"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 19 (2009), No. 2, 221 - 240

THE CENTENNIAL OF CONVERGENCE IN MEASURE

LIVIU C. FLORESCU

Abstract. In this work we recall the classical definitions and results about the topology of convergence in measure, but we also present some recent ones.

1. Introduction

Convergence in measure or convergence in probability was encountered already in the papers of Borel and Lebesgue, but it was not until 1909 that Riesz (see [18]) introduced it as an independent kind of convergence of a sequence of measurable functions.

Suppose that $(\Omega, \mathcal{A}, \mu)$ is a finite positive measure space, $(u_n)_{n\in\mathbb{N}}$ is a real valued measurable sequence on Ω and that u is a measurable function, also. It is happens that $\lim\limits_{n\to\infty} \mu(|u_n - u| \geq \varepsilon) = 0$, for any positive number ε, we say that the sequence $(u_n)_{n\in\mathbb{N}}$ converges in measure to u.

In his memoir from 1909, Riesz discovered that from every sequence of measurable functions which converges in measure one can select an almost everywhere convergent subsequence; from this he obtained that, in order for a measurable sequence $(u_n)_{n\in\mathbb{N}}$ to be convergent in measure, it is necessary and sufficient that the relation $\lim\limits_{m,n\to\infty} \mu(|u_m - u_n| \geq \varepsilon) = 0$ hold for any $\varepsilon > 0$.

Keywords and phrases: convergence in measure, Fréchet metric, Ky Fan metrics, topology of convergence in measure
(2000)Mathematics Subject Classification: 28A20

In [11] Fréchet introduced the following metric for convergence in measure:

$$(u, v) \mapsto d(u, v) = \inf\{\varepsilon + \mu(|u - v| \geq \varepsilon) : \varepsilon > 0\}.$$

Relating to the previous result of Riesz, d is a complete metric on the space of measurable functions. Fréchet showed that the almost everywhere convergence cannot be defined by a metric (it is not even a topological convergence). Other metrics on the same space were introduced by Ky Fan and by Hoffmann-Jørgensen (see [14]).

Fréchet gave in [12] a necessary and sufficient condition for a set of measurable functions on $[0, 1]$ to be compact under the metric d of convergence in measure. The generalization to an arbitrary measure space is due to Šmulian ([20]).

It is Vitali who proved that the convergence in measure play an important role for characterizing the strong convergence in $L^1(\mu)$. Exactly, if $(u_n)_{n\in\mathbb{N}}$ is a sequence of integrable functions and if u is a measurable function then u is integrable and $(u_n)_{n\in\mathbb{N}}$ is strongly convergent to u in $L^1(\mu)$ if and only if $(u_n)_{n\in\mathbb{N}}$ is uniformly integrable and convergent in measure to u (see [3, 7]). Having in mind the Dunford-Pettis criterion which characterizes the weak compactness in $L^1(\mu)$, the Vitali's result say that $(u_n)_{n\in\mathbb{N}}$ converges strongly to u if and only if it converges in measure and weakly to u (see Th. IV.8.12 from [7]).

Many mathematicians generalized the concept of convergence in measure to the case of functions valued in a separable metric space (see [1, 4, 5, 6, 21]) and, with the help of Hoffmann-Jørgensen's characterization, to a generally topological case.

Assume that S is a topological space; we say that a sequence $(u_n)_{n\in\mathbb{N}}$ of measurable functions from Ω to S is convergent in measure to the measurable function $u : \Omega \to S$ if, for any bounded continuous mapping $f : S \to \mathbb{R}$,

$$\|f(u_n) - f(u)\|_1 = \int_\Omega |f(u_n) - f(u)| d\mu \to 0, \text{ i.e. } f(u_n) \xrightarrow[L^1(\mu)]{\|\cdot\|_1} f(u).$$

If S is a metrizable space and d is a metric compatible with the topology of S this is equivalent with $\mu(d(u_n, u) \geq \varepsilon) \longrightarrow 0$, for any $\varepsilon > 0$, therefore with the classical convergence in measure of $(u_n)_{n\in\mathbb{N}}$ to u.

Surprisingly, for a separable metric space S, the convergence in measure on S is preserved if we replace the strong convergence on $L^1(\mu)$ with the weak convergence (i.e. $f(u_n) \xrightarrow[L^1(\mu)]{w} f(u)$, for every bounded continuous mapping $f : S \to \mathbb{R}$). But this last convergence is not induced by a complete uniformity (see [8]).

In the case where S is a metrizable Suslin space (particularly, if S is a Polish space), Dudley showed that $(u_n)_{n\in\mathbb{N}}$ is convergent in measure to $u : \Omega \to S$ if and only if

$$\sup_{\|f\|_{BL}\leq 1} \left| \int_A [f(u_n) - f(u)]\, d\mu \right| \longrightarrow 0, \text{ for every } A \in \mathcal{A},$$

where

$$\|f\|_{BL} = \|f\|_\infty + \|f\|_L = \sup_{x\in S} |f(x)| + \sup_{d(x,y)\neq 0} \frac{|f(x) - f(y)|}{d(x,y)},$$

and d is a metric compatible on S.

Let $\mathcal{U}$ be the uniformity on the set of measurable functions for which a sub-base is $\{U_{A,\varepsilon} : A \in \mathcal{A}, \varepsilon > 0\}$, where

$$U_{A,\varepsilon} = \{(u,v) : \sup_{\|f\|_{BL}\leq 1} \left| \int_A [f(u) - f(v)]\, d\mu \right| < \varepsilon\}.$$

$\mathcal{U}$ is a no-complete uniformity compatible with the topology of convergence in measure. A completion of the space of all measurable functions by rapport of $\mathcal{U}$ is the space of Young measures on S (see [2, 9] for Young measures).

2. Definitions. General properties

Let $(\Omega, \mathcal{A}, \mu)$ be a positive finite measure space; a set $N \subseteq \Omega$ is a μ–null set if there exists $B \in \mathcal{A}$ such that $N \subseteq B$ and $\mu(B) = 0$.

The Lebesgue extension of $(\Omega, \mathcal{A}, \mu)$ is the measure space $(\Omega, \mathcal{A}^1, \mu^1)$ where $\mathcal{A}^1 = \{A \cup N : A \in \mathcal{A}, N \text{ is a } \mu\text{–null set}\}$ and $\mu^1(A \cup N) = \mu(A)$, for all $A \in \mathcal{A}$ and all μ–null set N.

The measure μ is complete if $\mathcal{A} = \mathcal{A}^1$ and $\mu = \mu^1$; in this paper we suppose this situation.

2.1. Definition. Let (S, τ_S) be a topological space; a function $u : \Omega \to S$ is μ–**measurable**, or simple **measurable**, if $u^{-1}(D) \in \mathcal{A}$, for very $D \in \tau_S$; $A \subseteq S$ is μ–measurable iff the characteristic function χ_A is μ–measurable (i.e. $A \in \mathcal{A}$).

Let $\mathcal{M}(S)$ be the set of all μ–measurable functions on Ω to S. This set can be partitioned into equivalence classes; two functions belonging to the same equivalence class if they coincide μ-almost everywhere (with except of a μ–null set). We note by $\mathbf{M}(S)$ the set of all equivalence classes and we identify any μ–measurable function of $\mathcal{M}(S)$ with the corresponding equivalence class of $\mathbf{M}(S)$ and with all μ–measurable functions on Ω to S that belong to this equivalence class.

Particularly, if S is a separable metric space, then u is measurable whenever $u^{-1}(B) \in \mathcal{A}$, for all open ball $B \subseteq S$. If $S = \mathbb{R}$ then $u : \Omega \to \mathbb{R}$ is measurable whenever $u^{-1}(A_a) \in \mathcal{A}$, for all $a \in \mathbb{R}$, where A_a may have one of the following forms: $(a, +\infty), [a, +\infty), (-\infty, a)$ or $(-\infty, a]$.

2.2. Theorem (see [21], 1.4.20). *Let S be a topological space and let $u : \Omega \to S$; if u is μ–measurable then $f \circ u$ is μ–measurable, for every continuous mapping $f : S \to \mathbb{R}$.*

Conversely, if S is a metric space and $f \circ u$ is μ–measurable, for every continuous mapping $f : S \to \mathbb{R}$ then u is μ–measurable.

2.3. Theorem. *Let $w = (u, v) : \Omega \to S \times T$ be a mapping of Ω into a product of topological spaces S and T. If w is μ–measurable then so are $u : \Omega \to S$ and $v : \Omega \to T$.*

Conversely, if u and v are measurable and every open set in $S \times T$ is a countable union of open sets $D \times G$, where D is open in S and G is open in T, then w is μ-measurable.

Proof. If w is μ–measurable then, from theorem 2.2, composing w with the projections of $S \times T$ on S or T, we obtain that both u and v are measurable.

Conversely, if u and v are measurable, then, for any open sets D, G in S, T respectively, we have $w^{-1}(D \times G) = u^{-1}(D) \times v^{-1}(G)$. Hence $w^{-1}(D \times G)$ is μ–measurable.

The measurability of $w^{-1}(U)$, for any open set $U \subseteq S \times T$, follows from the assumption made on the topology of $S \times T$. ∎

2.4. Remarks. (i) If S and T are two separable metric spaces then $w = (u, v) : \Omega \to S \times T$ is μ–measurable if and only if u and v are μ–measurable.

(ii) If (S, d) is a separable metric space and $u, v : \Omega \to S$ are two μ–measurable mappings then $d(u, v) : \Omega \to \mathbb{R}_+, t \mapsto d(u(t), v(t))$, is a μ–measurable mapping. Indeed, from (i), $w = (u, v) : \Omega \to S \times S$ is a μ–measurable mapping and, from the theorem 2.2, $d \circ w = d(u, v)$ is μ–measurable.

(iii) We recall Nedoma's pathology: if $\mathcal{A}$ is a σ–algebra on a set S with $\mathrm{card}(S) > \mathrm{card}(\mathbb{R})$, then the diagonal set $\{(x,x) : x \in S\}$ is not a member of the product σ–algebra $\mathcal{A}\otimes\mathcal{A}$. *In this case, a metric d on the S is not a measurable mapping if we equip $S \times S$ with the product σ–algebra $\mathcal{A}\otimes\mathcal{A}$, where $\mathcal{A}$ is the Borel σ–algebra on* (S,d); for more details see 21.8, page 550, from [19] and [13].

We remark that, in this case, S cannot be equiped with a metric d that makes S separable (if (S,d) is a separable metric space then $\mathrm{card}(S) \le \mathrm{card}(\mathbb{R})$).

Now we define the convergence in measure on $\mathbf{M}(S)$. This notion was introduced in 1909 by F. Riesz ([18]) in the case $S = \mathbb{R}$:
"Soient $f_1(x), f_2(x), \cdots, f(x)$ des fonctions mesurables, définies sur l'ensemble E; ε étant une quantité positive quelconque, nous désignerons par $m(n,\varepsilon)$ la mesure de l'ensemble $[|f(x) - f_n(x)| > \varepsilon]$; alors nous dirons que la suite $[f_n(x)]$ tend en mesure vers la fonction $f(x)$ si, quelque petite que soit la quantité ε, on a $\lim_{n=\infty} m(n,\varepsilon) = 0$."

2.5. Definition. Let $(\Omega, \mathcal{A}, \mu)$ be a positive finite measure space, let (S,d) be a separable metric space, let $(u_n)_n \subseteq \mathbf{M}(S)$ and let $u \in \mathbf{M}(S)$; we say that $(u_n)_n$ **converges in measure** to u, and write $u_n \xrightarrow{\mu} u$, if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\left(\{t \in \Omega : d(u_n(t), u(t)) \ge \varepsilon\}\right) = 0.$$

$(u_n)_n$ is **Cauchy in measure** if, for every $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \mu\left(\{t \in \Omega : d(u_m(t), u_n(t)) \ge \varepsilon\}\right) = 0.$$

Usually, we will note $\{t \in \Omega : d(u(t), v(t)) \ge \varepsilon\}$ by $(d(u,v) \ge \varepsilon)$.

So $(u_n)_n$ converges in measure to u ($(u_n)_n$ is Cauchy in measure) if, for every $\varepsilon > 0$, $\lim_{n\to\infty} \mu(d(u_n,u) \ge \varepsilon) = 0$ ($\lim_{m,n\to\infty} \mu(d(u_m,u_n) \ge \varepsilon) = 0$).

Even in the case of a non-separable metric space (S,d) and of non-measurable functions we still say that $(u_n)_n \subseteq S^\Omega$ **converges in measure** to $u \in S^\Omega$ (respectively, $(u_n)_n$ is **Cauchy in measure**) if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu^*(d(u_n,u) \ge \varepsilon) = 0$$

$$(\lim_{m,n\to\infty} \mu^*(d(u_m,u_n) \ge \varepsilon) = 0).$$

In the above relations $\mu^* : \mathcal{P}(\Omega) \to \overline{\mathbb{R}}_+$ is the outer measure associated to μ:

$$\mu^*(A) = \inf\{\mu(B) : A \subseteq B, B \in \mathcal{A}\}.$$

Note that $\mu^*(A) = \mu(A)$ if $A \in \mathcal{A}$.

Because of Nedoma's pathology (see (iii) of remark 2.4, generally, the set $(d(u_n, u) \geq \varepsilon) \notin \mathcal{A}$ and we must replace the measure with the outer measure.

In the 2.14 we will extend the convergence in measure to the general topological case.

2.6. Theorem (Riesz, 1909 - for the real-valued measurable mappings).

Let $(u_n)_n$ be a sequence of arbitrary mappings defined on Ω with values in the metric space (S, d).

1). If $(u_n)_n$ converges in measure to $u : \Omega \to S$ then there exists a subsequence $(u'_n)_n$ of $(u_n)_n$ that converges to u μ–almost uniformly (i. e., for each $\varepsilon > 0$, there is a set $A_\varepsilon \subseteq \Omega$ with $\mu^*(A_\varepsilon) < \varepsilon$ so that $(u'_n)_n$ converges to u uniformly on $\Omega \setminus A_\varepsilon$); $(u'_n)_n$ converges also μ–almost everywhere to u.

2). If $(u_n)_n$ converges to u μ–almost uniformly then $(u_n)_n$ converges in measure and μ–almost everywhere to u.

3). If (S, d) is a complete metric space and if $(u_n)_n$ is Cauchy in measure then there exist a function $u : \Omega \to S$ and a subsequence $(u'_n)_n$ of $(u_n)_n$ μ–almost uniformly convergent to u.

4). If (S, d) is complete metric space then $(u_n)_n$ is convergent in measure if and only if $(u_n)_n$ is Cauchy in measure.

Proof. 1). We suppose that, for any $\varepsilon > 0$, $\lim_n \mu^*(d(u_n, u) > \varepsilon) = 0$; then we can find a subsequence $(u'_n)_n$ of $(u_n)_n$ such that

$$\mu^*\left(d(u'_n, u) \geq \frac{1}{2^n}\right) < \frac{1}{2^n}, \text{ for every } n \in \mathbb{N}.$$

Let $A_n = \bigcup_{i=n}^{\infty} \left(d(u'_i, u) > \frac{1}{2^i}\right)$ and $A = \bigcap_{n=1}^{\infty} A_n$; then, for every $n \in \mathbb{N}$, $\mu^*(A_{n+1}) < \frac{1}{2^n}$ and $\mu^*(A) = 0$. For every $p \in \mathbb{N}$, $(u'_n)_n$ is convergent to u uniformly on $\Omega \setminus A_p$ and $(u'_n)_n$ is pointwise convergent to u on $\Omega \setminus A$.

2). For every $\theta > 0$ there exists $A_\theta \subseteq \Omega$ such that $\mu^*(A_\theta) < \theta$ and $(u_n)_n$ converges to u uniformly on $\Omega \setminus A_\theta$. For every $\varepsilon > 0$, let now

$n_0 \in \mathbb{N}$ such that $d(u_n(t), u(t)) \leq \varepsilon$, for every $n \geq n_0$ and for any $t \in \Omega \setminus A_\theta$. Therefore $(d(u_n, u) > \varepsilon) \subseteq A_\theta$ and so $(u_n)_n$ is convergent in measure to u.

If $p \in \mathbb{N}^*, \theta = \frac{1}{p}$ and if A_p is such that $\mu^*(A_p) < \frac{1}{p}$ and $u_n \xrightarrow[\Omega \setminus A_p]{u} u$ then $A = \bigcap_{p=1}^{\infty} A_p$ is a μ–null set and $u_n(t) \to u(t)$, for every $t \in \Omega \setminus A$.

3). If $(u_n)_n$ is Cauchy in measure then, as it is noted above, we can find a subsequence $(u'_n)_n$ of $(u_n)_n$ such that, for every $n \in \mathbb{N}$,

$$\mu^* \left(d(u'_{n+1}, u'_n) > \frac{1}{2^n} \right) < \frac{1}{2^n}.$$

For any $p \in \mathbb{N}$, let us note $A_p = \bigcup_{i=p}^{\infty} \left(d(u'_{i+1}, u'_i) > \frac{1}{2^i} \right)$ and $A = \bigcap_{p=1}^{\infty} A_p$.

Then $\mu^*(A_p) < \frac{1}{2^{p-1}}, \mu^*(A) = 0$ and, for every $t \in \Omega \setminus A$, $(u'_n(t))_n$ is a Cauchy sequence in the complete metric space (S, d). Fix a point $x_0 \in S$ and then define $u : \Omega \to S, u(t) = \begin{cases} \lim_n u'_n(t) & , t \in \Omega \setminus A, \\ x_0 & , t \in A. \end{cases}$

Then $(u'_n)_n$ converges to u μ–almost uniformly.

4). Obviously, every sequence convergent in measure is Cauchy in measure.

Conversely, if $(u_n)_n$ is Cauchy in measure then, from 3), there exist a function u and a subsequence $(u'_n)_n$ of $(u_n)_n$ μ–almost uniformly convergent to u; from 2) $(u'_n)_n$ converges in measure to u.

For any $\varepsilon > 0$ and for every $n \in \mathbb{N}$,

$$\mu^*(d(u_n, u) > \varepsilon) \leq \mu^*(d(u_n, u'_n) > \varepsilon) + \mu^*(d(u'_n, u) > \varepsilon)$$

and so $(u_n)_n$ is convergent in measure to u. ■

2.7. Theorem (see theorem 4.2.2 from [6]). *If (S, d) is a metric space and if $(u_n)_n \subseteq \mathcal{M}(S)$ converges to u μ–almost everywhere then $u \in \mathcal{M}(S)$.*

Proof. Let D be an open set in (S, d); if $u^{-1}(D) = \emptyset$ then $u^{-1}(D) \in \mathcal{A}$.

Suppose now that $u^{-1}(D) \neq \emptyset$.

For every $m \in \mathbb{N}^*$ let $F_m = \{y \in D : S(y, \frac{1}{m}) \subseteq D\}$, where $S(y, \frac{1}{m}) = \{z \in S : d(y, z) < \frac{1}{m}\}$. F_m is a closed set, for every $m \in \mathbb{N}^*$. Indeed, if $(y_p)_p \subseteq F_m$ and $y_p \to y$ then, for every $z \in S(y, \frac{1}{m}), d(y, z) < \frac{1}{m}$; there exists $p \in \mathbb{N}$ such that $d(y_p, z) < \frac{1}{m}$ so $z \in S(y_p, \frac{1}{m}) \subseteq D$. It follows that $S(y, \frac{1}{m}) \subseteq D$, hence $y \in F_m$.

Now let A be the μ–null set $\{t \in \Omega : u_n(t) \nrightarrow u(t)\}$; then

$$(*) \qquad u^{-1}(D) \setminus A = \bigcup_{m\in\mathbb{N}^*} \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} u_k^{-1}(F_m) \setminus A.$$

Indeed, for every $t \in u^{-1}(D) \setminus A, u(t) \in D$ and $u_n(t) \to u(t)$.There exist $m \in \mathbb{N}^*$ such that $u(t) \in F_m$ and there is $n \in \mathbb{N}^*$ such that $d(u_k(t), u(t)) < \frac{1}{2m}$, for every $k \geq n$. For every $y \in S(u_k(t), \frac{1}{2m}), d(y, u(t)) \leq d(y, u_k(t)) + d(u_k(t), u(t)) < \frac{1}{m}$. Therefore $S(u_k(t), \frac{1}{2m}) \subseteq S(u(t), \frac{1}{m}) \subseteq D$, so that $u_k(t) \in F_{2m}$. Hence $t \in u_k^{-1}(F_{2m})$, for every $k \geq n$.

Conversely, for every $t \in \bigcup_{m\in\mathbb{N}^*} \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} u_k^{-1}(F_m) \setminus A$, there exist $m, n \in \mathbb{N}^*$ such that, for every $k \geq n$, $u_k(t) \in F_m$ and $u_k(t) \to u(t)$. As F_m is a closed set, $u(t) \in F_m \subseteq D$. So, $t \in u^{-1}(D) \setminus A$.

Now, as $(u_n) \subseteq \mathcal{M}(S)$ and F_m are closed sets, $\bigcup_{m\in\mathbb{N}^*} \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} u_k^{-1}(F_m) \in \mathcal{A}$ and, from $(*)$, $u^{-1}(D)$ is a μ–measurable set, for every $D \in \tau_S$. ∎

2.8. Remarks. (i) From the above theorem, in the theorem 2.6, $u \in \mathcal{M}(S)$ whenever $(u_n)_n \subseteq \mathcal{M}(S)$.

(ii) Generally, for a topological space S, the result in the previous theorem is not valid (see the proposition 4.2.3 from [6]).

2.9. Theorem. *Let (S, d) be a metric space; a sequence $(u_n)_n$ is convergent in measure to u if and only if every subsequence $(u'_n)_n$ of $(u_n)_n$ has a subsubsequence $(u''_n)_n$ convergent to u μ–almost everywhere.*

Proof. Every subsequence $(u'_n)_n$ of a sequence $(u_n)_n$ convergent in measure to u is convergent in measure to u. From 1) of theorem 2.6 , $(u'_n)_n$ has a subsequence $(u''_n)_n$ convergent μ–almost everywhere to u.

Conversely, if $(u_n)_n$ does not converge in measure to u, then there exist an $\varepsilon > 0$ and a subsequence $(u'_n)_n$ such that $\mu^*(d(u'_n, u) > \varepsilon) \geq \varepsilon$, for every $n \in \mathbb{N}$; but this subsequence cannot have a subsubsequence μ–almost everywhere convergent to u. ∎

2.10. Proposition. *Let S and T be metric spaces and let $(u_n)_n \subseteq S^\Omega$ be a sequence convergent in measure to $u : \Omega \to S$. If $f : S \to T$ is a continuous mapping then $(f(u_n))_n \subseteq T^\Omega$ is convergent in measure to $f(u)$.*

Proof. Every subsequence $(u'_n)_n$ of $(u_n)_n$ has a subsubsequence $(u''_n)_n$ convergent to u μ–almost everywhere. Then $(f(u''_n))_n$ is μ–almost everywhere convergent to $f(u)$. Therefore, from the theorem 2.9, $(f(u_n))_n$ is convergent in measure to $f(u)$. ∎

2.11. Corollary. *Let S, T, V be three metric space and let $(u_n)_n \subseteq S^\Omega$ convergent in measure to $u \in S^\Omega$ and $(v_n)_n \subseteq T^\Omega$ convergent in measure to $v \in T^\Omega$. If $h : S \times T \to V$ is a mapping continuous on the product metric space $S \times T$ then $(h(u_n, v_n))_n \subseteq V^\Omega$ is convergent in measure to $h(u, v)$.*

From the characterization given in the theorem 2.9 the notion of convergence in measure depends only on the topology of S and is independent of the choice of metric d. Fréchet showed that the convergence almost everywhere is not a topological one, even on the subspace $\mathcal{M}(S) \subseteq S^\Omega$. For this reason the characterization presented in the theorem 2.9 is not a convenient tool to extend the convergence in measure from metric spaces to general topological spaces.

The following theorem presents some topological alternatives of definition 2.5 for convergence in measure.

2.12. Theorem (Hoffmann-Jørgensen, 1996 - see [14]). *Let (S, d) be a separable metric space, let $(u_n)_n \subseteq \mathcal{M}(S)$ and $u \in \mathcal{M}(S)$; the following statements are equivalents:*

(*i*) $u_n \xrightarrow{\mu} u$.

(*ii*) $\int_\Omega \Phi(u_n, u) d\mu \to 0$, *for every bounded continuous mapping* $\Phi : S \times S \to [0, +\infty)$ *with* $\Phi(x, x) = 0$, *for every* $x \in S$.

(*iii*) $\int_\Omega |f(u_n) - f(u)| d\mu \to 0$, *for every* $f \in C^b(S)$.

(*iv*) $\int_A f(u_n) d\mu \to \int_A f(u) d\mu$, *for every* $A \in \mathcal{A}$ *and every* $f \in C^b(S)$.

(*v*) $\mu((u_n, u) \in F) \to 0$, *for every closed set* $F \subseteq S \times S$ *with* $\mu((u, u) \in F) = 0$.

Proof. From (ii) of remark 2.4, the mapping $t \mapsto (u_n(t), u(t))$ is measurable and so $((u_n, u) \in F) \in \mathcal{A}$.

$\boxed{(i) \Longrightarrow (ii)}$ Let $\Phi : S \times S \to [0, +\infty)$ be a bounded continuous mapping with $\Phi(x, x) = 0$, for every $x \in S$. For every subsequence

$(u'_n)_n$ of $(u_n)_n$ there exists a subsubsequence $(u''_n)_n$ convergent to u μ–almost everywhere (see the theorem 2.9); so $(\Phi(u''_n, u))_n$ converges μ-a.e. to 0. From theorem of bounded convergence $\int_\Omega \Phi(u''_n, u)d\mu \to 0$. So, every subsequence of $\left(\int_\Omega \Phi(u_n, u)d\mu\right)_n \subseteq \mathbb{R}$ has a subsubsequence convergent to 0; then $\int_\Omega \Phi(u_n, u)d\mu \to 0$.

$\boxed{(ii) \Longrightarrow (iii)}$ For every $f \in C^b(S)$, the mapping $\Phi : S \times S \to [0, +\infty)$, defined by $\Phi(x, y) = |f(x) - f(y)|$, is a bounded continuous function on $S \times S$ and $\Phi(x, x) = 0$, for every $x \in S$. Therefore $\int_\Omega |f(u_n) - f(u)|d\mu \to 0$.

$\boxed{(iii) \Longrightarrow (iv)}$ This implication is obvious if we remark that (iii) says that $(f(u_n))_n$ is strongly convergent to $f(u)$ while (iv) that it is weakly convergent to $f(u)$ in $L^1(\mu)$.

$\boxed{(iv) \Longrightarrow (i)}$ a). First, we suppose that (S, d) is a compact metric space; so $C^b(S) = C(S)$. With the uniform convergence norm, $\|\cdot\|_\infty$, $C(S)$ is a separable Banach space.

The mapping $\varphi : \Omega \to C(S)$ defined by $\varphi(t) = d(u(t), \cdot)$, for every $t \in \Omega$, is a measurable function ($\varphi = d(\cdot, \cdot) \circ u$ and we use the theorem 2.2). Moreover

$$\|\varphi\|_1 = \int_\Omega \|\varphi(t)\|_\infty d\mu(t) = \int_\Omega \sup_{x \in S} d(u(t), x)d\mu(t) \le K \cdot \mu(\Omega) < +\infty,$$

where $K = \sup_{x,y \in S} d(x, y)$.

Hence $\varphi \in L^1(\mu, C(S))$. As the subspace of all μ–simple functions is dense in $L^1(\mu, C(S))$, for every $\varepsilon > 0$, there exists $\theta : \Omega \to C(S), \theta(t) = \sum_{k=1}^p \chi_{A_k}(t) \cdot f_k$ such that

$$\|\varphi - \theta\|_1 = \int_\Omega \sup_{x \in S} |\varphi(t)(x) - \theta(t)(x)|d\mu(t) < \frac{\varepsilon}{4} \tag{1}$$

(in the representation of θ, $\{A_1, \cdots, A_p\} \subseteq \mathcal{A}$ is a partition of Ω and $\{f_1, \cdots, f_p\} \subseteq C(S)$).

From (iv), $\sum_{k=1}^p \int_{A_k} f_k(u_n)d\mu \to \sum_{k=1}^p \int_{A_k} f_k(u)d\mu$ and so, there exists $n_0 \in \mathbb{N}$ such that, for every $n \ge n_0$,

$$\left| \sum_{k=1}^p \int_{A_k} f_k(u_n)d\mu - \sum_{k=1}^p \int_{A_k} f_k(u)d\mu \right| < \frac{\varepsilon}{2}. \tag{2}$$

Now, from (2) and (1)

$$\int_\Omega d(u(t),u_n(t))d\mu(t) \le \int_\Omega |d(u,u_n) - \sum_{k=1}^{p} \chi_{A_k}\cdot f_k(u_n)|d\mu +$$

$$+\left|\sum_{k=1}^{p}\int_{A_k} f_k(u_n)d\mu - \sum_{k=1}^{p}\int_{A_k} f_k(u)d\mu\right| + \left|\sum_{k=1}^{p}\int_{A_k} f_k(u)d\mu - \int_\Omega d(u,u)d\mu\right| \le$$

$$\le \int_\Omega \sup_{x\in S}|d(u(t),x)-\theta(t)(x)|\, d\mu(t) + \frac{\varepsilon}{2} + \int_\Omega |\theta(t)(u) - \varphi(t)(u)|d\mu <$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \int_\Omega \sup_{x\in S}|\varphi(t)(x) - \theta(t)(x)|d\mu < \varepsilon, \text{ for every } n \ge n_0.$$

Therefore $\int_\Omega d(u(t),u_n(t))d\mu(t) \to 0$.

But, for every $\varepsilon > 0$,

$$\int_\Omega d(u(t),u_n(t))d\mu(t) \ge \int_{(d(u,u_n)>\varepsilon)} d(u,u_n)d\mu \ge \varepsilon\cdot\mu(d(u,u_n)>\varepsilon)$$

and so $\mu(d(u,u_n)>\varepsilon) \to 0$, for every $\varepsilon > 0$. Hence $u_n \xrightarrow{\mu} u$.

b). Let now S be a separable metric space and let $j : S \to [0,1]^{\mathbb{N}}$ be the natural embedding of S in the Hilbert cube; we note by $\bar{S} = \overline{j(S)}$, the metric compactification of S and let $\bar{d}$ be a compatible metric on $\bar{S}$. We define the mapping $J : \mathcal{M}(S) \to \mathcal{M}(\bar{S})$ by $J(u) = j\circ u \in \mathcal{M}(\bar{S})$ (for the measurability of J see theorem 2.2). We remark that J is an injective function and that $u_n \xrightarrow{\mu} u$ in $\mathcal{M}(S)$ iff $J(u_n) \xrightarrow{\mu} J(u)$ in $\mathcal{M}(\bar{S})$ (we can use the subsequence characterization from the theorem 2.9).

The J is an homeomorphism between $\mathcal{M}(S)$ and $J(\mathcal{M}(S)) \subseteq \mathcal{M}(\bar{S})$.

For every $\bar{f} \in C^b(\bar{S})$ we define $f : S \to \mathbb{R}$ letting $f = \bar{f}\circ j$. Then $f \in C^b(S)$ and therefore

$$\int_\Omega |\bar{f}(j(u_n)) - \bar{f}(j(u))|d\mu = \int_\Omega |f(u_n) - f(u)|d\mu \to 0.$$

From a), $j\circ u_n = J(u_n) \xrightarrow{\mu} j\circ u = J(u)$ and so $u_n \xrightarrow{\mu} u$.

$\boxed{(ii) \Longrightarrow (v)}$ For every $F = \bar{F} \subseteq S\times S$ with $\mu((u,u)\in F) = 0$ and for every $a > 0$ we note by $F^a = \{(x,y)\in S\times S : e((x,y),F) < a\}$ the open e–ball of radius a and center F, where $e = d\times d$ is the product metric on $S\times S$; F^a is an open set in $S\times S$ and so there is a continuous

function Φ_a from $S \times S$ to $[0,1]$ such that $\Phi_a(x,y) = 1$ if $(x,y) \in F$ and $\Phi_a(x,y) = 0$ for $(x,y) \in (S \times S) \setminus F^a$.
We define now $\Phi : S \times S :\to \mathbb{R}_+, \Phi(x,y) = |\Phi_a(x,y) - \Phi_a(x,x)|$. Then Φ is a bounded continuous mapping on $S \times S$ with $\Phi(x,x) = 0$, for every $x \in S$.
From (ii), $\int_\Omega \Phi(u_n,u)d\mu \to 0$. Now, $\Phi_a(x,y) \leq \Phi(x,y) + \Phi_a(x,x)$ and then

$$\limsup_n \int_\Omega \Phi_A(u_n,u)d\mu \leq \lim_n \int_\Omega \Phi(u_n,u)d\mu + \int_\Omega \Phi_a(u,u)d\mu. \tag{3}$$

But

$$\int_\Omega \Phi_a(u,u)d\mu = \int_{((u,u)\in F^a)} \Phi_a(u,u)d\mu \leq \mu((u,u) \in F^a)$$

and we rewrite (3)

$$\limsup_n \int_\Omega \Phi_a(u_n,u)d\mu \leq \mu((u,u) \in F^a). \tag{4}$$

On the other hand,

$$\int_\Omega \Phi_a(u_n,u)d\mu = \int_{((u_n,u)\in F^a)} \Phi_a(u_n,u)d\mu \geq$$
$$\geq \int_{((u_n,u)\in F)} \Phi_a(u_n,u)d\mu = \mu((u_n,u) \in F)$$

and, from (4), we obtain

$$\limsup_n \mu((u_n,u) \in F) \leq \mu((u,u) \in F^a), \text{ for every } a > 0. \tag{5}$$

$\lim_{a\downarrow 0} \mu((u,u) \in F^a) = \mu((u,u) \in F) = 0$ and, from (5), $\lim_n \mu((u_n,u) \in F) = 0$.

$\boxed{(v) \Longrightarrow (ii)}$ For every bounded continuous mapping $\Phi : S \times S \to [0,+\infty)$ with $\Phi(x,x) = 0$ and for every $p \in \mathbb{N}^*, F_p = \Phi^{-1}[\frac{1}{p},+\infty)$ is a closed set in $S \times S$ and $\mu((u,u) \in F_p) = \mu(\emptyset) = 0$. From (v),

$$\mu((u_n,u) \in F_p) = \mu\left(\Phi(u_n,u) \geq \frac{1}{p}\right) \to 0. \tag{6}$$

On the other hand,
(7)

$$\int_\Omega \Phi(u_n,u)d\mu = \int_{(\Phi(u_n,u)<\frac{1}{p})} \Phi(u_n,u)d\mu + \int_{(\Phi(u_n,u)\geq\frac{1}{p})} \Phi(u_n,u)d\mu \leq$$

$$\leq \frac{1}{p} \cdot \mu(\Omega) + M \cdot \mu\left(\Phi(u_n, u) \geq \frac{1}{p}\right),$$

where M is such that $\Phi(x,y) \leq M$, for every $(x,y) \in S \times S$.

From (6) and (7)

$$\limsup_n \int_\Omega \Phi(u_n, u) d\mu \leq \frac{1}{p} \cdot \mu(\Omega), \text{ for every } p \in N^*$$

and therefore $\lim_n \int_\Omega \Phi(u_n, u) d\mu = 0$. ∎

2.13. Remark. The equivalence $(iii) \iff (iv)$ shows that the strong convergence of $(f(u_n))_n$ in $L^1(\mu)$ is equivalent with the weak convergence, for every $f \in C^b(S)$. We remark that

$$\sup_{A\in\mathcal{A}} \left| \int_A [f(u_n) - f(u)]\, d\mu \right| \leq \int_\Omega |f(u_n) - f(u)| d\mu \leq$$

$$\leq 2 \cdot \sup_{A\in\mathcal{A}} \left| \int_A [f(u_n) - f(u)]\, d\mu \right|.$$

So (iii) is equivalent with $\int_A f(u_n) d\mu \to \int_A f(u) d\mu$, uniformly with $A \in \mathcal{A}$. From (iv) this is equivalent with $\int_A f(u_n) d\mu \to \int_A f(u) d\mu$, pointwise with $A \in \mathcal{A}$.

Previous theorem allows us to extend in a topological sense the definition of convergence in measure to topological spaces.

2.14. Definition. Let S be a topological space; a sequence $(u_n)_n \subseteq \mathcal{M}(S)$ **converges in measure** to $u \in \mathcal{M}(S)$ if

$$\int_\Omega |f(u_n) - f(u)| d\mu \to 0, \text{ for every } f \in C^b(S),$$

i.e. if $f(u_n) \xrightarrow[L^1(\mu)]{\|\cdot\|_1} f(u)$, for every $f \in C^b(S)$.

3. Metrics on $\mathbf{M}(S)$

In [11] M. Fréchet introduced a metric compatible with the topology of convergence in measure on $\mathbf{M}(\mathbb{R})$. Other metrics were introduced by Ky Fan.

3.1. Theorem (see th. 9.2.2 of [6]). *Let (S,d) be a separable metric space and let $\alpha : \mathbf{M}(S) \times \mathbf{M}(S) \to \mathbb{R}_+$ defined by*

$$\alpha(u,v) = \inf\{\varepsilon > 0 : \mu(d(u,v) > \varepsilon) \leq \varepsilon\}.$$

Then:

(*i*) $0 \leq \alpha(u,v) \leq \mu(\Omega)$ *and* $\mu(d(u,v) > \alpha(u,v)) \leq \alpha(u,v)$.
(In the other words the infimum in definition of $\alpha(u,v)$ is attained.)

(*ii*) α *is a metric on* $\mathbf{M}(S)$.

(*iii*) $u_n \xrightarrow[\mathbf{M}(S)]{\mu} u \Leftrightarrow \alpha(u_n, u) \to 0$ *and*
$(u_n)_n$ *is Cauchy in measure in* $\mathbf{M}(S) \Leftrightarrow (u_n)$ *is* α*–Cauchy.*

Proof. (i) Let us remark that, if $\mu(d(u,v) > \varepsilon) \leq \varepsilon$ and if $\varepsilon < \eta$, then $\mu(d(u,v) > \eta) \leq \eta$.

If we suppose that $\alpha(u,v) > \mu(\Omega)$ then $\mu(d(u,v) > \mu(\Omega)) > \mu(\Omega)$ what is absurd.

Let now $\varepsilon_k \downarrow \alpha(u,v)$ such that $\mu(d(u,v) > \varepsilon_k) \leq \varepsilon_k$. Then, for every $j \leq k, \mu(d(u,v) > \varepsilon_k) \leq \varepsilon_j$. Letting $k \to +\infty$ in the last inequality, $\mu(d(u,v) > \alpha(u,v)) \leq \varepsilon_j$, for all j and so $\mu(d(u,v) > \alpha(u,v)) \leq \alpha(u,v)$.

(ii) α is symmetric and nonnegative.

$\alpha(u,v) = 0$ if and only if $d(u,v) = 0$ μ–almost everywhere so that iff $u = v$ μ–a.e.

Let now $u, v, w \in \mathbf{M}(S)$; for every $a > \alpha(u,w) + \alpha(v,w)$ there exist $b > \alpha(u,w)$ and $c > \alpha(v,w)$ such that $a = b + c$. Let $\varepsilon < b$ and $\eta < c$ such that $\mu(d(u,w) > \varepsilon) \leq \varepsilon$ and $\mu(d(v,w) > \eta) \leq \eta$. As $d(u,v) \leq d(u,w) + d(v,w)$, $(d(u,w) \leq \varepsilon) \cap (d(v,w) \leq \eta) \subseteq (d(u,v) \leq \varepsilon + \eta)$ or $(d(u,v) > \varepsilon + \eta) \subseteq (d(u,w) > \varepsilon) \cup (d(v,w) > \eta)$ from where $\mu(d(u,v) > \varepsilon + \eta) \leq \varepsilon + \eta$. Therefore $\alpha(u,v) < \varepsilon + \eta < b + c = a$ so that $\alpha(u,v) \leq \alpha(u,w) + \alpha(v,w)$.

Therefore α is a metric on $\mathbf{M}(S)$.

(iii) If $u_n \xrightarrow{\mu} u$ (respectively, $(u_n)_n$ is Cauchy in measure) then, for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\mu(d(u_n, u) > \varepsilon) < \varepsilon$, for any $n \geq n_0$ (respectively, $\mu(d(u_m, u_n) > \varepsilon) < \varepsilon$, for every $m, n \geq n_0$) so that $\alpha(u_n, u) \leq \varepsilon$, for every $n \geq n_0$ $(\alpha(u_m, u_n) \leq \varepsilon$, for every $m, n \geq n_0)$.

Therefore $\alpha(u_n, u) \to 0$ $(\alpha(u_m, u_n) \to 0)$.

Conversely, if $\alpha(u_n, u) \to 0$ (respectively, $\alpha(u_m, u_n) \to 0$) then, for every $\eta > 0$ there is $n_0 \in \mathbb{N}$ such that $\alpha(u_n, u) < \eta$, for any $n \geq n_0$ $(\alpha(u_m, u_n) < \eta$, for every $m, n \geq n_0)$.

So $\mu(d(u_n, u) > \eta) \leq \eta$, for any $n \geq n_0$ $(\mu(d(u_m, u_n) > \eta) \leq \eta$, for every $m, n \geq n_0)$ and therefore $u_n \xrightarrow{\mu} u$ ($(u_n)_n$ is Cauchy in measure).

■

3.2. Corollary. *If (S,d) is a separable complete metric space then $(\mathbf{M}(S),\alpha)$ is a complete metric space.*

The proof follows immediately from the previous theorem and (iv) of theorem 2.6.

3.3. Theorem. *Let (S,d) be a separable metric space and let $\beta,\gamma : \mathbf{M}(S)\times\mathbf{M}(S)\to\mathbb{R}_+$ defined, for every $u,v\in\mathbf{M}(S)$, by*

$$\beta(u,v)=\int_\Omega \frac{d(u,v)}{1+d(u,v)}d\mu.$$

$$\gamma(u,v)=\inf\{\varepsilon+\mu(d(u,v)>\varepsilon):\varepsilon>0\}.$$

Then

(i) *β and γ are metrics on $\mathbf{M}(S)$.*

(ii) $\dfrac{1}{1+\mu(\Omega)}\cdot\alpha^2\le\beta\le(1+\mu(\Omega))\cdot\alpha$ *and* $\alpha\le\gamma\le 2\cdot\alpha$.

Proof. (i) β and γ are both nonnegative and symmetric.

$\beta(u,v)=0$ iff $d(u,v)=0$, μ–a.e., hence iff $u=v$ μ-a.e.

If $\gamma(u,v)=0$ then there exists a sequence $(\varepsilon_n)_n\subseteq(0,+\infty)$ such that $\varepsilon_n+\mu(d(u,v)>\varepsilon_n)<\frac{1}{n}$, for every $n\in\mathbb{N}^*$. Therefore $\varepsilon_n<\frac{1}{n}$ so that $\mu(d(u,v)>\frac{1}{n})<\frac{1}{n}$ so that $\mu(d(u,v)\neq 0)=0$, from where $u=v,\mu$–a.e.

Let now $u,v,w\in\mathbf{M}(S)$; from $d(u,v)\le d(u,w)+d(v,w)$ we obtain

$$\frac{d(u,v)}{1+d(u,v)}\le\frac{d(u,w)+d(v,w)}{1+d(u,w)+d(v,w)}\le\frac{d(u,w)}{1+d(u,w)}+\frac{d(v,w)}{1+d(v,w)}.$$

By integrating on Ω, $\beta(u,v)\le\beta(u,w)+\beta(v,w)$.

For every $a>\gamma(u,w)+\gamma(v,w)$ let $b>\gamma(u,w)$ and $c>\gamma(v,w)$ such that $a=b+c$ and then let $\varepsilon,\eta>0$ such that $\varepsilon+\mu(d(u,v)>\varepsilon)<b$ and $\eta+\mu(d(u,v)>\eta)<c$. As in the proof of (ii) from theorem 3.1, $\mu(d(u,v)>\varepsilon+\eta)\le\mu(d(u,w)>\varepsilon)+\mu(d(v,w)>\eta)$ from where $\gamma(u,v)\le\varepsilon+\eta+\mu(d(u,v)>\varepsilon+\eta)\le b+c=a$ and we obtain the triangle inequality for γ.

(ii) For every $u,v\in\mathbf{M}(S)$ and for any $a>\alpha(u,v),\mu(d(u,v)>a)<a$.

$$\beta(u,v)=\int_{(d(u,v)>a)}\frac{d(u,v)}{1+d(u,v)}d\mu+\int_{(d(u,v)\le a)}\frac{d(u,v)}{1+d(u,v)}d\mu\le$$

$$\le\mu(d(u,v)>a)+\frac{a}{1+a}\cdot\mu(\Omega)<a+a\cdot\mu(\Omega)$$

so that $\dfrac{1}{1+\mu(\Omega)}\cdot\beta(u,v)<a$. Therefore $\beta(u,v)\le(1+\mu(\Omega))\cdot\alpha(u,v)$.

For every $a<\alpha(u,v),\mu(d(u,v)>a)<a$. Hence

$$\beta(u,v)\ge\int_{(d(u,v)>a)}\frac{d(u,v)}{1+d(u,v)}d\mu\ge\frac{a}{1+a}\cdot\mu(d(u,v)>a)>$$

$$>\frac{a^2}{1+a}>\frac{a^2}{1+\alpha(u,v)}>\frac{a^2}{1+\mu(\Omega)}$$

so that $\sqrt{[1+\mu(\Omega)]\cdot\beta(u,v)}>a$. Therefore $\dfrac{1}{1+\mu(\Omega)}\cdot\alpha^2(u,v)\ge\beta(u,v)$.

For every $a>\gamma(u,v)$ there is $\varepsilon>0$ such that $\varepsilon+\mu(d(u,v)>\varepsilon)<a$ from where $\varepsilon<a$ and $\mu(d(u,v)>a)\le\mu(d(u,v)>\varepsilon)<a$ and so $\alpha(u,v)\le a$. Therefore $\alpha(u,v)\le\gamma(u,v)$.

For every $a>\alpha(u,v),\mu(d(u,v)>a)<a$ therefore $\gamma(u,v)\le a+\mu(d(u,v)>a)<2a$ and so $\gamma(u,v)\le 2\cdot\alpha(u,v)$. ∎

3.4. Corollary.

(i) If S is a separable metric space then α,β and γ are uniformly equivalent metrics on $\mathbf{M}(S)$.

(ii) If S is a complete separable metric space then $(\mathbf{M}(S),\alpha),(\mathbf{M}(S),\beta)$ and $(\mathbf{M}(S),\gamma)$ are complete metric spaces.

3.5. Remark. α and β are called the Ky Fan metrics and γ is the metric introduced by Fréchet in [11].

In the particular case where $\Omega=[0,1]$ and $S=\mathbb{R}$, Fréchet characterizes the relatively compact subsets of $(\mathbf{M}(S),\gamma)$. In the case where S is a complete separable metric space, due to condition (ii) of the corollary 3.4, a subset $A\subseteq(\mathbf{M}(S),\gamma)$ will be relatively compact if and only if it is totally bounded. Using this remark it is proved in [7] the following characterization of compactness in the topology of convergence in measure.

3.6. Theorem (see IV.11.1 of [7]). *Let $(S,\|\cdot\|)$ be a separable Banach space; a subset $A\subseteq\mathbf{M}(S)$ is relatively compact in measure if and only if, for every $\varepsilon>0$ there exist a constant $K>0$ and a measurable partition $\{E_1,\cdots,E_n\}$ of Ω such that:*

1). $\mu(\|f\|\ge K)<\varepsilon,$ *for every* $f\in A$.

2). $\sup_{s,t\in E_i\setminus E_f}\|f(s)-f(t)\|<\varepsilon,$ *for every* $f\in A$ *and for any* $i=1,\cdots,n,$ *where* $E_f=(\|f\|\ge K)$.

3.7. Remark. The previous theorem was proved first by Fréchet in the case $\Omega = [0,1]$ and $S = \mathbb{R}$ (see [12]). The generalization to an arbitrary measure space is due to Šmulian ([20]).

3.8. Remark. Let (S,d) be a separable metric space; for every $f \in C^b(S)$ we define $d_f : \mathbf{M}(S) \times \mathbf{M}(S) \to \mathbb{R}_+$ by $d_f(u,v) = \|f(u) - f(v)\|_1 = \int_\Omega |f(u) - f(v)| d\mu$, for every $u, v \in \mathbf{M}(S)$. Then $\{d_f : f \in C^b(S)\}$ is a family of pseudometrics on $\mathbf{M}(S)$; from (iii) of the theorem 2.12, this family generates the topology of convergence in measure on $\mathbf{M}(S)$.

From the (iv) of 2.12, the same topology is generated by the family of pseudometrics $\{d_{A,f} : A \in \mathcal{A}, f \in C^b(S)\}$, where $d_{A,f} : \mathbf{M}(S) \times \mathbf{M}(S) \to \mathbb{R}_+$, $d_{A,f}(u,v) = \left| \int_A [f(u) - f(v)] d\mu \right|$.

In the particular case of a locally compact space another family of pseudometrics generating the topology of convergence in measure is $\{d_{A,f} : A \in \mathcal{A}, f \in C^0(S)\}$ where $C^0(S)$ is the set of all real continuous mappings on S vanishing at infinity. In [8] we show that the uniformity generated by this family is not complete.

Let (S,d) be a separable metric space; for every $f : S \to \mathbb{R}$ let $\|f\|_\infty = \sup_{x \in S} |f(x)| \in \overline{\mathbb{R}}_+$ and $\|f\|_L = \sup_{x \neq y} \dfrac{|f(x) - f(y)|}{d(x,y)} \in \overline{\mathbb{R}}_+$. We note by $BL(S,d)$ the Banach space of all functions $f : S \to \mathbb{R}$ for which $\|f\|_{BL} = \|f\|_L + \|f\|_\infty < +\infty$. We remark that $BL(S,d) \subseteq C^b(S)$.

3.9. Theorem (see theorem 6 and 8 in [4] and proposition 3.3.4 from [9]). *Let $(u_n)_n \subseteq \mathbf{M}(S)$ and let $u \in \mathbf{M}(S)$; the following two statements are equivalent:*

(*i*) $\int_A f(u_n) d\mu \to \int_A f(u) d\mu$, *for every $A \in \mathcal{A}$ and every $f \in C^b(S)$.*

(*ii*) $\sup_{\|f\|_{BL} \leq 1} \left| \int_A [f(u_n) - f(u)] d\mu \right| \to 0$, *for every $A \in \mathcal{A}$.*

3.10. Remark. From the previous theorem and (iv) of theorem 2.12 we can deduce that $u_n \xrightarrow{\mu} u$ in $\mathbf{M}(S)$ if and only if, for every $A \in \mathcal{A}$, $d_A(u_n, u) \to 0$, where $d_A(u,v) = \sup_{\|f\|_{BL} \leq 1} \left| \int_A [f(u_n) - f(u)] d\mu \right|$.

So, the topology of convergence in measure is generated also by the family of pseudometrics $\{d_A : A \in \mathcal{A}\}$.

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"Al.I.Cuza" University, Faculty of Mathematics,
Bd. Carol I, 11, 700506 Iaşi, ROMANIA,
e-mail: lflo@uaic.ro