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AN APPLICATION OF AN INTEGRAL OPERATOR USING BRIOT-BOUQUET DIFFERENTIAL SUPERORDINATION

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Abstract. The notion of differential superordination was introduced as a dual concept of differential subordination by the S. S. Miller and P. T. Mocanu. In this paper we give applications to Briot-Bouquet differential superordination and we prove a sandwich theorem, generalizing some results from [7] and [8].

1. INTRODUCTION

Let $\mathcal{H}$ (U) denote the class of analytic functions in the open unit disc

$$U = \{z \in \mathbf{C} : |z| < 1\}$$

For $n \in \mathbf{N}^*$ and $a \in \mathbf{C}$, let

$$\mathcal{H}[a,n] = \left\{ f \in \mathcal{H}\ (U) : \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots,\ z \in U \right\}.$$

Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

and normalized by $f(0) = f'(0) - 1 = 0$.

Let $\mathcal{Q}$ denote the set of functions q that are analytic and injective on the set $\bar{U} \setminus E(q)$, where

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Let $E(q)$ is called exception set. Denote by $\mathcal{Q}$ (a) the subclass $\mathcal{Q}$ for which $q(0) = a$.

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Miller and Mocanu [1] introduced the following notion of differential superordination, as the dual concept of differential subordination.

Recall the concepts of superordination and subordination. Let f and F be members of $\mathcal{H}$ (U) and let F be univalent in U. The function F is said to be superordinate to f, or f is subordinate to F, if there exists a function w analytic in U, with $w(0)=0$ and $|w(z)|<1$, $z \in U$, and such that $f(z)=F(w(z))$. In such a case we write $f \prec F$. If F is univalent, then $f \prec F$ if and only if $f(0)=F(0)$ and $f(U) \subset F(U)$.

In the paper [2] the authors introduced the dual concept of a differential superordination and obtain a conditions so that the Briot-Bouquet sandwich

$$h_1(z) \prec p(z)+\frac{z\,p'(z)}{\beta p(z)+\gamma} \prec h_2(z) \text{ implies that}$$

$$q_1(z) \prec p(z) \prec q_2(z) \tag{1}$$

To prove our main result we will need the following lemmas.

Lemma 1 [6]: *Let $f \in \mathcal{H}$ (U). Then f is convex in U if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re}\frac{z\,f''(z)}{f'(z)}+1>0,\ z \in U.$$

Lemma 2 [2]: *Let $\beta, \gamma \in \mathbf{C}$ and the function h be convex in U, with $h(0)=a$. Suppose that the differential equation*

$$q(z)+\frac{z\,q'(z)}{\beta q(z)+\gamma}=h(z) \tag{2}$$

has a univalent solution q that satisfies $q(0)=a$ and $q(z) \prec h(z)$. If $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$ and $p(z)+\dfrac{z\,p'(z)}{\beta p(z)+\gamma}$ is univalent in U, then

$$h(z) \prec p(z)+\frac{z\,p'(z)}{\beta p(z)+\gamma} \ \Rightarrow\ q(z) \prec p(z). \tag{3}$$

The function q is the best subordinant.

Lemma 3 [2]: *Let $\beta, \gamma \in \mathbf{C}$ and h_i be convex in U, with $h_i(0)=a$, $i=\overline{1,2}$. Suppose that the differential equation*

$$q_i(z)+\frac{z\,q'_i(z)}{\beta q_i(z)+\gamma}=h_i(z) \tag{4}$$

has a univalent solution q_i *that satisfies* $q_i(0)=a$, $i=\overline{1,2}$ *and* $q_i(z)\prec h_i(z)$, *for* $i=\overline{1,2}$. *If* $p\in\mathcal{H}[a,1]\cap\mathcal{Q}$ *and* $p(z)+\dfrac{z\,p'(z)}{\beta\,p(z)+\gamma}$ *is univalent in U, then*

$$h_1(z)\prec p(z)+\frac{z\,p'(z)}{\beta\,p(z)+\gamma}\prec h_2(z)\quad\Rightarrow\quad q_1(z)\prec p\left(z\right)\prec q_2(z). \qquad (5)$$

The functions q_1 *and* q_2 *are the best subordinant and best dominant, respectively.*

If $\beta,\gamma\in\mathbf{C}$, then (4) has univalent (convex) solutions given by

$$q_i(z)=\left[\frac{\beta+\gamma}{z^{y}}\int_0^z h_i(t)\,t^{\gamma-1}dt\right]^{\frac{1}{\beta}},\ i=\overline{1,2}. \qquad (6)$$

In this case we obtain the following sandwich theorem.

If $\beta=\gamma=1$ then

$$q_i(z)=\frac{2}{z}\int_0^z h_i(t)\,dt,\ i=\overline{1,2}. \qquad (7)$$

Lemma 4 [2]: *Let* $\beta,\gamma\in\mathbf{C}$ *and let* q *be univalent in U, with* $q(0)=a$. *Set* h *be convex in U, with* $h(0)=a$. *Suppose that the differential equation*

$$h(z)=q(z)+\frac{z\,q'(z)}{\beta q(z)+\gamma} \qquad (8)$$

and suppose that

(i) $\mathrm{Re}\left[\beta q(z)+\gamma\right]>0$, *and*

(ii) $\dfrac{z\,q'(z)}{\beta q(z)+\gamma}$ *is starlike.*

If $p\in\mathcal{H}[a,1]\cap\mathcal{Q}$ *and* $p(z)+\dfrac{z\,p'(z)}{\beta\,p(z)+\gamma}$ *is univalent in U, then*

$$h(z)\prec p(z)+\frac{z\,p'(z)}{\beta\,p(z)+\gamma}\ \Rightarrow q(z)\prec p\left(z\right). \qquad (9)$$

and q *is the best subordinant.*

2. MAIN RESULTS

Theorem 1: *Let* $a \in \mathbf{R}$. *If* $a \in A_0 := (-\infty,-4.5116) \cup (-3.6179,-2) \cup (0.7571,+\infty)$, *then the function* $h(z) = z + a + \dfrac{z}{z+a+1}$, $z \in U$ *is convex.*

Proof.

The function

$$h(z) = z + a + \frac{z}{z+a+1}, z \in \overline{U}. \tag{10}$$

is well-defined if and only if $|a+1| > 1$.

To show that the function h is convex, we use the characterization of convex functions given by Lemma 1. We prove that

i) $h'(0) \neq 0$; (11)

ii) $\mathrm{Re}\dfrac{z\,h''(z)}{h'(z)} + 1 > 0$, $z \in U$. (12)

Taking the derivative with respect to z in both members of (10) we obtain

$$h'(z) = 1 + \frac{a+1}{(z+a+1)^2}, \ z \in U. \tag{13}$$

Equation (13) implies

$$h''(z) = -\frac{2(a+1)}{(z+a+1)^3}, \ z \in U. \tag{14}$$

Then

$$h'(0) = 1 + \frac{1}{a+1} = \frac{a+2}{a+1}.$$

Since $a \neq -1$, then we have $h'(0) \neq 0$

ii) By using (13) and (14) in relation (12), we obtain

$$\mathrm{Re}\frac{z\,h''(z)}{h'(z)} + 1 = \mathrm{Re}\frac{-\dfrac{2z(a+1)}{(z+a+1)^3}}{1+\dfrac{a+1}{(z+a+1)^2}} + 1 =$$

$$= \mathrm{Re}\frac{-2z(a+1)}{\left[(z+a+1)^2 + (a+1)\right](z+a+1)} + 1$$

Denote $b = a+1$. Then $|b| > 1$. We have

$$\operatorname{Re}\frac{z\,h''(z)}{h'(z)}+1=\operatorname{Re}\frac{-2\,z\,b}{\left[(z+b)^2+b\right](z+b)}+1=$$
$$=\operatorname{Re}\frac{-2\,z\,b}{(z^2+2bz+b^2+b)(z+b)}+1 \tag{15}$$

Take $z=e^{i\theta}=\cos\theta+i\sin\theta$, $\theta\in[0,2\pi]$ in relation (15). We obtain

$$\operatorname{Re}\frac{z\,h''(z)}{h'(z)}+1=\operatorname{Re}\frac{-2\,e^{i\theta}\,b}{(e^{2i\theta}+2\,b\,e^{i\theta}+b^2+b)(e^{i\theta}+b)}+1=$$
$$=\operatorname{Re}\frac{-2\,(\cos\theta+i\sin\theta)\,b}{\left[(\cos 2\theta+i\sin 2\theta)+2\,b\,(\cos\theta+i\sin\theta)+b^2+b\right]}$$
$$\frac{1}{\left[(\cos\theta+i\sin\theta)+b\right]}+1$$

where $z=e^{i\theta}$.

Denoting $\cos\theta=t$, for each $t\in[-1,1]$ we obtain for $z=e^{i\theta}$:

$$\operatorname{Re}\frac{z\,h''(z)}{h'(z)}+1=$$
$$=\frac{-2bt(2t^2-1+2bt+b^2+b)(t+b)-4b(1-t^2)(t+b)^2}{\left[(2t^2-1+2bt+b^2+b)^2+(1-t^2)(2t+2b)^2\right]\left[(t+b)^2+1-t^2\right]}+$$
$$+\frac{4bt(t+b)(1-t^2)-2b(1-t^2)(2t^2-1+2bt+b^2+b)}{\left[(2t^2-1+2bt+b^2+b)^2+(1-t^2)(2t+2b)^2\right]\left[(t+b)^2+1-t^2\right]}+1=$$

$\dfrac{G(t)}{H(t)}$, where we denoted

$G(t)=(8b^3+8b^2)t^3+(12b^4+12b^3+12b^2)t^2+(6b^5+6b^4+12b^3-6b^2+6b)t+$
$(b^6+2b^5+4b^4-6b^3+2b^2+1)$

and

$H(t)=(8b^3+8b^2)t^3+(12b^4+12b^3+12b^2+4b)t^2+(6b^5+8b^4+14b^3+6b)t+$
$(b^6+2b^5+4b^4+4b^2-2b+1)$.

Denote $F(t)=\dfrac{G(t)}{H(t)}$. The denominator of $F(\cos\theta)=\dfrac{G(\cos\theta)}{H(\cos\theta)}$ is $H(\cos\theta)=\left(\left|e^{i\theta}+b\right|\cdot\left|(e^{i\theta}+b)^2+b\right|\right)^2>0$.

We will find a set $B \subset R$ such that $G(t) > 0$ for every $t \in [-1,1]$, whenever $b \in B$.

$G(1) = P(b)$, where $P(b) = b^6 + 8b^5 + 22b^4 + 26b^3 + 16b^2 + 6b + 1$.

The real roots of $P(b)$ are:

$$b_1 = -3{,}5115..., b_2 = -2.6180..., b_3 = -1, b_4 = -0.3819... .$$

$G(-1) = Q(b)$, where $Q(b) = b^6 - 4b^5 + 10b^4 - 14b^3 + 12b^2 - 6b + 1$

The real roots of $Q(b)$ are $b_5 = 0.2847..$ and $b_6 = 1$.

Taking into account that $|b| > 1$, we see that $G(1) > 0$ and $G(-1) > 0$ if and only if $b \in (-\infty, b_1) \cup (b_2, -1) \cup (1, +\infty) =: B$. In the following we assume that $b \in B$.

To determine the set of values of b for which $G(t) > 0$, for each $t \in [-1,1]$, we apply the change of variable $u = \dfrac{1-t}{1+t}$, $t \in (-1,1]$ and determine the set of values of b for which $G_1(u) = G\left(\dfrac{1-u}{1+u}\right) > 0$, for all $u \geq 0$. Then

$$\begin{aligned}
&G_1(u) = \\
&(b^6\text{-}4b^5\text{+}10b^4\text{-}14b^3\text{+}12b^2\text{-}6b\text{+}1)u^3\text{+} \\
&(3b^6\text{-}6b^4\text{-}18b^3\text{+}24b^2\text{-}6b\text{+}3)u^2\text{+} \\
&(3b^6\text{+}12b^5\text{+}6b^4\text{-}42b^3\text{-}36b^2\text{+}6b\text{+}3)\,u\text{+} \\
&(b^6\text{+}8b^5\text{+}22b^4\text{+}26b^3\text{+}16b^2\text{+}6b\text{+}1)
\end{aligned}$$

Since the first coefficient is $Q(b) > 0$ and the last coefficient is $P(b) > 0$, it suffices to have $S(b) := 3b^6\text{-}6b^4\text{-}18b^3\text{+}24b^2\text{-}6b\text{+}3 > 0$ and $T(b) := 3b^6\text{+}12b^5\text{+}6b^4\text{-}42b^3\text{-}36b^2\text{+}6b\text{+}3 > 0$, where $b \in B$.

The real roots of $S(b)$ are $b_7 = 1, b_8 = 1{,}7570..$ and the real roots of $T(b)$ are $b_9 = -1, b_{10} = -0{,}2449.., b_{11} = 0{,}3187.., b_{12} = 1{,}7106..$

It follows that for $b \in (-\infty, b_1) \cup (b_2, -1) \cup (b_8, +\infty) =: B_0$ we get $G(t) > 0$ for every $t \in [-1,1]$, hence $\operatorname{Re}\dfrac{zh''(z)}{h'(z)} + 1 > 0$ whenever $|z| = 1$.

Then h is convex whenever $b \in B_0$, hence whenever $a \in A_0$.

Remark 1: For $b=2$ we get $F(t)=\dfrac{96t^3+336t^2+372t+153}{96t^2+344t^2+444t+205}$, the expression from [8].

Theorem 2: *Let* $a\in A_0$ *and* h *be convex in U, with* $h(0)=a$. *Suppose that the differential equation*

$$h(z)=q(z)+\frac{z\,q'(z)}{q(z)+1},\ z\in U. \tag{16}$$

has a univalent solution $q(z)=z+a$ *that satisfies* $q(0)=a$ *and* $q(z)\prec h(z)$. *If* $f\in\mathcal{A}$ *and* $\dfrac{z\,f'(z)}{f(z)}$ *is univalent,* $\dfrac{z\,F'(z)}{F(z)}\in\mathcal{H}[a,1]\cap\mathcal{Q}$ *and*

$$h(z)\prec\frac{z\,f'(z)}{f(z)},\ z\in U, \tag{17}$$

then

$$q(z)\prec\frac{z\,F'(z)}{F(z)},\ z\in U, \tag{18}$$

where

$$F(z)=\frac{2}{z}\int_0^z f(t)dt. \tag{19}$$

Proof.

According to Theorem 1, for $a\in A_0$ the function h is convex.

Since h is convex and the function $q(z)=z+a$ is a univalent solution of equation (16), we obtain

$$h(z)=z+a+\frac{z}{z+a+1}$$

which satisfies $h(0)=a$ and $q(z)\prec h(z)$, $z\in U$. From equation (18) we have

$$z\,F(z)=2\int_0^z f(t)dt,\ z\in U. \tag{20}$$

Taking the derivative with respect to z in both members of (20) we obtain

$$z\,F'(z)+F(z)=2\,f(z). \tag{21}$$

Hence

$$F(z)\left[\frac{z\,F'(z)}{F(z)}+1\right]=2\,f(z),\ z\in U. \tag{22}$$

The function p defined by

$$p(z)=\frac{zF'(z)}{F(z)}, \tag{23}$$

is analytic in U and $p(0)=1$. Then

$$F(z)\left[p(z)+1\right]=2f(z)\ ,\ z\in U\ . \tag{24}$$

Hence we obtain: $F'(z)\left[p(z)+1\right]+F(z)p'(z)=2f'(z)$, which can be written as

$$\left[p(z)+1\right]F(z)\left[\frac{zF'(z)}{F(z)}+\frac{z\,p'(z)}{p(z)+1}\right]=2z\,f'(z) \tag{25}$$

From equations (24) and (25) ,

$$\frac{zF'(z)}{F(z)}+\frac{z\,p'(z)}{p(z)+1}=\frac{z\,f'(z)}{f(z)} \tag{26}$$

Using (23) in equation (26), we obtain:

$$p(z)+\frac{z\,p'(z)}{p(z)+1}=\frac{z\,f'(z)}{f(z)}. \tag{27}$$

Equation (27) and subordination (17) imply

$$h(z)\prec p(z)+\frac{z\,p'(z)}{p(z)+1}\quad z\in U\ . \tag{28}$$

Applying Lemma 2 it follows that

$$q(z)\prec p(z)=\frac{zF'(z)}{F(z)}\ \Rightarrow\ z+a\prec\frac{zF'(z)}{F(z)}.$$

Using the conditions from Lemma 4 and Theorem 2 we can write the following.

Corollary 2: *Let* $a\in A_0$**.** *If* $f_1, f_2\in\mathcal{A}$ *and* $\frac{z\,f_1'(z)}{f_1(z)}$ *and* $\frac{z\,f_2'(z)}{f_2(z)}$ *are univalent,* $\frac{zF_1'(z)}{F_1(z)},\frac{zF_2'(z)}{F_2(z)}\in\mathcal{H}[a,1]\cap\mathcal{Q}$ *and*

$$\frac{z\,f_1'(z)}{f_1(z)}\prec z+a+\frac{z}{z+a+1}\prec\frac{z\,f_2'(z)}{f_2(z)},\ z\in U \tag{29}$$

then

$$\frac{zF_1'(z)}{F_1(z)}\prec z+a\prec\frac{zF_2'(z)}{F_2(z)}\ ,\ z\in U\ , \tag{30}$$

where

$$F_i(z)=\frac{2}{z}\int_0^z f_i(t)dt\ ,\ i=\overline{1,2}\,. \tag{31}$$

If $a=1$ in Theorem 2 then we obtain the result of G. I. Oros [7] i.e. :

Corollary 3: *Let h be convex in U, with* $h(0)=1$*, defined by*

$$h(z)=z+1+\frac{z}{2+z},\ z\in U. \tag{32}$$

If $f\in\mathcal{A}$ *and* $\frac{z\,f'(z)}{f(z)}$ *is univalent,* $\frac{z\,F'(z)}{F(z)}\in\mathcal{H}[1,1]\cap\mathcal{Q}$ *and*

$$h(z)\prec\frac{z\,f'(z)}{f(z)},\ z\in U, \tag{33}$$

then

$$q(z)=z+1\prec\frac{z\,F'(z)}{F(z)},\ z\in U, \tag{34}$$

where F is given by (19).

Corollary 4: *If* $f_1, f_2\in\mathcal{A}$ *and* $\frac{z\,f_1'(z)}{f_1(z)}$ *and* $\frac{z\,f_2'(z)}{f_2(z)}$ *is univalent,* $\frac{z\,F_1'(z)}{F_1(z)},\frac{z\,F_2'(z)}{F_2(z)}\in\mathcal{H}[1,1]\cap\mathcal{Q}$ *and*

$$\frac{z\,f_1'(z)}{f_1(z)}\prec z+1+\frac{z}{2+z}\prec\frac{z\,f_2'(z)}{f_2(z)},\ z\in U, \tag{35}$$

then

$$\frac{z\,F_1'(z)}{F_1(z)}\prec z+1\prec\frac{z\,F_2'(z)}{F_2(z)},\ z\in U, \tag{36}$$

where F_i, $i=\overline{1,2}$ *is given by* (31).

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