$\mu(A) = \begin{cases} [0,1], & \text{if } A \neq \emptyset, \\ \{0\}, & \text{if } A = \emptyset. \end{cases}$$

One can easily check that $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ is a multisubmeasure.

Let $A = \{x, y\}$. There is $B = \{x\} \subseteq A$, such that $\mu(B) \supsetneq \{0\}$ and $\mu(A \backslash B) = \mu(\{y\}) = [0,1] \supsetneq \{0\}$. So A is not an atom of μ.

Now, for every $C \in \mathcal{C}$, we have $\mu(C) = \{0\}$ for $C = \emptyset$ and $\mu(C) = [0,1] = \mu(A)$ for $C \neq \emptyset$, which shows that A is a pseudo-atom of μ.

So, there are pseudo-atoms of a multisubmeasure, which are not atoms.

Proposition 4.4. *Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ and $A \in \mathcal{C}$. If A is a pseudo-atom of μ, then A is also a pseudo-atom of $|\mu|$.*

Proof. Let $B \in \mathcal{C}$, $B \subset A$. Since A is a pseudo-atom of μ, it follows $\mu(B) = \{0\}$ or $\mu(B) = \mu(A)$. It results $|\mu(B)| = 0$ or $|\mu(B)| = |\mu(A)|$. So, A is a pseudo-atom of $|\mu|$. □

Remark 4.5. I. The converse of Proposition 4.4 is not valid as we can see in the following example. Let $T = \{a, b, c\}$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_0(\mathbb{R})$ defined, for every $A \in \mathcal{C}$, by:

$$\mu(A) = \begin{cases} \{0,1,2\}, & A \in \mathcal{P}_0(T) \text{ and } A \neq \{a,b\} \\ \{0,2\}, & A = \{a,b\} \\ \{0\}, & A = \emptyset. \end{cases}$$

It follows $|\mu(A)| = \begin{cases} 2, & A \neq \emptyset \\ 0, & A = \emptyset \end{cases}$, for every $A \in \mathcal{C}$.

Let $A = \{a, b\} \in \mathcal{C}$. We prove that A is a pseudo-atom of $|\mu|$. Let $B \in \mathcal{C}, B \subset A$. If $B = \emptyset$, then $|\mu(B)| = 0$. If $B \neq \emptyset$, then $|\mu(B)| = |\mu(A)|$. So, A is a pseudo-atom of $|\mu|$.

Now we show that the above set A is not a pseudo-atom of μ. Consider $B = \{a\} \in \mathcal{C}$. Then $B \subset A$ and we have that $\mu(B) \neq \{0\}$ and $\mu(B) \neq \mu(A)$.

II. Let $\nu : \mathcal{C} \to [0, \infty)$ be monotone, with $\nu(\emptyset) = 0$ and let μ be the monotone set multifunction induced by ν (i.e. $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$). Then $A \in \mathcal{C}$ is a pseudo-atom of μ if and only if A is a pseudo-atom of $|\mu| = \nu$.

Theorem 4.6. (Gavriluţ and Croitoru [7]) *Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a multisubmeasure and $A, B \in \mathcal{C}$ pseudo-atoms of μ. Then the following statements result:*

I. If $C \in \mathcal{A}$, $C \subseteq A$ has $\mu(C) \supsetneq \{0\}$, then C is a pseudo-atom of μ and $\mu(C) = \mu(A)$.

II. If $\mu(A \cap B) \supsetneq \{0\}$, then $A \cap B$ is a pseudo-atom of μ and $\mu(A \cap B) = \mu(A) = \mu(B)$.

III. If $\mu(A \cap B) = \{0\}$, then $A \backslash B$ and $B \backslash A$ are pseudo-atoms of μ and $\mu(A \backslash B) = \mu(A), \mu(B \backslash A) = \mu(B)$.

IV. If $\mu(A) \neq \mu(B)$, then $\mu(A \cap B) = \{0\}$, $\mu(A \backslash B) = \mu(A)$ and $\mu(B \backslash A) = \mu(B)$.

V. There exist mutual disjoint sets $C_1, C_2, C_3 \in \mathcal{C}$, with $A \cup B = C_1 \cup C_2 \cup C_3$, such that, for every $i \in \{1, 2, 3\}$, either C_i is a pseudo-atom of μ, or $\mu(C_i) = \{0\}$.

VI. If $\mu(A \cap B) \supsetneq \{0\}$ and $\mu(A \backslash B) = \mu(B \backslash A) = \{0\}$, then $A \cap B$ is a pseudo-atom of μ and $\mu(A \triangle B) = \{0\}$.

Remark 4.7. Suppose $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is a multisubmeasure.

I. According to Theorem 4.6 we may suppose that two pseudo-atoms of μ are either pairwise disjoint or coincide if we do not consider a $\{0\}$-multisubmeasure set.

II. If A_i is a pseudo-atom of μ, for every $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}^*$, then, by induction from Theorem 4.6-V, we can write $\bigcup\limits_{i=1}^{n} A_i = (\bigcup\limits_{j=1}^{m} B_j) \cup E$, where $\{B_j\}_{j=1}^m$ and E are pairwise disjoint sets of $\mathcal{C}$, $\{B_j\}_{j=1}^m$ are pseudo-atoms of μ and E is a $\{0\}$-multisubmeasure set.

Now we present some decomposition results using pseudo-atoms of fuzzy multisubmeasures with respect to the Hausdorff topology (using some ideas of Wu and Sun [12]).

Theorem 4.8. *Suppose $\mathcal{C}$ is a σ-ring and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is a fuzzy multisubmeasure. Then there exist at most countable pairwise disjoint pseudo-atoms $(A_n)_{n \in \mathbb{N}^*}$ of μ which satisfy the conditions:*

(i) $|\mu(A_n)| \geq |\mu(A_{n+1})|, \forall n \in \mathbb{N}^*$,
(ii) $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$,
(iii) $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}^*$, *such that* $|\mu(\bigcup_{k=n_0}^{\infty} A_k)| < \varepsilon$.

Proof. Let $\mathcal{U}_m = \{B \in \mathcal{C} | B$ is a pseudo-atom of μ and $\frac{1}{m} \leq |\mu(B)| < \frac{1}{m+1}\}$, for every $m \in \mathbb{N}^*$. Then $\mathcal{U}_m$ contains at most finite number of sets and by Remark 4.7, we may suppose that they are pairwise disjoint. Suppose, on the contrary, there are infinite pairwise disjoint sets $(B_n)_{n\in\mathbb{N}} \subset \mathcal{U}_m$. So, we have $|\mu(B_n)| \geq \frac{1}{m}$, for every $n \in \mathbb{N}$. Since μ is exhaustive (according to Remark 3.8-III and Theorem 3.9-II), it follows $\lim\limits_{n\to\infty} |\mu(B_n)| = 0$, which is false. Hence, there exist at most finite pairwise disjoint pseudo-atoms in $\mathcal{U}_m$, for every $m \in \mathbb{N}^*$ and denote all of them by $\{A_n\}_{n=1}^{\infty}$. Evidently, (i) is satisfied. Since (A_n) are pairwise disjoint and μ is exhaustive, it results (ii). We remark that $\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} A_k = \emptyset$. Since μ is o-continuous, it follows $\lim\limits_{n\to\infty} |\mu(\bigcup_{k=n}^{\infty} A_k)| = 0$, which proves (iii). □

Theorem 4.9. *Let $\mathcal{C}$ be a σ-ring and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ a fuzzy multisubmeasure. Then there exists a sequence (A_n) of pairwise disjoint pseudo-atoms of μ, such that for every $E \in \mathcal{C}$ and every $\varepsilon > 0$, there are a subsequence (A_{i_n}) of (A_n), $p_1, p_2 \in \mathbb{N}$ and $B \in \mathcal{C}$ which is pairwise disjoint of the sets (A_{i_n}), satisfying the conditions:*

(i) $E = (\bigcup\limits_{k=1}^{p_1} A_{i_k}) \cup (\bigcup\limits_{k=p_1+1}^{p_2} A_{i_k}) \cup (\bigcup\limits_{k=p_2+1}^{\infty} A_{i_k}) \cup B$,
(ii) $|\mu(A_{i_k})| > \varepsilon$, *for every* $k \in \{1, \ldots, p_1\}$,
(iii) $|\mu(A_{i_k})| \leq \varepsilon$, *for every* $k \in \{p_1 + 1, \ldots, p_2\}$,
(iv) $|\mu(\bigcup\limits_{k=p_2+1}^{\infty} A_{i_k})| \leq \varepsilon$,
(v) B *contains no pseudo-atoms of* μ.

Proof. From Theorem 4.6, we may suppose that all the pseudo-atoms which are contained in E are the subsequence (A_{i_n}) of (A_n) in Theorem 4.8. By Theorem 4.8-(iii), there exists $p_2 \in \mathbb{N}$ such that (iv) results. According to Theorem 4.8-(i), there exists $p_1 \in \mathbb{N}$ such that (ii) and (iii) follow. Now, denoting $B = E \setminus \bigcup\limits_{n=1}^{\infty} A_{i_n}$, (i) and (v) are obtained. □

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