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SOME CURVATURE PROPERTIES IN RANDERS SPACES

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Abstract. In this paper we get a condition for Randers spaces to be simultaneously with scalar flag curvature and with constant E-curvature.

1. INTRODUCTION

Let us consider a real differentiable manifold of dimension n. Denote by (TM, τ, M) the tangent bundle of M. Let $F^n = (M, F(x, y))$ be a Finsler space where $F : TM \to R$ is its fundamental function and the Hessian given by

(1.1) $$g_{ij}(x, y) = \frac{1}{2}\frac{\partial_2 F}{\partial y^i \partial y^j},$$

called the fundamental tensor field of F^n is positive defined.

The Finsler metric *F* induces a vector field

(1.2) $$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$$

on *TM*, defined by

(1.3) $$G^i = \frac{1}{4} g^{il}(x, y)\left\{\left[F^2\right]_{x^k y^l}(x, y) y^k - \left[F^2\right]_{x^l}(x, y)\right\}.$$

Any vector field in the above form (1.2) with the homogeneity property

(1.4) $$G^i(x, \lambda y) = \lambda^2 G^i(x, y),\ \lambda > 0$$

is called a *spray* and G^i are called the *spray coefficients*.

For a vector $y \in T_x M - \{0\}$, the *Riemann curvature*

(1.5) $$R_y = R^i_k(x, y) dx^k \otimes \frac{\partial}{\partial x^i} : T_x M \to T_x M$$

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is defined by

(1.6) $$R^i_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

For a flag $P = span\{y,u\} \subset T_xM$ with the flagpole y, the *flag curvature* $K = K(P,y)$ is defined by

(1.7) $$K(P,y) = \frac{g_y(u, K_y(u))}{g_y(y,y)g_y(u,u) - g_y(y,u)^2},$$

where $g_y = g_{ij}(x,y)dx^i \otimes dx^j$.

We say that a Finsler metric F is of scalar curvature if for any $y \in T_xM$, the flag curvature $K = K(x,y)$ is a scalar function. If $K = constant$, then F is said to be of constant flag curvature.

The *volume form* of F is expressed by

(1.8) $$dV_F = \sigma_F(x)dx^1 ... dx^n$$

where

(1.9) $$\sigma_F = \frac{Vol(B^n)}{Vol\left\{(y^i) \in R^n \,\middle|\, F\left(y^i \frac{\partial}{\partial x^i}|_x\right) < 1\right\}}.$$

The *S-curvature* is defined by

(1.10) $$S = \frac{\partial G^i}{\partial y^i} - y^i \frac{\partial}{\partial x^i}(\ln \sigma_F(x)).$$

A Finsler metric F is said to have *isotropic S-curvature* if there is a scalar function $c = c(x)$ on M such that

(1.11) $$S = (n+1)cF.$$

For a vector $y \in T_xM - \{0\}$, we define a symmetric bilinear form on T_xM

$$E_y = E_{ij}(x,y)dx^i \otimes dx^j,$$

with

(1.12) $$E_{ij}(x,y) = \frac{1}{2}\frac{\partial^3 G^m}{\partial y^m \partial y^i \partial y^j}(x,y),$$

called *E-curvature.*

An equivalent expression for E_{ij} is

(1.13) $$E_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 S}{\partial y^i \partial y^j}(x,y).$$

A Finsler metric F is said to have *isotropic E-curvature* if there is a scalar function $c = c(x)$ on M such that

(1.14) $$E_{ij} = \frac{n+1}{2} cF^{-1} h_{ij},$$

where

(1.15) $$h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j}.$$

2. E-CURVATURE AND S-CURVATURE PROPERTIES IN RANDERS SPACES

A *Randers metric* is a Finsler metric

(2.1) $$F(x,y) = \alpha(x,y) + \beta(x,y),$$

where $\alpha(x,y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta(x,y) = b_i(x)y^i$ is a 1-form on M.

Define $b_{i|j}$ by

$$b_{i|j}\theta^j = db_i - b_j\theta_i^j,$$

where

$$\theta^i = dx^i \text{ and } \theta_i^j = \Gamma_{ik}^j dx^k$$

denote the Levi-Civita connection forms of α.

We use the notation from [5]:

(2.2) $$\begin{aligned} r_{ij} &= \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}) \\ s_j^i &= a^{ih}s_{hj} \\ s_j &= b_i s_j^i, \quad e_{ij} = r_{ij} + b_i s_j + b_j s_i \\ e_{00} &= e_{ij}y^i y^j \\ s_0 &= s_i y^i \\ s_0^i &= s_j^i y^j \end{aligned}$$

According to [5], the spray coefficients G^i of F are related to the spray coefficients $\overline{G^i}$ of α by

$$G^i = \overline{G^i} + \left(\frac{e_{00}}{2F} - s_0\right) y^i + \alpha s_0^i \tag{2.3}$$

and the volume form σ_F of F is related to the volume form σ_α of α by

$$\sigma_F = \left(1 - \|\beta\|_\alpha^2\right)^{\frac{n+1}{2}} \sigma_\alpha . \tag{2.4}$$

In a Randers space we have a formula for S-curvature:

$$S = (n+1)\left(\frac{e_{00}}{2F} - (s_0 + \rho_0)\right), \tag{2.5}$$

where

$$\rho = ln\sqrt{1 - \|\beta\|_\alpha^2}\ ,\ d\rho = \rho_i dx^i\text{ , i.e }\rho_i = -\frac{b_j b_{j|i}}{1 - \|\beta\|_\alpha^2}\text{ and }\rho_0 = \rho_i y^i .$$

Then we have already known the following

Lemma 2.1 [2] *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. For a scalar function $c = c(x)$ on M the following are equivalent:*

i) $S = (n+1)cF$ *;*

ii) $e_{00} = 2c\left(\alpha^2 - \beta^2\right)$.

Lemma 2.2 [2] *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. For a scalar function $c = c(x)$ on M the following are equivalent:*

i) $E = \frac{n+1}{2} cF^{-1} h$;

ii) $e_{00} = 2c\left(\alpha^2 - \beta^2\right)$.

From the two lemmas we have

Theorem 2.1 [2] *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. For a scalar function $c = c(x)$ on M the following are equivalent:*

i) $S = (n+1)cF$ *;*

ii) $E = \frac{n+1}{2} cF^{-1} h$.

From [3] we also have

Theorem 2.2 *Let (M, F) be an n-dimensional Finsler manifold of scalar flag curvature $K(x,y)$. Suppose that F has an isotropic S-curvature,*

$S=(n+1)cF$, *with* $c=c(x)$ *a scalar function on M. Then there is a scalar function* $\sigma(x)$ *on M such that*

(2.6)
$$K=3\frac{c_{x^m}y^m}{F(x,y)}+\sigma(x).$$

From Theorem 2.1 and Theorem 2.2 we immediately get

Theorem 2.3 *Let* $F=\alpha+\beta$ *be a Randers metric on an n-dimensional manifold M of scalar flag curvature* $K(x,y)$. *Suppose that F has an isotropic E-curvature* , $E=\frac{n+1}{2}cF^{-1}h$, *with* $c=c(x)$ *a scalar function on M. Then there is a scalar function* $\sigma(x)$ *on M such that*

$$K=3\frac{c_{x^m}y^m}{F(x,y)}+\sigma(x).$$

3. RANDERS SPACES WITH SCALAR FLAG CURVATURE AND ISOTROPIC E-CURVATURE

Theorem 3.1 *Let* $F=\alpha+\beta$ *be a Randers metric on an n-dimensional manifold M of scalar flag curvature* $K(x,y)$. *Suppose that F has an isotropic E-curvature* , $E=\frac{n+1}{2}cF^{-1}h$, *with* $c=c(x)$ *a scalar function on M. Then there is a scalar function* $\sigma(x)$ *on M such that*

$$\left(3\frac{\left(c_{x^m}y^m\right)_{|l}}{F}+\sigma_{|l}\right)h_k^i+\left(3\frac{\left(c_{x^m}y^m\right)_{|k}}{F}+\sigma_k\right)h_l^i+$$
$$+\left(3\frac{\left(c_{x^m}y^m\right)_{|m}}{F}+\sigma_m\right)\left(\delta_k^iF_{;l}-\delta_l^iF_{;k}\right)$$
$$=h_k^i\frac{\left(c_{x^l}\right)_{|m}F-\left(c_{x^m}y^m\right)_{|m}F_{;l}}{F^2}-h_l^i\frac{\left(c_{x^k}\right)_{|m}F-\left(c_{x^m}y^m\right)_{|m}F_{;k}}{F^2}.$$

Proof.

From the assumption F is of scalar flag curvature we have

(3.1)
$$R_{jk}=K(x,y)h_{jk}$$

and

(3.2) $$R_{ikl} = K(x,y)h_{ikl} + \frac{1}{3}\left(h_{ik}K_{;l} - h_{il}K_{;k}\right),$$

where

$$K_{;l} = \frac{\partial K}{\partial y^l}$$

and

$$h_{ikl} = g_{ik}F_{;l} - g_{il}F_{;k}.$$

Contracting (3.2) with g^{ij} we get

(3.3) $$R^j_{kl} = K\left(g_{ik}g^{ij}F_{;l} - g_{il}g^{ij}F_{;k}\right) + \frac{1}{3}\left(h^j_k K_{;l} - h^j_l K_{;k}\right)$$

and then

(3.4)
$$\begin{aligned} R^j_{kl|m} &= K_{|m}\left(g_{ik}g^{ij}F_{;l} - g_{il}g^{ij}F_{;k}\right) \\ &+ K\left(\left(g_{ik}g^{ij}\right)_{|m}F_{;l} + g_{ik}g^{ij}F_{;l|m} - \left(g_{il}g^{ij}\right)_{|m}F_{;k} - g_{il}g^{ij}F_{;k|m}\right) \\ &+ \frac{1}{3}\left(h^j_{k|m}K_{;l} + h^j_k K_{;l|m} - h^j_{l|m}K_{;k} - h^j_l K_{;k|m}\right). \end{aligned}$$

We know that

(3.5) $$h^j_{k|m} = \left(h_{ik}g^{ij}\right)_{|m} = h_{ik|m}g^{ij} + h_{ik}\left(g^{ij}\right)_{|m} = 0.$$

Plugging (3.5) in (3.4) we obtain

(3.6) $$R^j_{kl|m} = K_{|m}\left(g_{ik}g^{ij}F_{;l} - g_{il}g^{ij}F_{;k}\right) + \frac{1}{3}\left(h^j_k K_{;l|m} - h^j_l K_{;k|m}\right).$$

For the hh-curvature R^i_{jkl} we have the following Bianchi identities:

(3.7) $$R^i_{jkl|m} + R^i_{jlm|k} + R^i_{jmk|l} = 0,$$

or, contracting with y^j

(3.8) $$R^i_{kl|m} + R^i_{lm|k} + R^i_{mk|l} = 0.$$

Contracting (3.8) with y^m results

(3.9) $$R^i_{kl|m}y^m + R^i_{l|k} + R^i_{k|l} = 0.$$

From (3.1) we get

(3.10) $$R^i_k = Kh^i_k$$

and

$$R^i_{k|l} = K_{|l} h^i_k. \tag{3.11}$$

From (3.6), (3.9) and (3.11) we obtain

$$\begin{aligned} &K_{|l} h^i_k + K_{|k} h^i_l + K_{|m} y^m \left(g_{jk} g^{ij} F_{;l} - g^{ji} g_{jl} F_{;k} \right) \\ &+ \frac{1}{3} y^m K_{;l|m} h^i_k - \frac{1}{3} y^m h^i_l K_{;l|m} = 0, \end{aligned} \tag{3.12}$$

or, equivalent

$$\begin{aligned} &K_{|l} h^i_k + K_{|k} h^i_l + K_{|m} y^m \left(\delta^i_k F_{;l} - \delta^i_l F_{;k} \right) \\ &+ \frac{1}{3} \left(K_{;l|m} h^i_k - h^i_l K_{;l|m} \right) y^m = 0. \end{aligned} \tag{3.13}$$

From Theorem 2.3 , there is scalar function $\sigma(x)$ on M such that

$$K = 3 \frac{c_{x^m} y^m}{F(x,y)} + \sigma(x).$$

Replacing

$$\begin{aligned} K_{|l} &= 3 \frac{\left(c_{x^m} y^m \right)_{|l}}{F} + \sigma_{|l} \\ K_{;l} &= 3 \frac{c_{x^l} F - c_{x^m} y^m F_{;l}}{F^2} \\ K_{;l|m} &= 3 \frac{\left(c_{x^l} \right)_m F - \left(c_{x^m} y^m \right)_{|m} F_{;l}}{F^2} \end{aligned} \tag{3.14}$$

in (3.13) we obtain the conclusion.

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