"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 19 (2009), No. 2, 275 - 288

ON TOTALLY-MEASURABLE FUNCTIONS

A. GAVRILUŢ AND A. CROITORU

Abstract. In this paper, we study different basic problems concerning real valued functions which are totally-measurable with respect to the variation of a (multi)submeasure. As applications, special considerations on their relation with Gould type integrability and additional problems are given (e.g., a Fatou lemma type, the Banach structure of a $\mathcal{L}^p$ space).

1. Introduction

In the last years, the non-additive case and the set-valued case received a special attention because of their applications in mathematical economics, decision theory, artificial intelligence, statistics or theory of games.

Particularly, problems concerning measurability and set-valued integrability were of a great interest.

In this paper, we shall deal with a special type of measurability, called here, totally-measurability in variation. It was successfully used in the study of (set-valued) Gould type integrals [4, 5], [7, 8], [10], [11], [17-19], which have remarkable properties. Among other results, we shall present relationships between totally-measurability in variation and Gould integrability.

Keywords and phrases: set multifunction, multisubmeasure, variation, totally-measurable in variation, atom, finitely purely atomic, $\mathcal{L}^p$
2000 Mathematics Subject Classification: 28B20, 28C15.

Results concerning $\mathcal{L}^p$ spaces and a Fatou type lemma are also given. An important class of real valued functions which are totally-measurable in the variation of a special set multifunction is provided.

2. Basic notions and results

Let $(X, \|\cdot\|)$ be a real normed space with the origin 0, $\mathcal{P}_0(X)$ the family of non-empty subsets of X, $\mathcal{P}_f(X)$ the family of nonvoid closed subsets of X, $\mathcal{P}_{bf}(X)$ the family of nonvoid closed bounded subsets of X, $\mathcal{P}_{kc}(X)$ the family of nonvoid compact convex subsets of X and h the Hausdorff pseudometric on $\mathcal{P}_f(X)$, which becomes a metric on $\mathcal{P}_{bf}(X)$.

It is known that $h(M, N) = \max\{e(M, N), e(N, M)\}$, where

$e(M, N) = \sup\limits_{x\in M} d(x, N)$, for every $M, N \in \mathcal{P}_f(X)$ is the excess of M over N and

$d(x, N)$ is the distance from x to N with respect to the distance induced by the norm of X.

We denote $|M| = h(M, \{0\}) = \sup\limits_{x\in M} \|x\|$, for every $M \in \mathcal{P}_0(X)$.

For every $M, N \in \mathcal{P}_0(X)$, let $M + N = \{x + y | x \in M, y \in N\}$.

By $\overline{M}$ we mean the closure of M with respect to the topology induced by the norm of X.

On $\mathcal{P}_0(X)$ we consider the Minkowski addition " $\overset{\bullet}{+}$ " [12], defined by:

$$M \overset{\bullet}{+} N = \overline{M + N}, \text{ for every } M, N \in \mathcal{P}_0(X).$$

Let T be an abstract nonvoid set, $\mathcal{P}(T)$ the family of all subsets of T and $\mathcal{C}$ a ring of subsets of T.

By $i = \overline{1, n}$ we mean $i \in \{1, 2, \ldots, n\}$, for $n \in \mathbb{N}^*$, where $\mathbb{N}$ is the set of all naturals and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. We also denote $\mathbb{R}_+ = [0, +\infty)$ and $\overline{\mathbb{R}}_+ = [0, +\infty]$.

First, we recall some classical notions. Among others, they were intensively studied, for instance in [15].

Definition 2.1. A set function $m : \mathcal{C} \to \overline{\mathbb{R}}_+$, with $m(\emptyset) = 0$, is said to be:

I) *monotone* if $m(A) \leq m(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

II) *superadditive* if $m(\bigcup_{i\in I} A_i) \geq \sum_{i\in I} m(A_i)$, for every sequence of pairwise disjoint sets $(A_i)_{i\in I} \subset \mathcal{C}$, with $\bigcup_{i\in I} A_i \in \mathcal{C}$, $I \subseteq \mathbb{N}$.

III) *subadditive* if $m(A \cup B) \leq m(A) + m(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

IV) *a submeasure* (in the sense of Drewnowski [3]) if m is monotone and subadditive.

V) *order-continuous* (shortly, *o-continuous*) if $\lim_{n\to\infty} m(A_n) = 0$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $\bigcap_{n=1}^{\infty} A_n = \emptyset$ (denoted by $A_n \searrow \emptyset$).

Examples 2.2. Suppose $\nu_1, \nu_2 : \mathcal{C} \to \mathbb{R}_+$ are finitely additive set functions.

Then $m_1, m_2 : \mathcal{C} \to \mathbb{R}_+$ defined for every $A \in \mathcal{C}$ by

$m_1(A) = \frac{\nu_1(A)}{1+\nu_1(A)}$, $m_2(A) = \sqrt{\nu_1(A)}$

are submeasures.

Also, $m : \mathcal{C} \to \mathbb{R}_+$ defined for every $A \in \mathcal{C}$ by $m(A) = \max\{\nu_1(A), \nu_2(A)\}$ is a submeasure, too.

We introduce now several notions in the set-valued case.

Suppose $\mu : \mathcal{C} \to \mathcal{P}_0(X)$, with $\mu(\emptyset) = \{0\}$ is an arbitrary set multifunction.

Definition 2.3. We consider:

I) the extended real valued set function $|\mu| : \mathcal{C} \to \overline{\mathbb{R}}_+$ defined for every $A \in \mathcal{C}$ by $|\mu|(A) = |\mu(A)|$.

II) *the variation* $\overline{\mu}$ of μ defined for every $A \in \mathcal{P}(T)$ by

$\overline{\mu}(A) = \sup\{\sum_{i=1}^{n} |\mu(A_i)|\}$, where the supremum is extended over all finite families of pairwise disjoint sets $\{A_i\}_{i=\overline{1,n}} \subset \mathcal{A}$, with $A_i \subseteq A$, for every $i \in \{1, \ldots, n\}$.

III) μ is said to be *of finite variation on* $\mathcal{C}$ if $\overline{\mu}(A) < \infty$, for every $A \in \mathcal{C}$.

Definition 2.4. μ is said to be:

I) *monotone* if $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

II) a *multimeasure* if $\mu(A \cup B) = \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

III) a *multisubmeasure* [4] if μ is monotone and
$\mu(A \cup B) \subseteq \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$
(or, equivalently, for every $A, B \in \mathcal{C}$).

IV) *h-σ-subadditive* if $|\mu(\bigcup\limits_{n=1}^{\infty} A_n)| \leq \sum\limits_{n=1}^{\infty} |\mu(A_n)|$, for every sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$.

V) *order-continuous* (shortly, *o-continuous*) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \searrow \emptyset$.

Remarks 2.5. I) If μ is $\mathcal{P}_f(X)$-valued, then in Definition 2.4 - II), III) it usually appears the Minkowski addition instead of the classical addition because the sum of two closed sets is not, generally, a closed set.

II) $\overline{\mu}$ is monotone and superadditive on $\mathcal{P}(T)$.
[4] If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is a multi(sub)measure, then $\overline{\mu}$ is finitely additive on $\mathcal{C}$ and $|\mu|$ is a submeasure.

III) Every monotone multimeasure is, particularly, a multisubmeasure. Evidently, the converse is not, generally, valid.

IV) [7] Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a multisubmeasure of finite variation. The following statements are equivalent:
i) μ is h-σ-subadditive;
ii) μ is order-continuous;
iii) $\overline{\mu}$ is σ-additive on $\mathcal{C}$.

Examples 2.6. I) Let $m_1, m_2 : \mathcal{C} \to \mathbb{R}_+$, with $m_1(\emptyset) = 0$ and $m_2(\emptyset) = 0$.
If m_1 is finitely additive and m_2 is a submeasure (finitely additive, respectively), then the set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, defined by

$$\mu(A) = [-m_1(A), m_2(A)], \text{ for every } A \in \mathcal{C},$$

is a multisubmeasure (a multimeasure, respectively).

II) Let $\mathcal{C}$ be a ring of subsets of T, $m : \mathcal{C} \to \mathbb{R}_+$ a finitely additive set function and $\mu : \mathcal{C} \to \mathcal{P}_{bf}(\mathbb{R})$ the set multifunction defined for every $A \in \mathcal{C}$ by

$$\mu(A) = \begin{cases} [-m(A), m(A)], & \text{if } m(A) \leq 1 \\ \\ [-m(A), 1], & \text{if } m(A) > 1 \end{cases}.$$

One can easily check that μ is a multisubmeasure.

III) Let $T = 2\mathbb{N} = \{0, 2, 4, \ldots\}$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined for every $A \in \mathcal{C}$ by

$$\mu(A) = \begin{cases} \{0\}, & \text{if } A = \emptyset \\ \frac{1}{2}A \cup \{0\}, & \text{if } A \neq \emptyset, \end{cases}$$

where $\frac{1}{2}A = \{\frac{x}{2} |\ x \in A\}$. Then μ is a multisubmeasure.

In [6, 9], the notions of an atom and of non-atomicity for set multifunctions were introduced and studied. In what follows, we recall the notion of an atom of a set multifunction and, using it, we extend to the set valued case the notion of a finitely purely atomic set function introduced and studied by Chiţescu in [1-2].

Definition 2.7. I) A set $A \in \mathcal{C}$ is said to be an *atom* of μ if $\mu(A) \supsetneq \{0\}$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = \{0\}$ or $\mu(A \backslash B) = \{0\}$.

II) We say that μ is *finitely purely atomic* if there is a finite disjoint family $(A_i)_{i=\overline{1,n}} \subset \mathcal{C}$ of atoms of μ so that $T = \bigcup\limits_{i=1}^{n} A_i$.

3. Totally-measurability in variation

In this section we present some remarkable properties of real valued functions which are totally-measurable with respect to the variation of a (multi)submeasure.

In the sequel, $\mathcal{A}$ will be an algebra of subsets of T, $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ a set multifunction of finite variation, with $\mu(\emptyset) = \{0\}$ and $f : T \to \mathbb{R}$ an arbitrary function.

Definition 3.1. i) *A partition* of a set $A \in \mathcal{A}$ is a finite family $P = \{A_i\}_{i=\overline{1,n}}$ of pairwise disjoint sets of $\mathcal{A}$ such that $\bigcup\limits_{i=1}^{n} A_i = A$.

ii) Let $P = \{A_i\}_{i=\overline{1,n}}$ and $P' = \{B_j\}_{j=\overline{1,m}}$ be two partitions of T.
P' is said to be *finer than* P, denoted $P \leq P'$ (or $P' \geq P$), if for every $j = \overline{1,m}$, there exists $i_j = \overline{1,n}$ so that $B_j \subseteq A_{i_j}$.

iii) The *common refinement* of two partitions $P = \{A_i\}_{i=\overline{1,n}}$ and $P' = \{B_j\}_{j=\overline{1,m}}$ is the partition $P \wedge P' = \{A_i \cap B_j\}_{\substack{i=\overline{1,n} \\ j=\overline{1,m}}}$.

Obviously, $P \wedge P' \geq P$ and $P \wedge P' \geq P'$.

We denote by $\mathcal{P}$ the class of all partitions of T and if $A \in \mathcal{A}$ is fixed, by $\mathcal{P}_A$, the class of all partitions of A.

Definition 3.2. I) f is said to be *totally-measurable in variation* (shortly, *t.m.v.*) *on* $(T, \mathcal{A}, \mu)$ if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of T such that:

$$(*) \qquad \begin{cases} a)\ \overline{\mu}(A_0) < \varepsilon \ \text{ and} \\ b)\ \sup\limits_{t,s \in A_i} |f(t) - f(s)| = osc(f, A_i) < \varepsilon, \ \forall i = \overline{1,n}. \end{cases}$$

II) f is said to be *totally-measurable in variation (*shortly*, t.m.v.) on* $B \in \mathcal{A}$ if the restriction $f|_B$ of f to B is totally measurable in variation on $(B, \mathcal{A}_B, \mu_B)$, where $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$ and $\mu_B = \mu|_{\mathcal{A}_B}$.

Remark 3.3. Totally-measurability in variation is hereditary, i.e., if f is t.m.v. on T, then f is t.m.v. on every $A \in \mathcal{A}$.

We consider now the extended real valued set function $\widetilde{\mu}$ defined by $\widetilde{\mu}(A) = \inf\{\overline{\mu}(B); A \subseteq B, B \in \mathcal{A}\}$, for every $A \in \mathcal{P}(T)$.

From now on, suppose $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is a multisubmeasure of finite variation.

Remarks 3.4. [7] I) $\widetilde{\mu}(A) = \overline{\mu}(A)$, for every $A \in \mathcal{A}$.

II) $\widetilde{\mu}$ is a submeasure on $\mathcal{P}(T)$. If, moreover, μ is h-σ-subadditive, then $\widetilde{\mu}$ is σ-subadditive on $\mathcal{P}(T)$.

Definition 3.5. We say that *a property* (P) *holds* μ*-almost everywhere (*shortly, μ*-ae*) if the property (P) is valid on $T\backslash A$, with $\widetilde{\mu}(A) = 0$.

In [5, 13, 14, 18, 19] several types of convergences for sequences of functions are introduced and studied. Here we shall use the following:

Definition 3.6. If $f_n : T \to \mathbb{R}$, for every $n \in \mathbb{N}$, then the sequence of functions $(f_n)_n$ *converges in submeasure to* f (denoted by $f_n \xrightarrow{\mu} f$) if for every $\delta > 0$, $\lim\limits_{n\to\infty} \widetilde{\mu}(B_n(\delta)) = 0$, where

$$B_n(\delta) = \{t \in T; |f_n(t) - f(t)| \geq \delta\}.$$

Proposition 3.7. ([5, 7]) *I) If* $f, g : T \to \mathbb{R}$ *are bounded t.m.v. functions, then:*

i) $f + g$, λf *are t.m.v., for every* $\lambda \in \mathbb{R}$*;*

ii) f^2 *and* fg *are t.m.v.*

II) If for every $n \in \mathbb{N}$, $f_n : T \to \mathbb{R}$ *is bounded, t.m.v. and* (f_n) *is convergent in submeasure to a bounded function* $f : T \to \mathbb{R}$*, then* f *is t.m.v.*

Using the definition of totally-measurability in variation, one can easily obtain by standard proofs the following:

Proposition 3.8. *Suppose* $f, g : T \to \mathbb{R}$ *are bounded functions and* $p \in [1, +\infty)$ *is arbitrarily. Then:*

I) If f *is t.m.v., then* $|f|^p$ *is t.m.v.;*

II) If $|f|^p$ *and* $|g|^p$ *are t.m.v., then* $|f + g|^p$ *is t.m.v.*

Proposition 3.9. [7] *Let* $f : T \to \mathbb{R}$ *be a bounded function and* $A \in \mathcal{A}$.

If $\{A_i\}_{i=\overline{1,p}} \subset \mathcal{A}$ *is an arbitrary partition of* A*, then* f *is t.m.v. on* $\bigcup\limits_{i=1}^{p} A_i$ *if and only if the same is* f *on every* $A_i, i = \overline{1,p}$.

In the following, we obtain an important class of functions which are totally measurable in the variation of a special multisubmeasure:

Theorem 3.10. *Suppose* (T, ρ) *is a compact metric space,* $\mathcal{B}$ *is the Borel* δ*-ring generated by the compact subsets of* T*,* $f : T \to \mathbb{R}$

is continuous on T and $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is a finitely purely atomic R_l'-regular [6] multisubmeasure. Then f is t.m.v. on T.

Proof. According to Proposition 3.9, it is sufficient to establish the t.m.v. of f on an arbitrary fixed atom A_0 of μ.

Because μ is a R_l'-regular multisubmeasure, by [6], there is an unique $a_0 \in A_0$ so that $\mu(A_0 \backslash \{a_0\}) = \{0\}$.

Consider arbitrary $\varepsilon > 0$. Since f is continuous in a_0, there is $\delta_\varepsilon > 0$ so that for every $t \in A_0$, with $\rho(t, a_0) < \delta_\varepsilon$, we have $|f(t) - f(a_0)| < \frac{\varepsilon}{3}$.

Let $B_\varepsilon = \{t \in A_0; \rho(t, a_0) < \delta_\varepsilon\} = A_0 \cap B(a_0, \delta_\varepsilon)$, where $B(a_0, \delta_\varepsilon)$ is the open ball of center a_0 and radius δ_ε.

Then $B_\varepsilon \in \mathcal{B}$ and since A_0 is an atom, we have $\mu(B_\varepsilon) = \{0\}$ or $\mu(A_0 \backslash B_\varepsilon) = \{0\}$. Also, one can easily check that $\mathcal{B}$ is an algebra.

If $\mu(B_\varepsilon) = \{0\}$, then since $a_0 \in B_\varepsilon$, we get $\mu(\{a_0\}) = \{0\}$. But $\mu(A_0 \backslash \{a_0\}) = \{0\}$, so $\mu(A_0) = \{0\}$, a contradiction. So, we have $\mu(A_0 \backslash B_\varepsilon) = \{0\}$.

Now, one can easily observe that the partition $P_{A_0} = \{A_0 \backslash B_\varepsilon, B_\varepsilon\}$ assures the t.m.v. of f.

Theorem 3.11. *Suppose $\mathcal{A}$ is a σ-algebra, $m : \mathcal{A} \to \mathbb{R}_+$ is an o-continuous submeasure of finite variation and $(f_n)_{n \in \mathbb{N}^*}$ is a sequence of uniformly bounded t.m.v. functions $f_n : T \to \mathbb{R}$.*

If for every $t \in T$, there exists $\lim\limits_{n \to \infty} f_n(t) = f(t)$, then f is t.m.v.

Proof. I. We prove that g defined for every $t \in T$ by $g(t) = \inf\limits_{n \in \mathbb{N}^*} f_n(t)$, is t.m.v.

One can easily check that for every $t, s \in T$, the following inequality holds:

$$|g(t) - g(s)| \leq \sup_{n \in \mathbb{N}^*} |f_n(t) - f_n(s)|. \ (1)$$

Because for every $n \in \mathbb{N}^*$, f_n is t.m.v., then for every $\varepsilon > 0$, there is a partition $P_\varepsilon^n = \{A_j^n\}_{j=\overline{0,p_n}} \in \mathcal{P}$ so that $\overline{m}(A_0^n) < \frac{\varepsilon}{2^{n+1}}$ and

$$\sup_{t,s \in A_j^n} |f_n(t) - f_n(s)| < \frac{\varepsilon}{2^{n+1}}, \quad j = \overline{1, p_n}. \ (2)$$

Consider $A_0 = \bigcup\limits_{n=1}^{\infty} A_0^n \in \mathcal{A}$. Because m is an o-continuous submeasure of finite variation, then $\overline{m}$ is σ-additive on $\mathcal{A}$, so,

$$\overline{m}(A_0) \leq \sum_{n=1}^{\infty} \overline{m}(A_0^n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} < \frac{\varepsilon}{2}.$$

On the other hand,

$$\begin{aligned} cA_0 &= \bigcap_{n=1}^{\infty} cA_0^n = \bigcap_{n=1}^{\infty} (A_1^n \cup A_2^n \cup ... \cup A_{p_n}^n) = \\ &= (A_1^1 \cup A_2^1 \cup ... \cup A_{p_1}^1) \cap (A_1^2 \cup A_2^2 \cup ... \cup A_{p_2}^2) \cap ... \\ &= \bigcup_{(i_n) \in \prod_{n=1}^{\infty} I_n} (A_{i_1}^1 \cap A_{i_2}^2 \cap \ldots \cap A_{i_n}^n \cap \ldots), \end{aligned}$$

where $I_n = \{1, 2, \ldots, p_n\}$, for every $n \in \mathbb{N}^*$.

Denote the last reunion by $\bigcup\limits_{n=1}^{\infty} B_n$. Now let $C_n = \bigcup\limits_{k=1}^{n} B_k$ and $D_n = cA_0 \backslash C_n$, for every $n \in \mathbb{N}^*$.

We observe that $B_n \cap B_m = \emptyset$ whenever $n \neq m$, $\bigcup\limits_{n=1}^{\infty} C_n = \bigcup\limits_{n=1}^{\infty} B_n = cA_0$ and $D_n \searrow \emptyset$.

Since $\overline{m}$ is o-continuous, there is $n_0(\varepsilon) = n_0 \in \mathbb{N}^*$ such that $\overline{m}(cA_0 \backslash (\bigcup\limits_{i=1}^{n_0} B_i)) < \frac{\varepsilon}{2}$. Because $\overline{m}(A_0) < \frac{\varepsilon}{2}$, we get $\overline{m}(c(\bigcup\limits_{i=1}^{n_0} B_i)) < \varepsilon$.

From (1) and (2), for every $i \in \{1, \ldots, n_0\}$ we have:

$$\sup_{t,s \in B_i} |g(t) - g(s)| \leq \sup_{t,s \in B_i} \{\sup_{n \in \mathbb{N}^*} |f_n(t) - f_n(s)|\} < \frac{\varepsilon}{2}.$$

If we now consider the partition $P_\varepsilon = \{c(\bigcup\limits_{i=1}^{n_0} B_i), B_1, ..., B_{n_0}\}$, we obtain that g is t.m.v.

II. By I, the function h defined for every $t \in T$ by $h(t) = \sup\limits_{n \in \mathbb{N}^*} f_n(t)$, is also t.m.v.

III. By I and II we immediately have the conclusion.

4. Totally-measurability in variation and Gould integrability

As in the previous section, $\mathcal{A}$ is an algebra of subsets of T, $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ a set multifunction of finite variation with $\mu(\emptyset) = \{0\}$ and $f : T \to \mathbb{R}$ an arbitrary bounded function.

In what follows, $\sigma_{f,\mu}(P)$ (or, if there is no doubt, $\sigma_f(P), \sigma_\mu(P)$ or $\sigma(P)$) denotes $\overset{\bullet}{\sum}_{i=1}^{n} f(t_i)\mu(A_i)$, for every partition $P = \{A_i\}_{i=\overline{1,n}}$ of T and every $t_i \in A_i, i = \overline{1,n}$.

Definition 4.1. [4, 5, 7, 8, 17-19] I) The function f is said to be *μ-integrable on T* if the net $(\sigma(P))_{P\in(\mathcal{P},\le)}$ is convergent in $(\mathcal{P}_f(X), h)$, where P, the set of all partitions of T, is ordered by the relation " $\le$ " given in Definition 3.1.

If $(\sigma(P))_{P\in(\mathcal{P},\le)}$ is convergent, then its limit is called *the integral of f on T with respect to μ*, denoted by $\int_T f d\mu$.

II) If $B \in \mathcal{A}, f$ is said to be *μ-integrable on B* if the restriction $f|_B$ of f to B is μ-integrable on $(B, \mathcal{A}_B, \mu_B)$.

Remarks 4.2. I) f is μ-integrable on T if and only if there exists a set $I \in \mathcal{P}_{bf}(X)$ such that for every $\varepsilon > 0$, there exists a partition P_ε of T, so that for every other partition of T, $P = \{A_i\}_{i=\overline{1,n}}$, with $P \ge P_\varepsilon$ and every choice of points $t_i \in A_i, i = \overline{1,n}$, we have $h(\sigma(P), I) < \varepsilon$.

II) If $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$, then $\int_T f d\mu \in \mathcal{P}_{kc}(X)$.

III) [7] If $m : \mathcal{A} \to \mathbb{R}_+$, with $m(\emptyset) = 0$ is arbitrary, of finite variation, and $f : T \to \mathbb{R}$ a bounded function, consider:

i) the set multifunction $\mu : \mathcal{A} \to \mathcal{P}_f(\mathbb{R})$, defined by $\mu(A) = \{m(A)\}$, for every $A \in \mathcal{A}$.

Then f is m-integrable on T if and only if there is $I \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists a partition P_ε of T, so that for every other partition of T, $P = \{A_i\}_{i=\overline{1,n}}$, with $P \ge P_\varepsilon$ and every choice of points $t_i \in A_i, i = \overline{1,n}$, we have $|\sigma(P) - I| = |\sum_{i=1}^{n} f(t_i)m(A_i) - I| < \varepsilon$.

Here, $I = \int_T f d\mu$.

ii) the set multifunction $\mu : \mathcal{A} \to \mathcal{P}_f(\mathbb{R})$, defined by $\mu(A) = [0, m(A)]$, for every $A \in \mathcal{A}$.

Then on every $A \in \mathcal{A}$, f is m-integrable if and only if f is μ-integrable and, in this case,

$$\int_A f d\mu = [0, \int_A f dm].$$

IV) The Gould type integral [18, 19], if it exists, is unique and has remarkable properties. Among them, homogenity and additivity with respect to the function f and the set multifunction μ, additivity with respect to the set, monotonicity with respect to the function f, to the set multifunction μ, and to the set.

In what follows, we point out some relationships between integrability and totally-measurability in variation for various set (multi)functions:

Theorem 4.3. [17] *Suppose $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$ is a multimeasure. If f is t.m.v. on T, then f is μ-integrable on T.*

For multisubmeasures, in general, such a result does not hold. Although, if we deal with finitely purely atomic multisubmeasures, totally-measurability in variation implies integrability:

Theorem 4.4. [7] *Suppose $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is a finitely purely atomic multisubmeasure. If f is t.m.v. on T, then f is μ-integrable on T.*

By Theorem 3.10 and Theorem 4.4, we obtain:

Corollary 4.5. Suppose (T, ρ) is a compact metric space, $\mathcal{B}$ is the Borel δ-ring generated by the compact subsets of T, $f : T \to \mathbb{R}$ is continuous on T and $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is a finitely purely atomic R_l'-regular multisubmeasure. Then f is μ-integrable on T.

From now on, suppose $m : \mathcal{A} \to \mathbb{R}_+$ is a submeasure of finite variation.

In this case, as we shall see from the following result, totally measurability in variation is equivalent to integrability:

Proposition 4.6. [10] The following statements are equivalent:
i) f is m-integrable;
ii) f is t.m.v.
Moreover, in this case, $\int_T f dm = \int_T f d\overline{m}$
(where $\int_T f d\overline{m}$ is the Gould integral [11]).

In the following, let us consider the space $\mathcal{L}^p = \{f : T \to \mathbb{R};\ f$ is bounded on T and $|f|^p$ is m-integrable on $T\}$.

Proposition 4.7. [8] i) $\mathcal{L}^p$ is a linear space.

ii) The function $||\cdot|| : \mathcal{L}^p \to \mathbb{R}_+$, defined for every $f \in \mathcal{L}^p$ by $||f|| = (\int_T |f|^p dm)^{\frac{1}{p}}$, is a semi-norm.

Theorem 4.8. (Fatou Lemma) [7] *Suppose $\mathcal{A}$ is a σ-algebra, $m : \mathcal{A} \to \mathbb{R}_+$ is a submeasure of finite variation so that $\widetilde{m}$ is o-continuous on $\mathcal{P}(T)$ and $(f_n)_{n\in\mathbb{N}}$ is a sequence of uniformly bounded, t.m.v. functions $f_n : T \to \mathbb{R}$. Then*

$$\int_T \liminf_n f_n dm \leq \liminf_n \int_T f_n dm.$$

By the Fatou lemma, using the ideas of [16], in [8] the following result is obtained:

Corollary 4.9. Let $\mathcal{A}$ be a σ-algebra and suppose m is finitely purely atomic and $\widetilde{m}$ is o-continuous on $\mathcal{P}(T)$. Then $||\cdot|| : \mathcal{L}^p \to \mathbb{R}_+$ is a norm and $\mathcal{L}^p$ is a Banach space.

References

[1] Chiţescu, I. - **Finitely purely atomic measures and $\mathcal{L}^p$-*spaces***, An. Univ. Bucureşti Şt. Natur. 24 (1975), 23-29.

[2] Chiţescu, I. - **Finitely purely atomic measures: coincidence and rigidity properties,** Rend. Circ. Mat. Palermo (2) 50 (2001), no. 3, 455-476.

[3] Drewnowski, L. - **Topological rings of sets, continuous set functions, Integration**, I, II, III, Bull. Acad. Polon. Sci. Sér. Math. Astron. Phys. 20 (1972), 269-276, 277-286.

[4] Gavriluţ, A. - **A Gould type integral with respect to a multisubmeasure**, Math. Slovaca, 58 (2008), no 1, 1-20.

[5] Gavriluţ, A. - **On some properties of the Gould type integral with respect to a multisubmeasure,** An. Şt. Univ. Iaşi 52 (2006), no 1, 177-194.

[6] Gavriluţ, A. - **Non-atomicity and the Darboux property for fuzzy and non-fuzzy Borel/Baire multivalued set functions**, Fuzzy Sets and Systems 160 (2009), 1308-1317.

[7] Gavriluţ, A., Croitoru, A. - **Classical theorems for a Gould type integral**, submitted for publication.

[8] Gavriluţ, A., Croitoru, A. - **$\mathcal{L}^p$ spaces generated by the Gould integrability**, submitted for publication.

[9] Gavriluţ, A., Croitoru, A. - **Non-atomicity for fuzzy and non-fuzzy multivalued set functions**, Fuzzy Sets and Systems, 160 (2009), 2106-2116.

[10] Gavriluţ, A., Petcu, A. - **A Gould type integral with respect to a submeasure**, An. Şt. Univ. Iaşi, Tomul LIII, 2007, f. 2, 351-368.

[11] Gould, G.G. - **On integration of vector-valued measures,** Proc. London Math. Soc. 15 (1965), 193-225.

[12] Hu, S. Papageorgiou, N.S. - **Handbook of Multivalued Analysis**, vol. I, Kluwer Acad. Publ., Dordrecht, 1997.

[13] Li, J. - **Order continuous of monotone set function and convergence of measureable functions sequence**, Applied Mathematics and Computation 135 (2003), 211-218.

[14] Li, J. - **A note on the null-additivity of the fuzzy measure**, Fuzzy Sets and Systems 125(2002), 269-271.

[15] Pap, E. - **Null-additive Set Functions**, Kluwer Academic Publishers, Dordrecht-Boston-London, 1995.

[16] Precupanu, A.M. – **Analiză Matematică. Funcţii Reale** (in Romanian), Edit. Didactică şi Pedagogică, Bucureşti, 1976.

[17] Precupanu, A., Croitoru, A. - **A Gould type integral with respect to a multimeasure**, *I*, An. Şt. Univ. Iaşi, 48 (2002), 165-200.

[18] Precupanu, A., Gavriluţ, A., Croitoru, A. - **A fuzzy Gould type integral**, submitted for publication.

[19] Precupanu, A., Gavriluţ, A., Croitoru, A. - **On a generalized Gould type set valued integral**, VMIA Conference, Eichstätt, Germany, 2008.

Faculty of Mathematics, "Al.I. Cuza" University,
Carol I Bd. 11, Iaşi, 700506, România,
gavrilut@uaic.ro, croitoru@uaic.ro