"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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A GENERALIZATION OF ORLICZ-SOBOLEV CAPACITY IN METRIC MEASURE SPACES

MARCELINA MOCANU

Abstract. Given a Banach function space B and a metric measure space X, we investigate continuity and regularity properties of the $B-$capacity, that we introduced in [13] by means of a Sobolev-type space $N^{1,B}(X)$. It was proved that $B-$capacity is an outer measure, which represents the correct gauge for distinguishing between two functions in $N^{1,B}(X)$ [13] . In the case when B is reflexive we show that $B-$capacity is continuous on increasing sequences of arbitrary subsets of X. Assuming that B has absolutely continuous norm, that every function in B is dominated by a semicontinuous function in B and that continuous functions are dense in $N^{1,B}(X)$, we prove that $B-$capacity is outer regular. As consequences of this outer regularity we obtain the continuity of $B-$capacity on decreasing sequences of compact subsets of X and the coincidence between the $B-$capacity and another usual capacity.

1. **Introduction**

In this paper (X, d) is a metric space endowed with a Borel regular measure μ, which is finite and positive on balls.

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Sobolev-type spaces based on Banach function spaces have been introduced in [13] as a generalization of Orlicz-Sobolev spaces on metric measure spaces studied in [17], that in turn generalize the Newtonian spaces introduced in [15], [16]. Newtonian spaces turned out to be a remarkable tool in analysis and nonlinear potential theory on metric measure spaces.

Capacity is a set function arising in potential theory as the abstract analogue of the physical concept of electrostatic capacity. The notion of capacity is important in understanding the behavior of Sobolev functions, since capacity takes the place of measure in Egorov and Lusin type theorems for these functions [7] and in estimates used in studying solutions of PDEs.

In the classical potential theory the capacity admits a variational characterization as the minimum of an energy functional for functions achieving particular boundary values. This characterization allows the definition of several capacities using various energy functionals. For example, an absolute Sobolev $p-$capacity of a set $E \subset \mathbb{R}^n$ is defined by $cap_p(E) = \inf \int_{\mathbb{R}^n} (|u|^p + \|\nabla u\|^p)\, d\mu : u \in W^{1,p}(\mathbb{R}^n), u \geq 1$ in a neighborhood of E. This concept of capacity has been first extended to metric measure spaces in [9], where the generalization of the Sobolev space $W^{1,p}(\mathbb{R}^n)$ is the Hajłasz-Sobolev space $M^{1,p}(X)$ introduced in [5]. Another extension to metric measure spaces, based on Newtonian spaces, of the concept of Sobolev capacity has been given in [16]. A more general concept of capacity, based on Orlicz-Sobolev spaces, has been introduced and studied in [17], as follows. Given an Orlicz-Sobolev space $N^{1,\Psi}(X)$ with the norm $\|\cdot\|_{N^{1,\Psi}(X)}$, the corresponding $\Psi-$capacity of a set $E \subset X$ is defined by $Cap_\Psi(E) = \inf \left\{ \|u\|_{N^{1,\Psi}(X)} : u \in N^{1,\Psi}(X),\, u \geq 1 \text{ on } E \right\}$. It is proved in [17] that sets of zero $\Psi-$capacity are removable for $N^{1,\Psi}$.

In nonlinear potential theory a Sobolev capacity is said to be a Choquet capacity on X if it is an outer regular outer measure, continuous on arbitrary increasing sequences of sets and continuous on decreasing sequences of compact sets. In [9] and [10] it was proved that the Hajłasz-Sobolev $p-$capacity is a Choquet capacity. In [4] a relative Sobolev $p-$capacity based on the Newtonian space $N^{1,p}$ is studied, being shown that this is a Choquet capacity.

The notion of $B-$capacity, based on a Sobolev-type space $N^{1,B}(X)$, with B a Banach function space, has been introduced in [13], as a generalization of the above notion of $\Psi-$capacity. For every Banach function space B the $B-$capacity is an outer measure, which represents the correct gauge for distinguishing between two functions in $N^{1,B}(X)$ [13]. A characterization of sets of zero $B-$capacity, as well as a partial result on the outer regularity of $B-$capacity, extending results from [18] have also been proved in [13] .

In this paper we prove, under quite general assumptions for B and $N^{1,B}(X)$, that $B-$capacity is a Choquet capacity. In the case when B is reflexive we show that $B-$capacity is continuous on increasing sequences of arbitrary subsets of X. Assuming that B has absolutely continuous norm, that every function in B is dominated by a semicontinuous function in B and that continuous functions are dense in $N^{1,B}(X)$, we prove that $B-$capacity is outer regular. As consequences of this outer regularity we obtain the continuity of $B-$capacity on decreasing sequences of compact subsets of X and the coincidence between the $B-$capacity and another usual capacity. In conclusion, assuming that B is a reflexive Banach function space with absolutely continuous norm, that every function in B is dominated by a semicontinuous function in B and that continuous functions are dense in $N^{1,B}(X)$, it follows that $B-$capacity is a Choquet capacity.

We deal only with absolute capacity, although some of our results could be extended to relative capacity.

2. **Preliminaries**

We recall the concept of Banach function space, an unifying axiomatic framework for Orlicz and Lorentz spaces, as presented in [2].

Banach function spaces are Banach spaces of measurable functions, in which the norm is related to the underlying measure in an appropriate way, allowing an interplay between functional-analytic and measure/theoretic techniques .

Let (X, Σ, μ) be a complete $\sigma-$finite measure space and let $\boldsymbol{M}^+(X)$ be the collections of all measurable functions $f : X \to [0, +\infty]$.

Definition 1. [2]

A function $N : \boldsymbol{M}^+(X) \to [0, \infty]$ is called a *Banach function norm* if, for all f, g, f_n $(n \geq 1)$ in $\boldsymbol{M}^+(X)$, for all constants $a \geq 0$ and for all measurable sets $E \subset X$, the following properties hold:

(P1) $N(f) = 0 \Leftrightarrow f = 0$ $\mu-$a.e.; $N(af) = aN(f)$; $N(f+g) \leq N(f) + N(g)$.

(P2) If $0 \leq g \leq f$ $\mu-$a.e., then $N(g) \leq N(f)$.

(P3) If $0 \leq f_n \uparrow f$ $\mu-$a.e., then $N(f_n) \uparrow N(f)$.

(P4) If $\mu(E) < \infty$, then $N(\chi_E) < \infty$.

(P5) If $\mu(E) < \infty$, then $\int_E f \; d\mu \leq C_E N(f)$, for some constant $C_E \in (0, +\infty)$ depending only on E and ρ.

The collection B of the $\mu-$measurable functions $f : X \to [-\infty, +\infty]$ for which $N(|f|) < \infty$ is called a *Banach function space* on X. For $f \in B$ define

$$\|f\|_B = N(|f|).$$

By (P5), every function in B is locally integrable, hence finite $\mu-$a.e. in X, since μ is $\sigma-$finite.

The Fatou property (P3) implies, for $f_n \in B$ $(n \geq 1)$ the following convergence properties [2, Theorem 1.4, Theorem 1.7] :

(C1) (Strong Fatou property) If $0 \leq f_n \uparrow f$ $\mu-$a.e, then either $f \notin B$ and $\|f_n\|_B \uparrow \infty$, or $f \in B$ and $\|f_n\|_B \uparrow \|f\|_B$.

(C2) (Fatou's lemma-lower semicontinuity of the $B-$norm) If $f_n \to f$ $\mu-$a.e. and if $\liminf_{n\to\infty} \|f_n\|_B < \infty$, then $f \in B$ and $\|f\|_B \leq \liminf_{n\to\infty} \|f_n\|_B$.

(C3) If $\sum_{n=1}^{\infty} \|f_n\|_B < \infty$, then the series $\sum_{n=1}^{\infty} f_n$ converges in B to a function $f \in B$ and $\|f\|_B \leq \sum_{n=1}^{\infty} \|f_n\|_B$.

(C4) If $f_n \to f$ in B, then a subsequence $(f_n)_{n\geq 1}$ converges pointwise $\mu-$a.e. to f.

It is known that $(B, \|\cdot\|_B)$ is a complete normed space, see [2, Theorem 1.6].

In the following, $(B, \|\cdot\|_B)$ is a Banach function space corresponding to a metric measure space (X, d, μ). We assume throughout the paper that μ is an outer regular Borel measure, positive and finite on balls.

Let Γ_{rec} be the family of all rectifiable curves in X. The $B-$*modulus of a family* Γ of curves in X is defined by

$$M_B(\Gamma) = \inf \|\rho\|_B \,,$$

where the infimum is taken over all Borel functions $\rho : X \to [0, +\infty]$ with

$$\int_\gamma \rho \, ds \geq 1 \tag{2.1}$$

for all $\gamma \in \Gamma_{rec}$.

If a nonnegative Borel function satisfies (2.1), it is said to be *admissible* for Γ. The family of all curves that are not rectifiable has zero $B-$modulus. It is said that a property holds for $B-$almost every curve if it holds for every curve except a family of curves of zero $B-$modulus. Several basic results regarding the modulus of a family of curves have been extended in [13] to this abstract setting.

In the definitions of Newtonian spaces and Orlicz-Sobolev spaces the substitute for the notion of length of the gradient is a notion of weak upper gradient.

Definition 2. *A Borel function $g : X \to [0, +\infty]$ is an upper gradient of a function $u : X \to \mathbb{R}$ if for every rectifiable path $\gamma : [0, 1] \to X$*

$$|u(\gamma(1)) - u(\gamma(0))| \leq \int_\gamma g ds. \tag{2.2}$$

Moreover, g is said to be a $B-$weak upper gradient ($B-$w.u.g.) of u if inequality (2.2) holds for $B-$ almost every compact rectifiable curve γ.

Denote by $\widetilde{N}^{1,B}(X)$ the collection of all real-valued functions $u \in B$ having a $B-$weak upper gradient $g \in B$. Then $\widetilde{N}^{1,B}(X)$ is a vector space. For $u \in \widetilde{N}^{1,B}(X)$ define

$$\|u\|_{1,B} = \|u\|_B + \inf_g \|g\|_B \,,$$

where the infimum is taken over all $B-$w.u.g. s $g \in B$ of u. Then $\|\cdot\|_{1,B}$ is a seminorm on $\widetilde{N}^{1,B}(X)$. The seminormed space $(\widetilde{N}^{1,B}(X), \|\cdot\|_{1,B})$ is turned into a normed space via the equivalence relation defined by: $u \sim v \Leftrightarrow \|u - v\|_{1,B} = 0$.

It follows that $N^{1,B}(X) = \widetilde{N}^{1,B}(X)/\sim$ is a vector space with the norm $\|u\|_{N^{1,B}} := \|u\|_{1,B}$.

If $B = L^p(X)$, $p \geq 1$ is a Lebesgue space or, more general, $B = L^\Psi(X)$ is an Orlicz space, then $N^{1,B}(X)$ is the Newtonian space introduced by Shanmugalingam [15], respectively the Orlicz-Sobolev space introduced by Tuominen [17].

We will use the following convergence result that is a substitute for Mazur's lemma in Sobolev-type spaces on metric measure spaces, generalizing Theorem 4.17 of [17].

Lemma 1. [13, Theorem 1] *Let $(u_j)_{j\geq 1}$ be a sequence of functions in B and $(g_j)_{j\geq 1}$ be a sequence in B of corresponding $B-$weak upper gradients. Assume that $u_j \to u$ and $g_j \to g$ weakly in B, for some $u, g \in B$. Then there are sequences $(\widetilde{u}_j)_{j\geq 1}$ and $(\widetilde{g}_j)_{j\geq 1}$ of convex combinations*

$$\widetilde{u}_j = \sum_{k=j}^{n_j} \lambda_{kj} u_k,\ \widetilde{g}_j = \sum_{k=j}^{n_j} \lambda_{kj} g_k,$$

where $\lambda_{kj} \geq 0$, $\sum_{k=j}^{n_j} \lambda_{kj} = 1$, such that $\widetilde{u}_j \to u$ and $\widetilde{g}_j \to g$ in B. In addition, g is a $B-$weak upper gradient of u.

The following lemma will be useful in building monotone sequences in $N^{1,B}(X)$. Its proof relies on the lattice properties of B and $N^{1,B}(X)$ and on the Fuglede lemma for $B-$modulus (see [13, Proposition 1, Remarks 2 and 3]).

Lemma 2. [13, Lemma 1] *Let $(u_i)_{i\geq 1}$ be a sequence in B, with corresponding $B-$upper gradients $(g_i)_{i\geq 1}$ in B, such that $(u_i(x))_{i\geq 1}$ is non-negative and bounded for each $x \in X$, $\sum_{i=1}^{\infty} \|u_i\|_B < \infty$ and $\sum_{i=1}^{\infty} \|g_i\|_B < \infty$. Define $v_j = \max_{1\leq i\leq j} u_i$, $w_i = \min_{1\leq i\leq j} u_i$, and $h_j = \max_{1\leq i\leq j} g_i$. Then:*

(a) For each $j \geq 1$, $v_j \in B$ has the $B-$w.u.g. $h_j \in B$;

(b) The pointwise limits $v := \lim_{j\to\infty} v_j$ and $w := \lim_{j\to\infty} w_j$ are well-defined and $v, w \in B$ with $\|w\|_B = 0$;

(c) The sequence $(h_j)_{j\geq 1}$ is convergent in B to some function $h \in B$ and $\|h\|_B \leq \lim_{j\to\infty} \|h_j\|_B$;

(d) h is a $B-$w.u.g. of v and w.

3. Continuity of $B-$capacity on increasing sequences of sets

We define a capacity with respect to the space $N^{1,B}(X)$, called *$B-$capacity.*

Definition 3. [13] *The $B-$capacity of a set $E \subset X$ is*

$$Cap_B(E) = \inf_{u \in \mathcal{A}(E)} \|u\|_{N^{1,B}},$$

where $\mathcal{A}(E) = \{u \in N^{1,B}(X) : u \geq 1 \text{ on } E\}$. The functions belonging to $\mathcal{A}(E)$ are said to be admissible for E.

The following properties of $B-$capacity have been proved in [13], generalizing Proposition 7.3 and Proposition7.4 of [17].

(a) $Cap_B(E) = \inf_{u \in \widetilde{\mathcal{A}}(E)} \|u\|_{N^{1,B}}$, where

$$\widetilde{\mathcal{A}}(E) = \{u \in A(E) : 0 \leq u \leq 1\}.$$

(b) $B-$capacity is an outer measure.

(c) $Cap_B(E) = 0$ if and only if $\mu(E) = 0$ and $M_B(\Gamma_E) = 0$.

(d) Two representatives of an equivalence class in $N^{1,B}(X)$ can differ only on a set of zero $B-$capacity. Conversely, modifying a function in $N^{1,B}(X)$ on a set of zero $B-$capacity we obtain a function in the same equivalence class in $N^{1,B}(X)$.

It is natural to look for assumptions on B and $N^{1,B}(X)$ under which $B-$capacity is a Choquet capacity. These assumptions have to be general enough to encompass the properties used in [18] in proving the Choquet properties of Orlicz-Sobolev capacity.

In the following we prove the continuity of $B-$capacity on increasing sequences of arbitrary subsets of X, provided that B is reflexive. Analogous continuity property have been proved in case $B = L^p(X)$ for Hajlasz-Sobolev capacity [10] and for relative Newtonian $p-$capacity [4]. The theorem below is new even for $B = L^{\Psi}(X)$.

Theorem 1. *If the Banach function space B is reflexive, then $B-$capacity is continuous on increasing sequences of arbitrary subsets of X:*

$$Cap_B\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i \to \infty} Cap_B(E_i),$$

whenever $E_1 \subset E_2 \subset ... \subset E_i \subset E_{i+1} \subset ... \subset X$.

Proof. Denote $E = \bigcup_{i=1}^{\infty} E_i$. The monotonicity of $B-$capacity implies $Cap_B(E) \geq L := \lim_{i\to\infty} Cap_B(E_i)$. In order to prove the reverse inequality, it suffices to assume that L is finite.

Let $\varepsilon > 0$. For each $i \geq 1$, let $u_i \in \widetilde{\mathcal{A}}(E_i)$ and let g_i be a $B-$w.u.g. of u_i, such that $\|u_i\|_B + \|g_i\|_B < Cap_B(E_i) + \varepsilon$. Then the sequences $(u_i)_{i\geq 1}$ and $(g_i)_{i\geq 1}$ are bounded in the normed space B. Since B is reflexive, we may assume, passing to subsequences, that $(u_i)_{i\geq 1}$ and $(g_i)_{i\geq 1}$ are weakly convergent in B to $u \in B$ and $g \in B$, respectively. By Mazur-type property Lemma 1, there are sequences $(\widetilde{u}_j)_{j\geq 1}$ and $(\widetilde{g}_j)_{j\geq 1}$ of convex combinations $\widetilde{u}_j = \sum_{k=j}^{n_j} \lambda_k u_k$, $\widetilde{g}_j = \sum_{k=j}^{n_j} \lambda_k g_k$,where $\lambda_k \geq 0$, $\sum_{k=j}^{n_j} \lambda_k = 1$, such that $\widetilde{u}_j \to u$ and $\widetilde{g}_j \to g$ in B. In addition, g is a weak $B-$w.u.g. of u.

We have

$$\|\widetilde{u}_j\|_B + \|\widetilde{g}_j\|_B \leq \sum_{k=j}^{n_j} \lambda_k \left(\|u_k\|_B + \|g_k\|_B\right) < \sum_{k=j}^{n_j} \lambda_k \left(Cap_B(E_k) + \varepsilon\right),$$

hence

$$\|\widetilde{u}_j\|_B + \|\widetilde{g}_j\|_B < L + \varepsilon \tag{3.1}$$

for all sufficiently large j.

Using the sequence $(\widetilde{u}_j)_{j\geq 1}$ we will build a function $w \in \widetilde{\mathcal{A}}(E)$ such that $\|w\|_{N^{1,B}} < L + \varepsilon$, using some ideas from the proof of [4, Theorem 3.2 (v)]. It follows that $Cap_B(E) < L + \varepsilon$, where $\varepsilon > 0$ is arbitrarly small, hence $Cap_B(E) \leq L$.

Passing to subsequences if necessary, we may assume that for every $i \geq 1$ the following inequality holds:

$$\|\widetilde{u}_{i+1} - \widetilde{u}_i\|_B + \|\widetilde{g}_{i+1} - \widetilde{g}_i\|_B < 2^{-i-1}\varepsilon. \tag{3.2}$$

Define $v_{j,k} = \sup_{j\leq i\leq k} \widetilde{u}_i$ for $1 \leq j \leq k$ and $w_j = \sup_{i\geq j} \widetilde{u}_i$ for $j \geq 1$. Then $w_j = \lim_{k\to\infty} v_{j,k} = \sup_{k\geq j} v_{j,k}$, where the convergence is pointwise.

Since $0 \leq u_i \leq 1$ on X for every $i \geq 1$, we have $0 \leq v_{j,k} \leq 1$ on X whenever $1 \leq j \leq k$, as well as $0 \leq w_j \leq 1$ on X for every $j \geq 1$. Fix $j \geq 1$. For every $j \leq k \leq n_j$ we have $u_k = 1$ on E_j, hence $\widetilde{u}_j(x) = 1$

on E_j.For every $x \in E$ there exists a positive integer $N = N(x)$ such that $x \in E_i$ for every $i \geq N$. For $j \geq N$ we have $w_j(x) \geq \widetilde{u}_N(x) = 1$, while for $j < N$ we have $w_j(x) \geq w_N(x) \geq 1$. It follows that $w_j = 1$ on E, for every $j \geq 1$.

By Lemma 2, a $B-$weak upper gradient of $v_{j.k}$ is $g_{j,k} := \sup\limits_{j \leq i \leq k} \widetilde{g}_i$, where $1 \leq j \leq k$.

Using the elementary inequality $\sup\limits_{j \leq i \leq k} a_i \leq a_j + \sum\limits_{i=j}^{k-1} |a_{i+1} - a_i|$, where a_i are non-negative reale numbers, for $1 \leq j \leq k$, we get

$$v_{j,k} \leq \widetilde{u}_j + \sum_{i=j}^{k-1} |\widetilde{u}_{i+1} - \widetilde{u}_i| \text{ on } X \tag{3.3}$$

and

$$g_{j,k} \leq \widetilde{g}_j + \sum_{i=j}^{k-1} |\widetilde{g}_{i+1} - \widetilde{g}_i| \text{ on } X.$$

Letting $k \to \infty$ in (3.3) we get $w_j \leq \widetilde{u}_j + \sum\limits_{i=j}^{\infty} |\widetilde{u}_{i+1} - \widetilde{u}_i|$ on X. Since $\|\widetilde{u}_j\|_B + \sum\limits_{i=j}^{\infty} \|\widetilde{u}_{i+1} - \widetilde{u}_i\|_B < +\infty$, the previous inequality shows that $w_j \in B$ and

$$\|w_j\|_B \leq \|\widetilde{u}_j\|_B + \sum_{i=j}^{\infty} \|\widetilde{u}_{i+1} - \widetilde{u}_i\|_B \,. \tag{3.4}$$

Define $G_j := \widetilde{g}_j + \sum\limits_{i=j}^{\infty} |\widetilde{g}_{i+1} - \widetilde{g}_i|$, $j \geq 1$. By (3.2), it follows that $G_j \in B$ for every $j \geq 1$ and

$$\|G_j\|_B \leq \|\widetilde{g}_j\|_B + \sum_{i=j}^{\infty} \|\widetilde{g}_{i+1} - \widetilde{g}_i\|_B \,. \tag{3.5}$$

We show that G_j is a $B-$weak upper gradient of w_j. There exists $\Gamma_0 \subset \Gamma_{rec}$ with $M_B(\Gamma_0) = 0$ such that for every $\gamma \in \Gamma_{rec} \setminus \Gamma_0$ and every $1 \leq j \leq k$, $|v_{j,k}(x) - v_{j,k}(y)| \leq \int\limits_\gamma g_{j,k} \, ds$, where x and y are the endpoints of γ. Then

$$|v_{j,k}(x) - v_{j,k}(y)| \leq \int_{\gamma} G_j ds$$

for every $\gamma \in \Gamma_{rec} \setminus \Gamma_0$ and all $k \geq j$. Letting k tend to infinity we get $|w_j(x) - w_j(y)| \leq \int_{\gamma} G_j ds$, for every $\gamma \in \Gamma_{rec} \setminus \Gamma_0$, hence G_j is a $B-$weak upper gradient of w_j.

Since $0 \leq w_j \leq 1$ on X, $w_j = 1$ on E, $w_j \in B$ and w_j has a $B-$weak upper gradient of w_j, it follows that $w_j \in \widetilde{\mathcal{A}}(E)$ and

$$Cap_B(E) \leq \|w_j\|_{N^{1,B}} \leq \|w_j\|_B + \|G_j\|_B .$$

By (3.4), (3.5), (3.2) and (3.1) we obtain

$$\|w_j\|_B + \|G_j\|_B \leq L + \varepsilon\left(1 + 2^{-j}\right).$$

for every $j \geq 1$.

Then $Cap_B(E) \leq L + \varepsilon\left(1 + 2^{-j}\right)$ for every $j \geq 1$ and all $\varepsilon > 0$. Letting j tend to infinity, then letting ε tend to zero, we get $Cap_B(E) \leq L$, q.e.d. ∎

Remark 1. *The Orlicz space $L^{\Psi}(X)$ is reflexive if and only if Ψ satisfies the growth conditions Δ_2 and ∇_2 simultaneously* [2]. *Recall that an Young function $\Psi : [0, \infty) \to [0, \infty)$ is said to satisfy*

a) the Δ_2-condition if there is a constant $C_2 > 0$ such that

$$\Psi(2t) \leq C_2 \Psi(t) \tag{3.6}$$

for every $t \geq 0$;

b) *the ∇_2-condition if there is a constant $C > 1$ such that* $\Psi(t) \leq \frac{1}{2C}\Psi(Ct)$ *for every $t \geq 0$.*[14], [17].

4. Outer regularity of $B-$capacity and consequences

The purpose of this section is to establish some assumptions on B and X, that imply the outer regularity of $B-$capacity and that are fulfilled in the case $B = L^p(X)$.

Definition 4. *The $B-$capacity is said to be outer regular (in other words, $B-$capacity is an outer capacity) if*

$$Cap_B(E) = \inf\{Cap(O) : O \ open, E \subset O \subset X\},$$

for every set $E \subset X$.

We need the notion of Banach function space with absolutely continuous norm.

Definition 5. [2, Lemma 3.4] *A function* f *in the Banach function space* B *is said to have absolutely continuous norm if* $\left\|f\chi_{E_n}\right\|_B \to 0$ *for every sequence* $E_n \subset X$, $n \geq 1$ *of measurable sets such that* $\limsup\limits_{n\to\infty} E_n$ *has measure zero. The Banach function space* B *is said to have absolutely continuous norm if each* $f \in B$ *has absolutely continuous norm.*

Remark 2. *Moreover, If* f *has absolutely continuous norm, then for every* $\varepsilon > 0$ *there is* $\delta(\varepsilon) > 0$ *such that* $\mu(A) < \delta(\varepsilon)$ *implies* $\|f\chi_A\|_B < \varepsilon$, *by* [2, Lemma 3.4],

Lemma 3. *If the Young function* Ψ *is doubling and* Ψ *is continuous at origin, then the Orlicz space* $L^\Psi(X)$ *has absolutely continuous norm. Moreover, if* Ψ *is doubling, then for each* $f \in L^\Psi(X)$ *and every* $\varepsilon > 0$ *there is* $\delta(\varepsilon) > 0$ *such that* $\mu(A) < \delta(\varepsilon)$ *implies* $\|f\chi_A\|_B < \varepsilon$.

Proof. Let Ψ be a doubling Young function. Let $f \in L^\Psi(X)$. Since Ψ is doubling, $\Psi(|f|) \in L^1(X)$.

Assume that $E_n \subset X$, $n \geq 1$ are measurable sets such that $\mu\left(\limsup\limits_{n\to\infty} E_n\right) = 0$. Then $\lim\limits_{n\to\infty} \chi_{E_n} = 0$ $\mu-$a.e., hence $\lim\limits_{n\to\infty} f\chi_{E_n} = 0$ $\mu-$a.e. Assuming that Ψ is continuous at origin,we get $\lim\limits_{n\to\infty} \Psi\left(\left|f\chi_{E_n}\right|\right) = 0$ $\mu-$a.e. But $0 \leq \Psi\left(\left|f\chi_{E_n}\right|\right) \leq \Psi(|f|)$ for every $n \geq 1$, hence $\lim\limits_{n\to\infty} \int\limits_X \Psi\left(\left|f\chi_{E_n}\right|\right) d\mu = 0$, by Lebesgue dominated convergence theorem. In every Orlicz space $L^\Psi(X)$, the convergence with respect to the Luxemburg norm $\|\cdot\|_{L^\Psi(X)}$ implies the convergence in $\Psi-$mean and for Ψ doubling the converse also holds. Then $\left\|f\chi_{E_n}\right\|_{L^\Psi(X)} \to 0$, which proves that f has absolutely continuous norm.

The following estimate follows from [18, Lemma 4], where Ψ satisfies (3.6) and $u \in L^{\Psi}(U)$, $\|u\|_{L^{\Psi}(U)} > 0$ and $I := \int\limits_U \Psi(|u|)\, d\mu$:

$$\|u\|_{L^{\Psi}(U)} \leq \max\{I, (C_2 I)^{1/\log_c C_2}\}.$$

Let $\varepsilon > 0$. Since $\Psi(|f|) \in L^1(X)$, it follows by the absolute continuity of Lebesgue integral that there exists $\delta(\varepsilon) > 0$ such that $\mu(A) < \delta(\varepsilon)$ implies $\int\limits_A \Psi(|f|)\, d\mu < \min\{\varepsilon, C_2^{-1}\varepsilon^{\log_2 C_2}\}$. Hence, by the above estimate, $\mu(A) < \delta(\varepsilon)$ implies $\|f\chi_A\|_{L^{\Psi}(X)} = \|f\|_{L^{\Psi}(A)} < \varepsilon$. ∎

Definition 6. *We say that a Banach function space B has the Vitali-Carathéodory property if every function in B is majorated by a lower semicontinuous function belonging to B: for every $f \in B$ there exists a semicontinuous function $g \in B$ such that $f \leq g$.*

Remark 3. *It is well-known that $L^1(X)$ has the Vitali-Carathéodory property if X is a Hausdorff locally compact topological space. Assume that X is a locally compact metric space, endowed with a Borel measure non-trivial and finite on balls. Then $L^p(X)$ has the Vitali-Carathéodory property for every p with $1 \leq p < \infty$,(see* [8, Lemma 2.3]*). Moreover, if Ψ is a strictly increasing, continuous and doubling Young function, then $L^{\Psi}(X)$ has the Vitali-Carathéodory property (see* [18]*, proof of Lemma 3).*

First we prove the outer regularity of $B-$capacity for sets of $B-$capacity zero, generalizing [18, Lemma 3].

Lemma 4. *Assume that the metric space X is proper (i.e. all closed balls in X are compact). Assume that B has absolutely continuous norm and B has Vitali-Carathéodory property. If $E \subset X$ has $B-$capacity zero, then*

$$Cap_B(E) = \inf\{Cap_B(O) : O \text{ open }, E \subset O \subset X\}.$$

Proof. Let $E \subset X$ with $cap_B(E) = 0$. Consider $\varepsilon > 0$. We prove that there exists an open set O so that $cap_B(O) < \varepsilon$. By Proposition 1, $cap_B(E) = 0$ implies $\mu(E) = 0$ and $M_B(\Gamma_E) = 0$, therefore the characteristic function χ_E of E admits the zero function as a $B-$weak upper gradient. There exists an upper gradient $g \in B$ of χ_E.Since B

has the Vitali-Carathéodory property, there exists a lower semicontinuous function $v \in B$ with $g \leq v$. Let $k > 0$ be a constant and let $\rho := v + k$. Then ρ is lower semicontinuous, $\rho > g$ and $\rho\chi_A \in B$ for every bounded measurable set $A \subset X$.

Assume first that E is bounded. Using Remark 2 and the outer regularity of the measure μ we find a bounded open set V such that $E \subset V$ and

$$\|\chi_V\|_B + \|\rho\chi_V\|_B < \varepsilon/2.$$

Define as in the proof of [18, Lemma 3] the function

$u(x) = 2\min\left\{1, \int\limits_\gamma \rho\, ds\right\}$, $x \in X$, where the infimum is taken over all rectifiable paths connecting x to $X \setminus V$. By [3, Lemma 3.4], u is lower semicontinuous, $2\rho\chi_V$ is an upper gradient of u on X and $u = 0$ on $X \setminus V$. It follows that $u \in N^{1,B}(X)$ and the set $U := \{x \in X : u(x) > 1\}$ is open. By the proof of [18, Lemma 3], $u = 2$ on E, hence $E \subset U$. Then we obtain $cap_B(U) \leq \|u\|_B + \|2\rho\chi_V\|_B$. The definition of u implies $0 \leq u \leq 2\chi_V$, hence $\|u\|_B \leq \|2\chi_V\|_B$ by the monotonicity property (P2) of the $B-$norm. It follows that $cap_B(U) \leq 2\left(\|\chi_V\|_B + \|\rho\chi_V\|_B\right)$, therefore $cap_B(U) < \varepsilon$.

If E is unbounded, fix $x_0 \in X$ and let $E_n := E \cap B(x_0, n)$ for each positive integer n. Then $E = \bigcup\limits_{n=1}^{\infty} E_n$ and $cap_B(E_n) = 0$, $n \geq 1$. By the first part of the proof, for each $n \geq 1$ there exists an open set $U_n \subset X$, with $E_n \subset U_n$ such that $cap_B(U_n) < \frac{\varepsilon}{2^{n+1}}$. Then $U_0 := \bigcup\limits_{n=1}^{\infty} U_n$ is an open set containing E with $cap_B(U_0) \leq \sum\limits_{n=1}^{\infty} cap_B(U_n) \leq \sum\limits_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2} < \varepsilon$. ∎

We need the following characterization of the outer regularity of $B-$capacity, that connects this property to the quasicontinuity of functions belonging to $N^{1,B}(X)$.

Lemma 5. [13, Theorem 3] *Assume that (X, d, μ) is a metric measure space, where μ is a Borel regular outer measure, positive and* *finite on balls, and let B be a Banach function space on X. If continuous functions are dense in $N^{1,B}(X)$, then the following properties are equivalent:*

(1) Every function in $N^{1,B}(X)$ is $B-$quasicontinuous (continuous on complements of open sets of arbitrarily small capacity);
(2) $Cap_B(E) = \inf\{Cap_B(U) : U$ open$, E \subset U \subset X\}$ for every $E \subset X$ such that $Cap_B(E) = 0$;
(3) $Cap_B(F) = \inf\{Cap_B(G) : G$ open$, F \subset G \subset X\}$ for every $F \subset X$.

The last two lemmas imply the following

Theorem 2. *Let (X, d) be a proper metric endowed with a Borel regular outer measure μ, which is positive and finite on balls. Assume that B is a Banach function space that has absolutely continuous norm and Vitali-Carathéodory property. Moreover, assume that continuous functions are dense in $N^{1,B}(X)$. Then:*

(1) *Every function in $N^{1,B}(X)$ is $B-$quasicontinuous.*
(2) *$B-$capacity is outer regular, that is*
$Cap_B(E) = \inf\{Cap(O) : O$ *open,* $E \subset O \subset X\}$ *for every* $E \subset X$.

Remark 4. *Applying the above theorem to $B = L^{\Psi}(X)$, where Ψ is a doubling Young function, we recover some recent results of* [18], *Theorem 1 and Corollary 1 (1).*

Next we prove two important consequences of the outer regularity of $B-$capacity: the continuity of $B-$capacity on decreasing sequences of compact subsets and the coincidence of two types of $B-$capacity.

Proposition 1. *Assume that (X, d, μ) is a metric measure space, where μ is a Borel regular outer measure, positive and finite on balls, and let B be a Banach function space on X. If $B-$capacity is outer regular, then for every decreasing sequence $(K_i)_{i\geq 1}$ of compact subsets of X we have $Cap_B\left(\bigcap_{i=1}^{\infty} K_i\right) = \lim_{i\to\infty} Cap_B(K_i)$.*

Proof. Let $(K_i)_{i\geq 1}$ be a decreasing sequence of compact subsets of X. Denote $K = \bigcap_{i=1}^{\infty} K_i$. By the monotonicity of $B-$capacity, $Cap_B(K) \leq \lim_{i\to\infty} Cap_B(K_i)$.

Let $O \subset X$ be an open set containing K. By compactness, we have $K_i \subset O$ for all sufficiently large i. The monotonicity of $B-$capacity yields $\lim_{i\to\infty} Cap_B(K_i) \leq Cap_B(O)$. Taking the infimum over all open sets $O \subset X$ containing K and using the outer regularity of $B-$capacity, we get $\lim_{i\to\infty} Cap_B(K_i) \leq Cap_B(K)$. ∎

Remark 5. *The above proposition generalizes Corollary 1 (2) from* [18].

Besides the $B-$capacity defined by $Cap_B(E) = \inf\{\|u\|_{N^{1,B}} : u \in N^{1,B}(X) : u \geq 1 \text{ on } E\}$ we introduce, following the definitions of Sobolev capacity from [9], [10], [4] the following alternative version of $B-$capacity:

$$Cap_B^*(E) = \inf\{\|u\|_{N^{1,B}} : u \in N^{1,B}(X) : u \geq 1 \text{ on a neighborhood of } E\}.$$

Denote $\mathcal{A}^*(E) = \{u \in N^{1,B}(X) : u \geq 1 \text{ on a neighborhood of } E\}$.

Since $\mathcal{A}^*(E) \subset \mathcal{A}(E)$, we have $Cap_B(E) \leq Cap_B^*(E)$ for every $E \subset X$.

Remark 6. *Cap_B^* is monotonically increasing. Outer regularity of Cap_B^* is a straightforward consequence of its definition. Obviouosly, $Cap_B(O) = Cap_B^*(O)$ for every open set $O \subset X$.*

If $Cap_B(E) = Cap_B^*(E)$ for every $E \subset X$, then Cap_B is outer regular. Conversely, we have

Proposition 2. *If the $B-$capacity Cap_B is outer regular, then $Cap_B(E) = Cap_B^*(E)$ for every $E \subset X$.*

Proof. Let$E \subset X$. Assume that Cap_B is outer regular. We have to prove that $Cap_B^*(E) \leq Cap_B(E)$.

Let $\varepsilon > 0$. There exist $O \subset X$ open such that $E \subset O$ and $Cap_B(O) < Cap_B(E) + \varepsilon$. Since

$$Cap_B^*(E) \leq Cap_B^*(O) = Cap_B(O) < Cap_B(E) + \varepsilon,$$

we have $Cap_B^*(E) < Cap_B(E) + \varepsilon$. Letting ε tend to zero, we get $Cap_B^*(E) \leq Cap_B(E)$, q.e.d. ∎

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University "Vasile Alecsandri" of Bacău, Department of Mathematics and Informatics, Spiru Haret 8, Bacău 600114, Romania
E-mail address: `mmocanu@ub.ro`