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A SET-VALUED LUSIN TYPE THEOREM

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Abstract. In this paper, we present a Lusin type theorem under a suitable type of measurability for regular multisubmeasures in Hausdorff topology.

1. Introduction

In classical measure theory, regularity is an important property of continuity. It connects measure theory and topology, aproximating general Borel sets by more tractable sets, such as, compact and/or open sets.

In the last years, non-additive regular (set-valued) measures were intensively studied (see Gavriluţ [5-9], Ha and Wang [11], Kawabe [13], Narukawa, Murofushi and Sugeno [14], Pap [15], Precupanu [16], Song and Li [18], Wu and Ha [19], Wu and Wu [20] etc.) due to their considerable applications in many fields, such as mathematical economics, theory of control, decision theory, physics, biology, nonatomic games, medicine etc.

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In this paper, we present a Lusin type theorem under a suitable type of measurability for regular multisubmeasures in Hausdorff topology. This type of measurability, called here totally-measurability in variation was used before, for instance in [17], in the study of a set-valued Gould type integral.

2. Terminology and notations

Let T be an abstract set, $\mathcal{C}$ a ring of subsets of T, X a real normed space, $\mathcal{P}_0(X)$ the family of all nonempty subsets of $X, \mathcal{P}_f(X)$ the family of closed, nonvoid sets of X, $\mathcal{P}_{bf}(X)$ the family of all bounded, closed, nonvoid sets of X and h the Hausdorff pseudometric on $\mathcal{P}_f(X)$.

As it is well-known, $h(M,N) = \max\{e(M,N), e(N,M)\}$, for every $M, N \in \mathcal{P}_f(X)$, where $e(M,N) = \sup\limits_{x\in M} d(x,N)$.

($d(x,N)$ is the distance from x to N induced by the norm of X). e is called *the excess* of M over N.

On $\mathcal{P}_{bf}(X)$, h becomes a metric [12].

We denote $|M| = h(M,\{0\})$, for every $M \in \mathcal{P}_f(X)$, where 0 is the origin of X.

We observe that $e(N,M) = h(M,N)$, for every $M, N \in \mathcal{P}_f(X)$, with $M \subseteq N$.

On $\mathcal{P}_0(X)$ we introduce the Minkowski addition " $\overset{\bullet}{+}$ " defined by:

$$M \overset{\bullet}{+} N = \overline{M+N}, \text{ for every } M, N \in \mathcal{P}_0(X),$$

where $M + N = \{x + y; x \in M, y \in N\}$ and $\overline{M+N}$ is the closure of $M+N$ with respect to the topology induced by the norm of X.

By $\mathbb{N}^*$ we mean $\mathbb{N}\backslash\{0\}$ and by $\overline{1,n}$, we mean $\{1, 2, ..., n\}$.

We also denote $\mathbb{R}_+ = [0, +\infty)$.

First, we recall the following classical notions. They are studied, for instance, in Pap [15] in the context of non-additive set functions.

Definition 2.1. A set function $m : \mathcal{C} \to \mathbb{R}_+$ is said to be:

I) *exhaustive* if $\lim\limits_{n\to\infty} m(A_n) = 0$, for every pairwise disjoint sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$.

II) *increasing convergent* if $\lim_{n\to\infty} m(A_n) = m(A)$, for every increasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \nearrow A$ (ie., $A_n \subset A_{n+1}$, for every $n \in \mathbb{N}^*$) and $A = \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$.

III) *decreasing convergent* if $\lim_{n\to\infty} m(A_n) = m(A)$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \searrow A$ (ie., $A_n \supset A_{n+1}$, for every $n \in \mathbb{N}^*$) and $A = \bigcap\limits_{n=1}^{\infty} A_n \in \mathcal{C}$.

III') *order-continuous (*shortly, *o-continuous*) if we have $\lim_{n\to\infty} m(A_n) = 0$, for every sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \searrow \emptyset$.

IV) *monotone* if $m(A) \leq m(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

V) a *submeasure* (in the sense of Drewnowski [4]) if $m(\emptyset) = 0$, m is monotone and *subadditive*, i.e., $m(A \cup B) \leq m(A) + m(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

We shall need the following notions in the set valued case:

Definition 2.2. If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is a set multifunction, by $|\mu|$ we mean the real extended valued set function defined by $|\mu|(A) = |\mu(A)|$, for every $A \in \mathcal{C}$.

Definition 2.3. A set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(X)$, with $\mu(\emptyset) = \{0\}$ is said to be:

I) a *multisubmeasure* ([5-10]) if

a) μ is *monotone* (ie. $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$) and

b) $\mu(A \cup B) \subseteq \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$) (or, equivalently, for every $A, B \in \mathcal{C}$).

II) a *multimeasure* if $\mu(A \cup B) = \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

Definition 2.4. A set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be:

I) *increasing convergent* (with respect to h) if $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every increasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \nearrow A \in \mathcal{C}$.

II) *decreasing convergent* (with respect to h) if $\lim\limits_{n\to\infty} h(\mu(A_n),\mu(A)) = 0$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \searrow A \in \mathcal{C}$.

III) *exhaustive* (with respect to h) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every pairwise disjoint sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$.

IV) *order-continuous (*shortly, *o-continuous*) (with respect to h) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \searrow \emptyset$.

Examples 2.5. I) Let $\nu_1,\ldots,\nu_p : \mathcal{C} \to \mathbb{R}_+$, be p finitely additive set functions, where $\mathcal{C}$ is a ring of subsets of an abstract space T.

One can easily prove that the set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, defined for every $A \in \mathcal{C}$ by:

$$\mu(A) = \overline{\bigcup_{\substack{B\subset A,\\ B\in\mathcal{C}}} \{\nu_1(B),\nu_2(B),\ldots,\nu_p(B)\}},$$

is a multisubmeasure.

II) If $\nu_1,\nu_2 : \mathcal{C} \to \mathbb{R}_+$, ν_1 is a finitely additive set function and ν_2 is a submeasure (finitely additive set function, respectively), then the set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, defined by

$$\mu(A) = [-\nu_1(A),\nu_2(A)], \text{ for every } A \in \mathcal{C},$$

is a multisubmeasure (monotone multimeasure, respectively).

Note that $|\mu(A)| = \max\{\nu_1(A),\nu_2(A)\}$, for every $A \in \mathcal{C}$.

Therefore, both ν_1 and ν_2 are order continuous (respectively, exhaustive) if and only if the same is μ.

Also, both ν_1 and ν_2 are increasing (respectively, decreasing) convergent if and only if the same is μ.

Remark 2.6. I) Definitions 2.4 I)-IV) generalize the classical ones from Definition 2.1.

Indeed, if $m : \mathcal{C} \to \mathbb{R}_+$ is a set function with $m(\emptyset) = 0$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ is defined by $\mu(A) = [0,m(A)]$, for every $A \in \mathcal{C}$, then μ is increasing convergent (decreasing convergent, exhaustive, o-continuous, respectively) if and only if the same is m.

The statements follow since $|\mu(A)| = m(A)$, for every $A \in \mathcal{C}$ and $h([0,a],[0,b]) = |a-b|$, for every $a,b \in \mathbb{R}_+$.

II) In Definition 2.3, if a set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ with $\mu(\emptyset) = \{0\}$ is single-valued, then the monotonicity of μ implies that $\mu(A) = \{0\}$, for every $A \in \mathcal{C}$. Therefore, the monotonicity finds a meaning only in the case when the set multifunction is not single-valued.

III) If $\mathcal{C}$ is finite, then any set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(X)$, with $\mu(\emptyset) = \{0\}$ is increasing convergent, decreasing convergent, exhaustive and o-continuous.

The following two notions are classic (see, for instance, Chiţescu [1, 2] for set functions), but here they are generalized for the set-valued case. Suppose $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is a set multifunction, with $\mu(\emptyset) = \{0\}$.

Definition 2.7. I) A set $A \in \mathcal{C}$ is said to be an *atom* of μ if $\mu(A) \supsetneqq \{0\}$ and for every $B \in \mathcal{C}$, with $B \subset A$, we have $\mu(B) = \{0\}$ or $\mu(A \backslash B) = \{0\}$.

II) μ is said to be *finitely purely atomic* if there is a finite disjoint family $(A_i)_{i=\overline{1,p}} \subset \mathcal{C}$ of atoms of μ so that $T = \bigcup\limits_{i=1}^{p} A_i$.

3. Regular multisubmeasures in Hausdorff topology

In this section, let T be a locally compact, Hausdorff space, $\mathcal{B}_0$ (respectively, $\mathcal{B}'_0$) the Baire δ-ring (respectively, σ-ring) generated by compact sets, which are G_δ (that is, countable intersections of open sets) and $\mathcal{B}$ (respectively, $\mathcal{B}'$) the Borel δ-ring (respectively, σ-ring) generated by the compact sets of T.

Note that $\mathcal{B}_0 \subset \mathcal{B}$, $\mathcal{B}_0 \subset \mathcal{B}'_0$ and $\mathcal{B} \subset \mathcal{B}'$.

By $\mathcal{K}$ we denote the family of compacts and by $\mathcal{D}$, the family of open sets of T.

Consider X a real normed space, $\mathcal{C}$ a ring of subsets of T, $A \in \mathcal{C}$ an arbitrary set and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ a set multifunction, with $\mu(\emptyset) = \{0\}$.

In [5-9] (inspired by [3, 16]), various types of regularity in Hausdorff topology were studied.

Here, we recall the following notion, which is consistent if, for instance, $\mathcal{C}$ is the ring (δ-ring, σ-ring, respectively) generated by the compact/compact, G_δ subsets of T.

Definition 3.1. [5] μ is said to be *regular* if for every set $A \in \mathcal{C}$ and every $\varepsilon > 0$, there is $K \in \mathcal{K} \cap \mathcal{C}, K \subset A$ so that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, with $B \subset A \backslash K$.

In what follows, suppose μ is **a multisubmeasure**.

Remark 3.2. [8] I) Every $K \in \mathcal{K} \cap \mathcal{C}$ is regular.
II) μ is regular if and only if for every $A \in \mathcal{C}$ and every $\varepsilon > 0$, there is $K \in \mathcal{K} \cap \mathcal{C}, K \subset A$ so that $|\mu(A \backslash K)| < \varepsilon$.

Theorem 3.3. [6] *i) If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is regular, then μ is o-continuous.*
ii) If $\mu : \mathcal{B}_0 \to \mathcal{P}_f(X)$, then μ is o-continuous if and only if μ is regular on $\mathcal{B}_0$.

4. Totally-measurability in variation and a Lusin type theorem

In what follows, without any special assumptions, suppose $\mathcal{A}$ is an algebra of subsets of an abstract space T, X is a real normed space, $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is a multisubmeasure and $f : T \to \mathbb{R}$ is a function.

Definition 4.1. A *partition* of T is a finite family $P = \{A_i\}_{i=\overline{1,n}} \subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset, i \neq j$ and $\bigcup\limits_{i=1}^{n} A_i = T$.

We denote by $\mathcal{P}$ the class of all partitions of T and if $A \in \mathcal{A}$ is fixed, by $\mathcal{P}_A$, the class of all partitions of A.

We consider *the variation* $\overline{\mu}$ of μ, defined by $\overline{\mu}(A) = \sup\{\sum\limits_{i=1}^{n} |\mu(A_i)|\}$, for every $A \in \mathcal{A}$,
where the supremum is extended over all finite partitions $\{A_i\}_{i=\overline{1,n}}$ of A.

Remark 4.2. [5] $\overline{\mu}(A) \geq |\mu(A)|$, for every $A \in \mathcal{A}$ and $\overline{\mu}$ is finitely additive on $\mathcal{A}$.
$|\mu|$ is a submeasure on $\mathcal{A}$.

Definition 4.3. f is said to be:

I) *totally-measurable in variation on* $(T, \mathcal{A}, \mu)$ if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of T such that:

a) $\overline{\mu}(A_0) < \varepsilon$ and

b) $\sup\limits_{t,s \in A_i} |f(t) - f(s)| < \varepsilon$, for every $i = \overline{1,n}$.

II) *totally-measurable in variation on* $B \in \mathcal{A}$ if the restriction $f|_B$ of f to B is totally- measurable in variation on $(B, \mathcal{A}_B, \mu_B)$, where $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$ and $\mu_B = \mu|_{\mathcal{A}_B}$.

Remark 4.4. If f is totally-measurable in variation on T, then f is totally-measurable in variation on every set $A \in \mathcal{A}$.

Proposition 4.5. [10] Let $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ and $\{A_i\}_{i=\overline{1,p}} \subset \mathcal{A}$ an arbitrary partition of a set $A \in \mathcal{A}$. Then f is totally-measurable in variation on A if and only if it is totally-measurable in variation on every $A_i, i = \overline{1,p}$.

In the following, suppose (T, d_1) is a locally compact, metric space.

Theorem 4.6. **(Lusin type)** *Let* $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ *be a regular multisubmeasure and* $f : T \to \mathbb{R}$ *totally-measurable in variation on* T.

Then for every $\varepsilon > 0$, *there is a compact set* $K_\varepsilon \in \mathcal{A}$ *so that* f *is "pseudo"-continuous on* K_ε *(i.e., for* $\varepsilon > 0$, $\exists \delta_\varepsilon > 0$, $\forall t, s \in K_\varepsilon$, $d_1(t,s) < \delta_\varepsilon \Rightarrow |f(t) - f(s)| < \varepsilon$*) and* $|\mu(T \backslash K_\varepsilon)| < \varepsilon$ *(that is,* f *is "pseudo"-quasi-continuous on* T*).*

Proof. Let $\varepsilon > 0$ be arbitrary.

Since f is totally measurable in variation on T, there is $P_\varepsilon = \{A_0, A_1, ..., A_p\} \in \mathcal{P}$ so that $\overline{\mu}(A_0) < \frac{\varepsilon}{2}$ and $\sup\limits_{t,s \in A_i} |f(t) - f(s)| < \varepsilon$, for every $i = \overline{1,p}$.

Because for every $i = \overline{1,p}$, A_i is regular, there is a compact set $K_i \in \mathcal{A}$ so that $K_i \subset A_i$ and $|\mu(A_i \backslash K_i)| < \frac{\varepsilon}{2^{i+1}}$.

Then

$$|\mu(\bigcup_{i=1}^{p}(A_i \backslash K_i))| \leq \sum_{i=1}^{p} |\mu(A_i \backslash K_i)| < \sum_{i=1}^{p} \frac{\varepsilon}{2^{i+1}} < \frac{\varepsilon}{2}.$$

Denote $K_\varepsilon = \bigcup\limits_{i=1}^{p} K_i$. Evidently, K_ε is a compact set of $\mathcal{A}$ and

$$|\mu((\bigcup\limits_{i=1}^{p} A_i)\backslash K_\varepsilon)| \leq |\mu(\bigcup\limits_{i=1}^{p} (A_i \backslash K_i))| < \frac{\varepsilon}{2}.$$

Because $|\mu(A_0)| \leq \overline{\mu}(A_0) < \frac{\varepsilon}{2}$, then

$$\begin{aligned} |\mu(T\backslash K_\varepsilon)| &= |\mu[A_0 \cup (\bigcup\limits_{i=1}^{p} A_i)\backslash K_\varepsilon]| \leq \\ &\leq |\mu(A_0)| + |\mu((\bigcup\limits_{i=1}^{p} A_i)\backslash K_\varepsilon)| < \varepsilon. \end{aligned}$$

It remains to prove that f is "pseudo"-continuous on K_ε.

Indeed, since $\sup\limits_{t,s\in A_1} |f(t) - f(s)| < \varepsilon$ and $\sup\limits_{t,s\in A_2} |f(t) - f(s)| < \varepsilon$, then f is "pseudo"-continuous on A_1 and A_2 and also on $K_1 \subset A_1$ and on $K_2 \subset A_2$.

Because $K_1 \cap K_2 = \emptyset$ and K_1, K_2 are compact, then $d_1(K_1, K_2)(\varepsilon) = \delta_\varepsilon > 0$.

Consequently, for $\varepsilon > 0$, there is $\delta_\varepsilon > 0$ such that for every $t, s \in K_1 \cup K_2$, with $d_1(t, s) < \delta_\varepsilon = \inf\limits_{u\in K_1, v\in K_2} d_1(u, v)$, we must have either $t, s \in K_1 \subset A_1$, or $t, s \in K_2 \subset A_2$, so $|f(t) - f(s)| < \varepsilon$, which means f is "pseudo"-continuous on $K_1 \cup K_2$.

Now, f is "pseudo"-continuous on the compact disjoint sets $K_1 \cup K_2$ and K_3.

We prove that f is "pseudo"-continuous on $K_1 \cup K_2 \cup K_3$.

Indeed, for $\varepsilon > 0$, there is $\delta'_\varepsilon = \min\{d_1(K_1, K_2)(\varepsilon), d_1(K_1 \cup K_2, K_3)(\varepsilon)\} > 0$.

Let be arbitrary $t, s \in (K_1 \cup K_2) \cup K_3$, with $d_1(t, s) < \delta'_\varepsilon$.

Then $d_1(t, s) < d_1(K_1 \cup K_2, K_3)(\varepsilon)$, so we have either $t, s \in K_1 \cup K_2$, or $t, s \in K_3$.

I) If $t, s \in K_3 \subset A_3$, then $|f(t) - f(s)| < \varepsilon$ and the proof finishes.

II) If $t, s \in K_1 \cup K_2$, since $d_1(t, s) < d_1(K_1, K_2)$, then we have either $t, s \in K_1 \subset A_1$, or $t, s \in K_2 \subset A_2$, so $|f(t) - f(s)| < \varepsilon$. Then f is "pseudo"-continuous on $K_1 \cup K_2 \cup K_3$.

Continuing this way, f is "pseudo"-continuous on $K_\varepsilon = \bigcup\limits_{i=1}^{p} K_i$.

From now on, suppose (T, d_1) **is a compact, metric space**. Then, according to [3], $\mathcal{B}_0 = \mathcal{B}$ is an algebra.

By Theorem 3.3 and Theorem 4.6, we get:

Corollary 4.7. **(Lusin type)** Let $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ be an o-continuous multisubmeasure and $f : T \to \mathbb{R}$ totally-measurable in variation on T.

Then for every $\varepsilon > 0$, there is a compact set $K_\varepsilon \in \mathcal{B}$ so that f is "pseudo"-continuous on K_ε and $|\mu(T \backslash K_\varepsilon)| < \varepsilon$.

Proposition 4.8. [10] Suppose $f : T \to \mathbb{R}$ is continuous on T and $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is a finitely purely atomic regular multisubmeasure. Then f is totally-measurable in variation on T.

By Corollary 4.7 and Proposition 4.8, we immediately have:

Corollary 4.9. Suppose $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is a regular finitely purely atomic multisubmeasure. Let $f : T \to \mathbb{R}$.

Then, in this case, for f, totally-measurability in variation stands between continuity and "pseudo"-quasi-continuity on T:

f continuous $\Rightarrow$ f totally-measurable in variation $\Rightarrow$ f "pseudo"-

quasi-continuous.

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